Q: Why do we require that the function $f(x)$ is <u>decreasing</u> in order to use the Integral Test?

A: So that we do not run into the following problem:

Consider the series $\sum_{n=1}^{\infty} \frac{|\sin n\pi|}{n}$

Since $\sin n\pi = 0$ for n - integer, this is just a fancy way of writing:

$$\sum_{n=1}^{\infty} \frac{|\sin n\pi|}{n} = \sum_{n=1}^{\infty} \frac{0}{n} = \sum_{n=1}^{\infty} 0 = 0 \qquad \text{(Conv)}$$

which is, of course, convergent.

But if we look at the corresponding integral, than we should look at

$$\int_1^{\infty} \frac{|\sin \pi x|}{x}\,dx$$

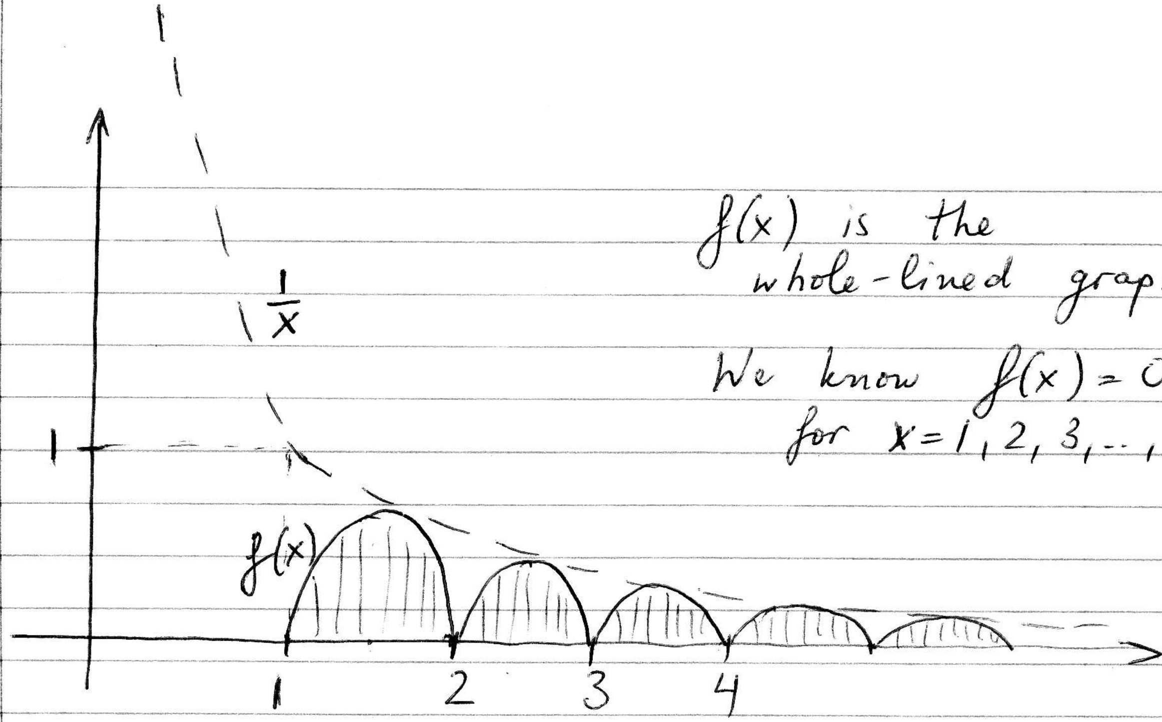
(we have $f(n) = \frac{|\sin n\pi|}{n} \Rightarrow$

$f(x) = \frac{|\sin x\pi|}{x}$

Notice that of course if x is <u>not</u> integer, than $\sin \pi x$ is <u>not</u> 0.)

Also, $f(x)$ is <u>not</u> decreasing, in fact it looks like:

$\nearrow$



$f(x)$ is the whole-lined graph.

We know $f(x)=0$ for $x=1,2,3,\dots,n,\dots$

The integral $\int_1^\infty \frac{|\sin \pi x|}{x}\,dx = \int_1^\infty f(x)\,dx$

is the area under the graph of $f(x)$.

It takes some calculation, but one can check that $\int_1^\infty \frac{|\sin \pi x|}{x}\,dx = +\infty$,

i.e., the integral is Divergent.

So, because $f(x)$ is <u>not</u> decreasing, it so happened that $\int_1^\infty f(x)\,dx = \text{Div}$, but $\sum_{n=1}^{\infty} f(n)$ is Convergent.

In other words, in order to apply the Integral Test, <u>check</u> if $f(x)$ is decreasing.