MAT 132 Calculus II

Final Exam Topics

This is the description of problems that may appear on the final exam. (The actual practice questions can be found in the textbook or past homework or exams.) The goal of this list is to help you to focus on essential skills and to streamline your exam preparation. The list below is quite large; not all questions below will be included, those that will be included may contain fewer parts, and the questions and their parts might be arranged differently; for example, there may be a true-false question based on these topics. Topics not listed below will not be included in the test.

1. Determine whether a given sequence

- is increasing, decreasing, or neither (for simple sequences only; you must understand the concept, but calculations will not be required)
- converges or diverges
- if converges, find limit

2. Determine whether a given series $\sum a_n$ is convergent or divergent. Use one of the following convergence tests (listed here for your reviewing convenience, won't be listed on exam).

- divergence test
- alternating series test
- ratio test
- comparison/limit comparison
- integral test
- is it one of the standard series: geometric, p-series?

3. For a given power series

- find the radius of convergence
- indicate on the x-axis where the series converges, and where it diverges
- indicate points that require further testing
- check whether the series converges at those points (or at some other points)

4. For a given function, find the Taylor series centered at a given point (say at $x = 0$ or $x = 3$). You may be asked to use any method, or specifically one of the following.

- use the Taylor series formula
- use algebraic manipulations with geometric series

5. Use series to

- compute limits
- express an integral as series

6. Compute given integrals (definite or indefinite). Use the following methods (listed for your reviewing convenience; no hints will be given on the test, except perhaps for trigonometry).

- substitution
  (including trig sub such as $x = \sin u$). If trig sub is on the test, hints and useful trig formulas will be provided.
- integration by parts
  (applies to functions such as $x^2 \cos 3x$, $x^3 e^{-x}$, $e^{-x} \sin x$ ("boomerang"), inverse functions such as $\ln x$, $\tan^{-1} x$)
- partial fractions
  (for functions like $\dfrac{x-1}{x^2-5x+6}$ where the denominator factors, or $\dfrac{x-1}{x^2+4}$ or $\dfrac{5}{x^2+9}$ where it doesn't)
- trig identities for working with trig integrals such as $\int \sin^3 x\,dx$ (relevant formulas and hints will be given)

7. Determine whether a given improper integral ($\int_0^b f(x)\,dx$ or $\int_a^\infty f(x)\,dx$) is convergent or divergent. Use one the following methods (or any of them).

- compute $\int_a^b f(x)\,dx$ and take the limit
- use comparison

8. Areas and arclength

- Sketch the region enclosed by two given graphs and given horizontal/vertical lines. Express its area as an integral and compute it
- Express the length of a given curve as an integral.

9. Compute the volume of a given solid of revolution, or of some other solid such as a pyramid.

- make a sketch of the solid
- decide how it should be sliced, illustrate on the picture
- use the slices to express the volume as an integral
- compute the integral to find volume

10. Given a differential equation,

- determine whether a given function is a solution
- determine whether a given function is a solution for a given initial value problem
- for a family of functions (such as $\cos kx$), determine which values of the parameter k give solutions

11. Given a direction field corresponding to a differential equation

- sketch the graph of solution of the given initial value problem (say $y(0) = 1$)
- determine whether the solution with, say, $y(0) = 0$ is increasing, decreasing, or neither; find all initial values that give solutions that increase/decrease
- find all equilibrium solutions

- given several equations and several direction fields, determine which field corresponds to which equation

You will not be asked to sketch direction fields on the test, but you must be able to work with them.

12. Given a differential equation of the form $y' = f(x, y)$, use the Euler method with the given number of steps to approximate a solution $y(x)$ with given initial value. (If this question is on the test, only very simple formulas will be given, and the Euler method will be done with very few steps. As calculators are not allowed, calculations will be easy, but you have to understand the meaning of the Euler method.)

13. Given a separable equation such as, say, $y' = (x + 1)(y^2 - 4)$ or $y' = y^2 - 2y$,

- find all equilibrium solutions
- find the general solution for the equation
- find the solution of the given initial value problem

14. For a given second-order differential equation of the form $ay'' + by' + cy = 0$

- write the characteristic equation
- solve the characteristic equation (are the roots real or complex? is it two different roots or a repeated root?)
- find the general solution of the differential equation
- find the solution for a given initial value problem, such as $y(0) = 2, y'(0) = 1$.

Midterm I questions

1. **(a)** Let $a_n = \dfrac{2+3n}{2n+1}$.

Does the sequence $\{a_n\}$ converge? If it does, find the limit $\lim_{n\mapsto\infty} a_n$.
Is the series $\sum_{n=0}^{\infty} a_n$ convergent or divergent? Why?

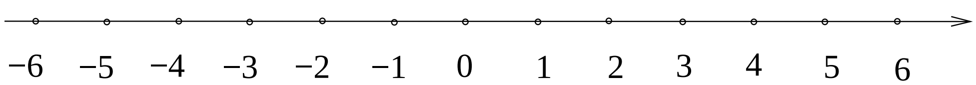
Is the series $\sum_{n=0}^{\infty}(-1)^n a_n$ convergent or divergent? Why?

(b) Let $b_n = \dfrac{n^2+2n}{n^3}$.
Does the sequence $\{b_n\}$ converge? If it does, find the limit $\lim_{n\mapsto\infty} a_n$.
Is the series $\sum_{n=0}^{\infty} b_n$ convergent or divergent? Why?

2. **(a)** Find the radius of convergence for the series

$$\sum_{n=0}^{\infty} \frac{(x-2)^n}{3^n(1+3n)}.$$

Indicate on the axis below where the series converges.
Indicate where the series diverges.
Put question marks over the points which require more testing.



(b) Is this series convergent for $x = 3$? For $x = -3$? Why?

(c) Is this series convergent for $x = -1$? Why?

3. Find the Taylor series decomposition for $f(x) = 1/x$ around $a = -2$. Use any method.

4. **(a)** Compute the limit $\displaystyle\lim_{x\mapsto 0} \frac{1-2x^2-\cos 2x}{x^4}$

(b) Express the indefinite integral $\displaystyle\int \frac{1-x^2-\cos 2x}{x^4}\,dx$ as a power series.

5. Please circle **ALL** correct answers for each of the following questions. You do not have to show any work or explain your reasoning. There may be several correct answers for some of the questions.

(a) The sequence

$$3, 2, 1, -1, -2, -3, \frac{1}{2}, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{5}, \ldots, \frac{1}{n}, -\frac{1}{n}, \ldots,$$

(A) decreases (B) increases (C) converges to 0 (D) diverges

(b) The Ratio Test can detect convergence/divergence for the series

(A) $\sum_{n=1}^{\infty} \frac{n-1}{n+1}$ (B) $\sum_{n=1}^{\infty} \frac{1}{2^n}$ (C) $\sum_{n=1}^{\infty} \frac{1}{n}$

(c) The series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

(A) diverges because $\frac{1}{\sqrt{n}} > \frac{1}{n}$, and the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

(B) converges because $\frac{1}{\sqrt{n}} = \frac{1}{n^{1/2}}$, and the series $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ converges.

(C) diverges because $\frac{1}{\sqrt{n}} > \frac{1}{n^2}$, and the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

(D) converges because $\frac{1}{\sqrt{n}} > \frac{1}{n^2}$, and the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

(d) The series $5x - \frac{5^3x^3}{3!} + \frac{5^5x^5}{5!} - \frac{5^7x^7}{7!} + \ldots$ is the Maclaurin series for

(A) $\sin 5x$ (B) $5 \sin x$ (C) $\sin^5 x$ (D) none of them

Midterm II questions

1. Compute the following integrals.

(a) $\displaystyle\int \frac{4x\,dx}{x^2+2x-3}$ **(b)** $\displaystyle\int e^{-x}\sin x\,dx$ **(c)** $\displaystyle\int_0^1 \frac{e^y+1}{e^y+y}\,dy$ **(d)** $\displaystyle\int_0^{1/2} \sin^{-1} x\,dx$

2. **(a)** Determine whether the integral $\displaystyle\int_0^{+\infty} xe^{-x^2}\,dx$ is convergent or divergent. If convergent, evaluate the integral.

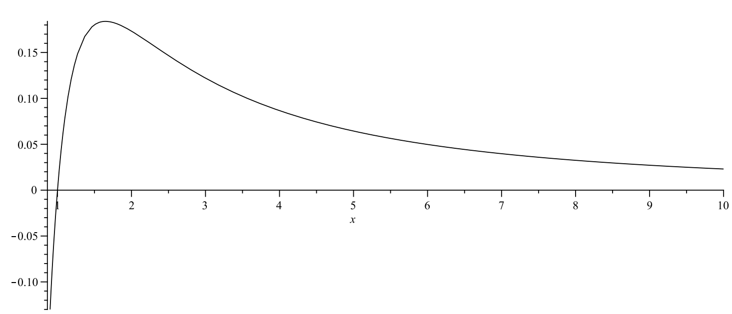
(b) Determine whether the integral $\displaystyle\int_1^{\infty} \frac{1+\sin^4 x}{x^4+5x^3+2}$ is convergent or divergent. You do not have to evaluate the integral.

(c) Determine whether the integral $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}+x^2}$ is convergent or divergent. You do not have to evaluate the integral.

3. Compute the following two integrals using any method. (There are several solutions.) It is possible, although not necessary, to use a trigonometric substitution, $u = 3\sin x$ or $u = 3\cos x$. If you choose this strategy, some trig formulas from the reference page will be useful. *You do not have to use trig substitution if you can solve the problem by other means.*

(a) $\displaystyle\int_0^3 \sqrt{9-x^2}\,dx$ **(b)** $\displaystyle\int_0^3 x\sqrt{9-x^2}\,dx$

4. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ converges or diverges.
Use the Integral test.
The graph of the function $f(x) = \dfrac{\ln x}{x^2}$ is given below.
Draw a picture representing the sum of the series on this diagram.
Explain how the Integral test works for this example.



5. Consider the region of the (x, y)-plane enclosed by the curves $y = \sin \pi x$ and $y = x^2 - x$.
Make a clear sketch of this region. Label all relevant points.
Compute the area of this region.