Products of two symmetries of order two

Oleg Viro

September 16, 2015

Plane Isometries

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Lemma. *A plane isometry is determined by its restriction to any three non-collinear points.* $\square$

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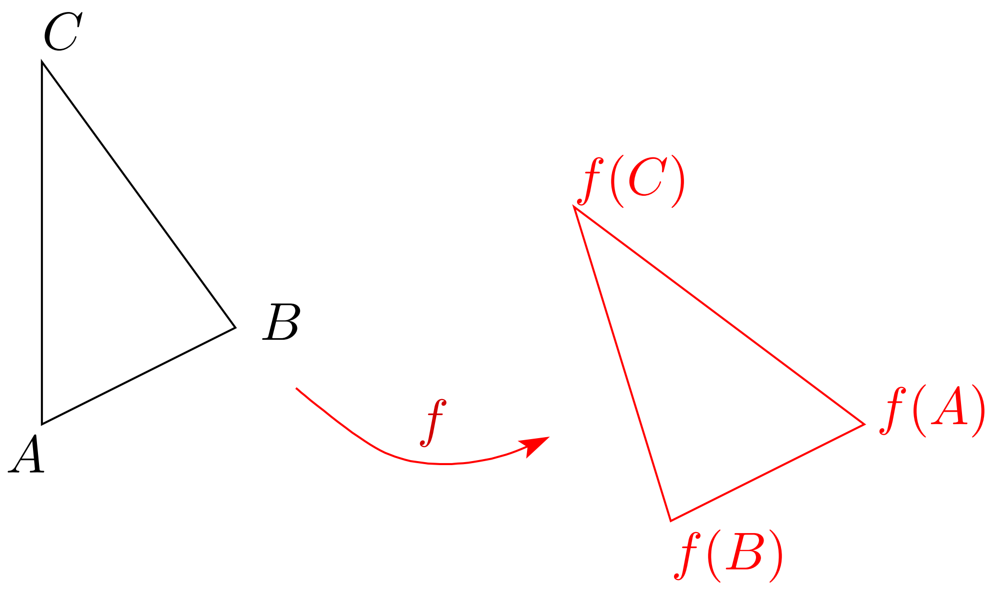
Proof of Theorem. Given an isometry:



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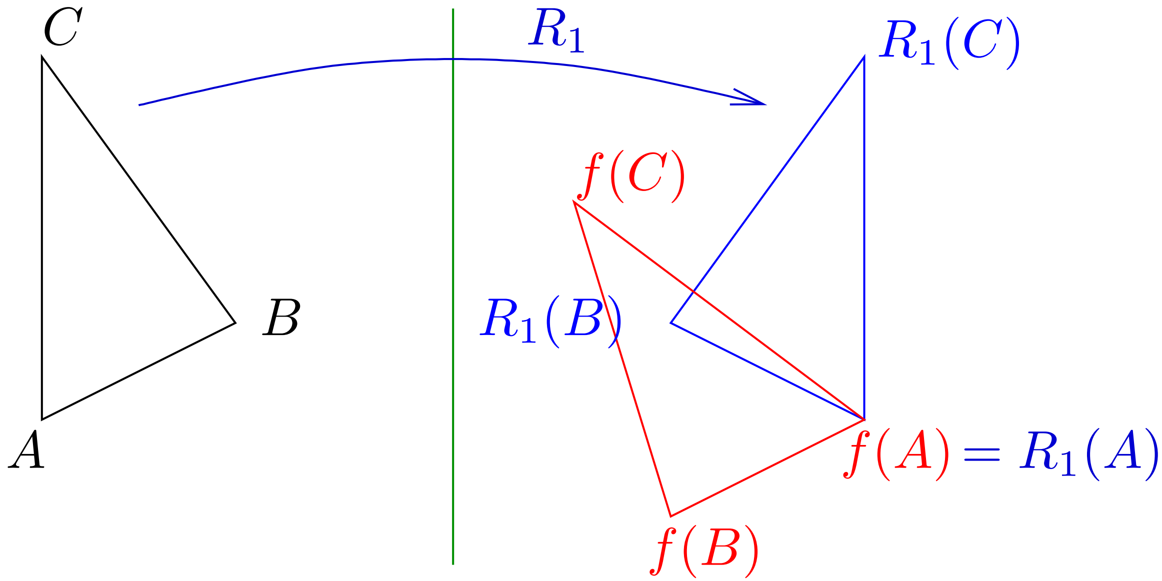

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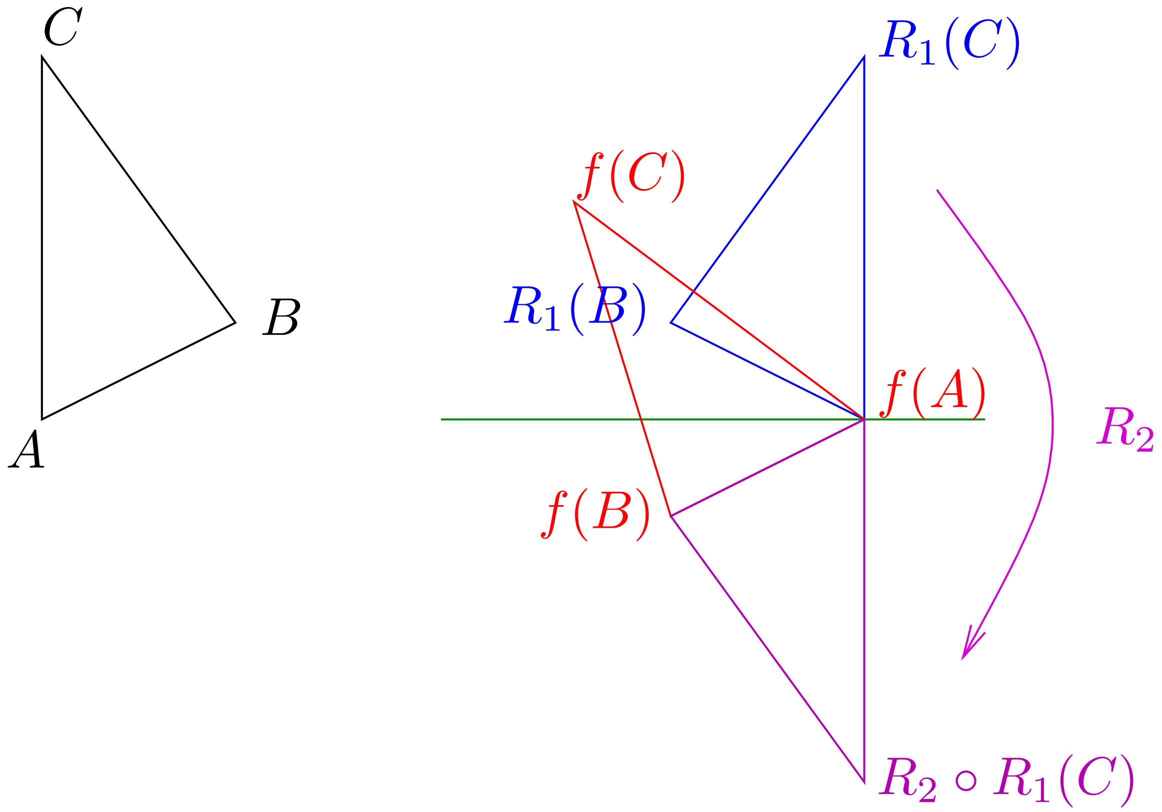

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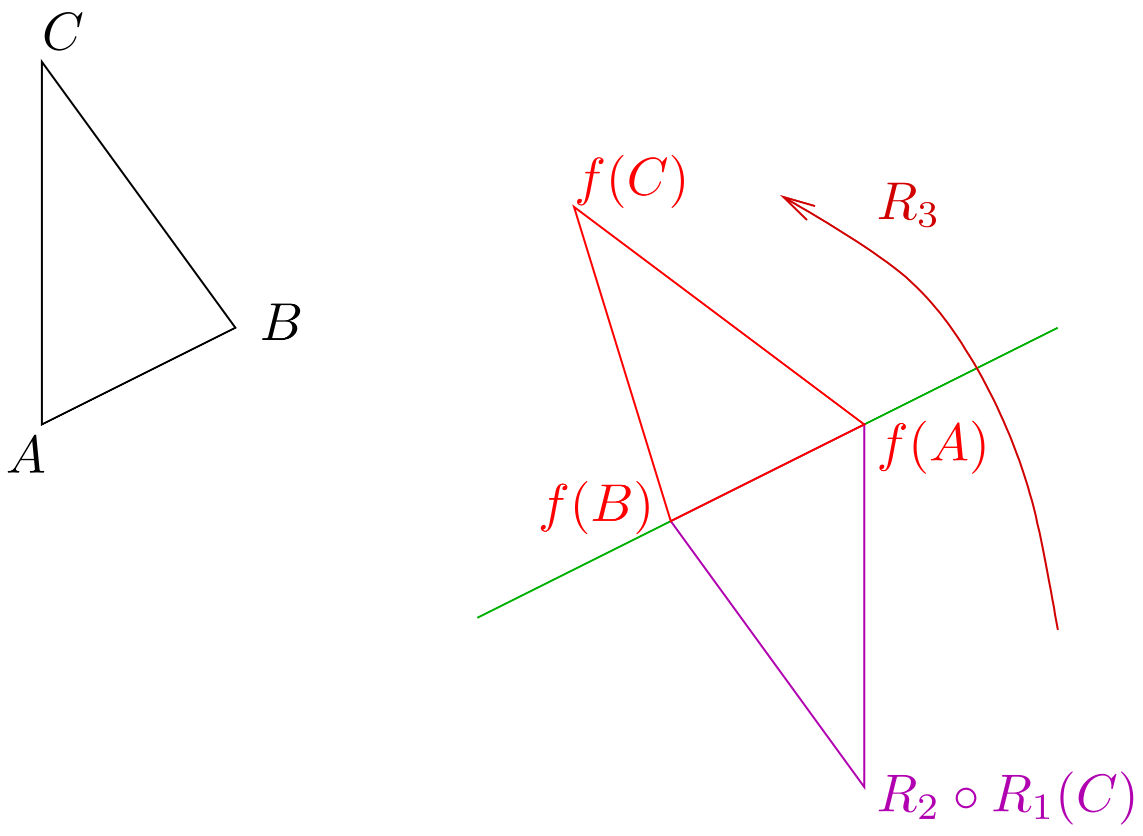

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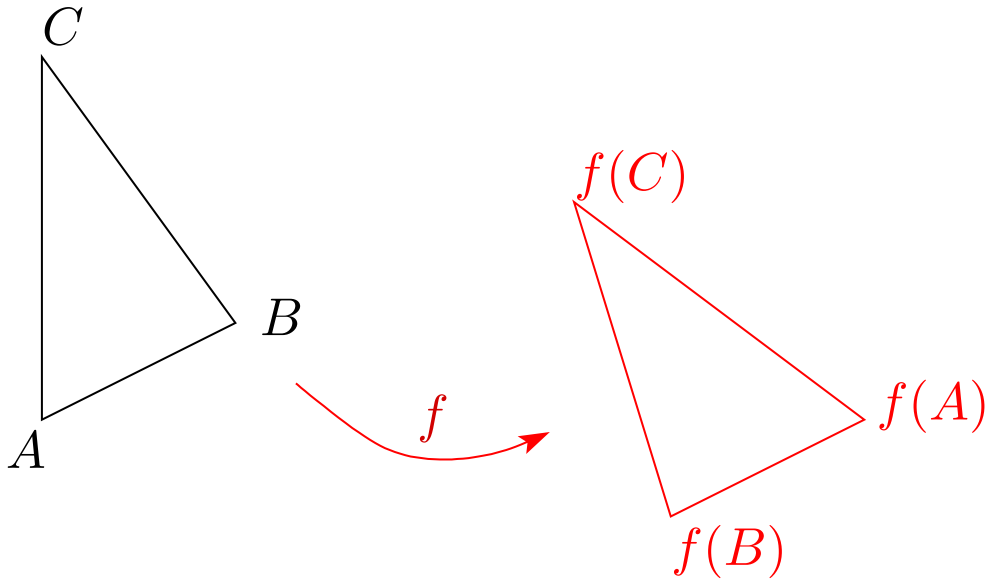
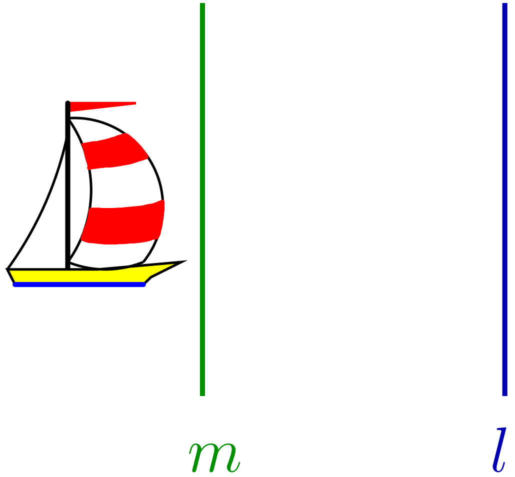


We are done. $\square$

Compositions of reflections in parallel lines



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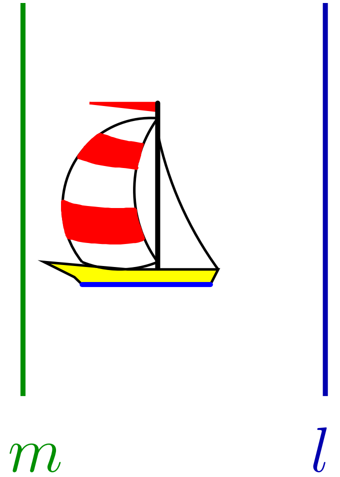

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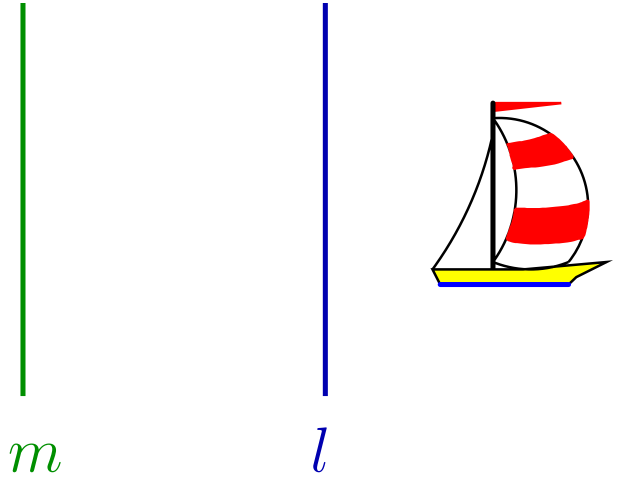

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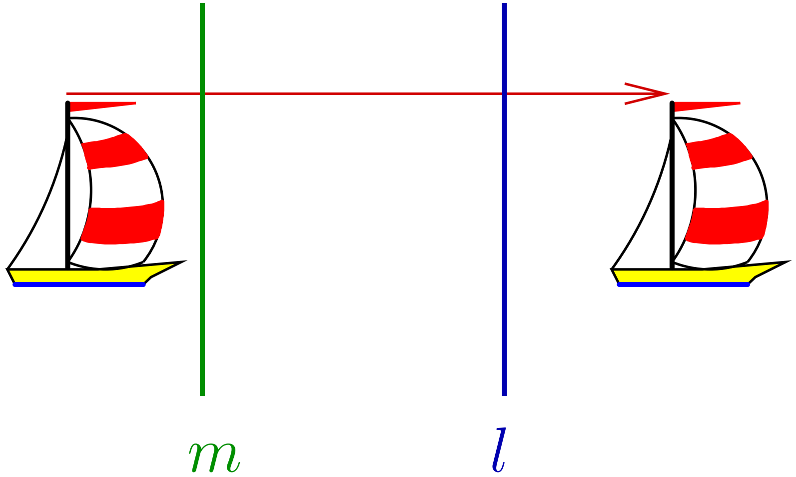

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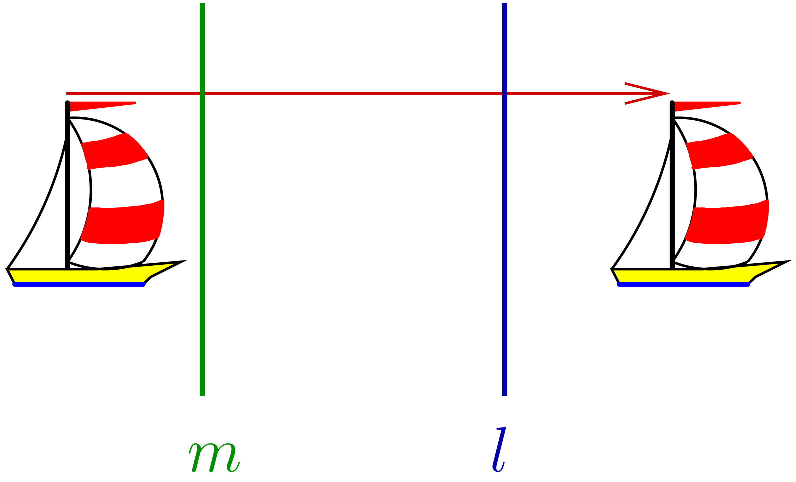

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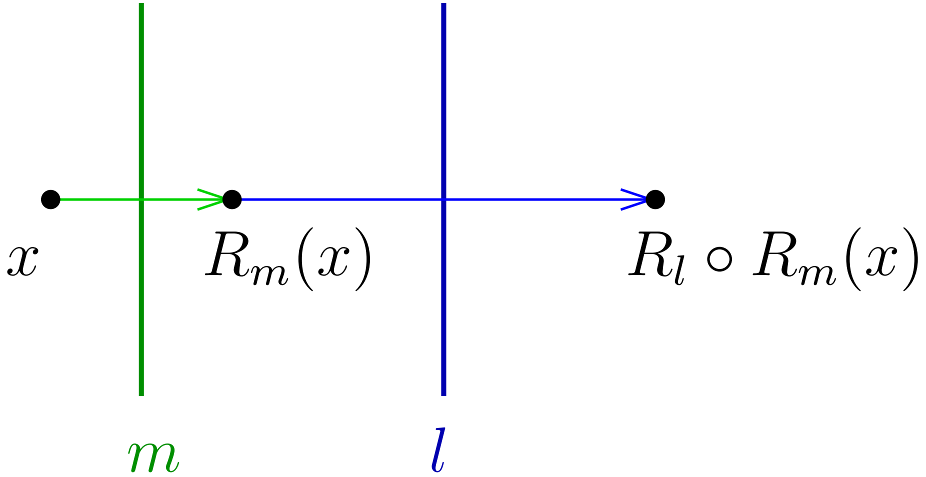

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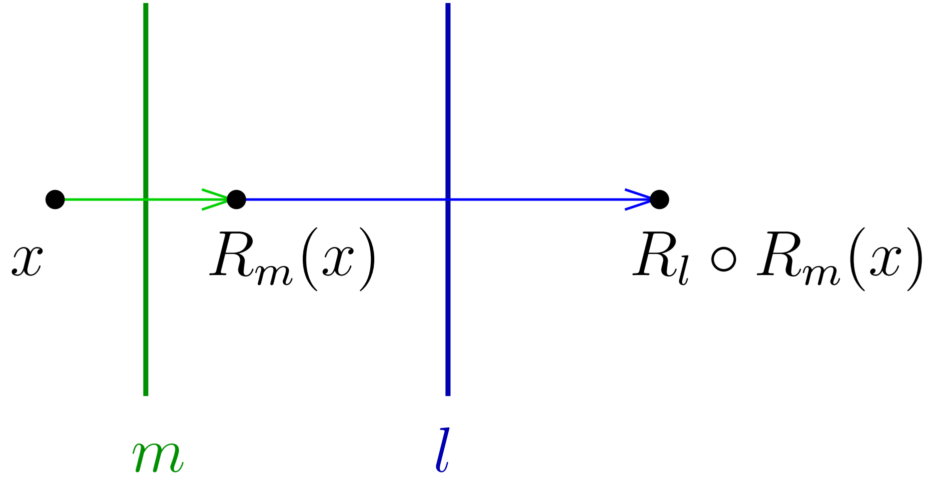

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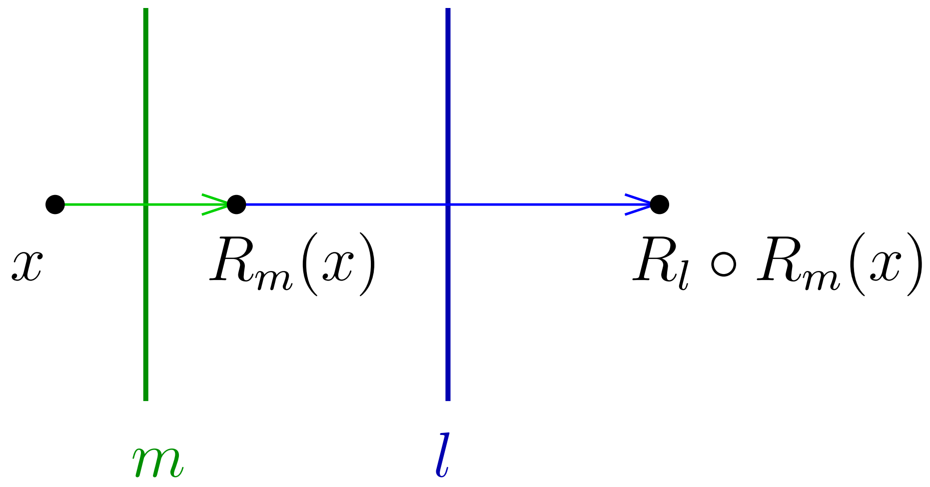

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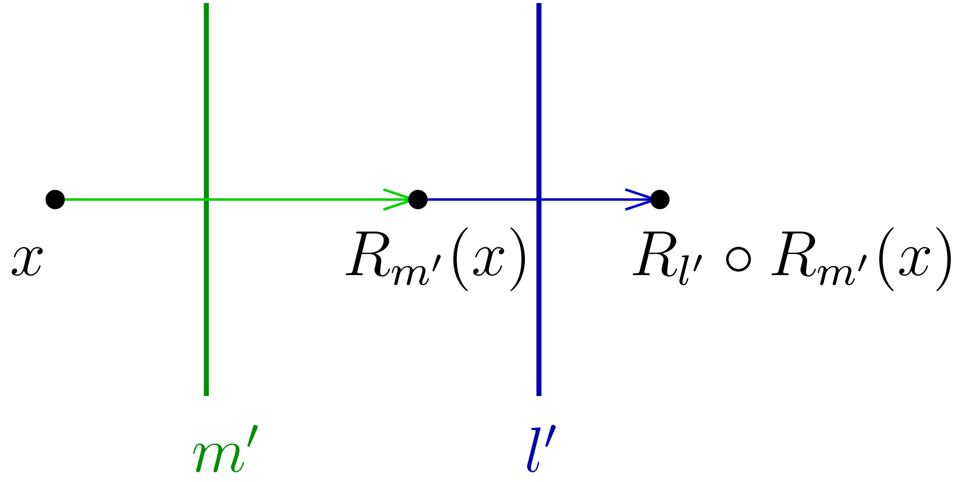

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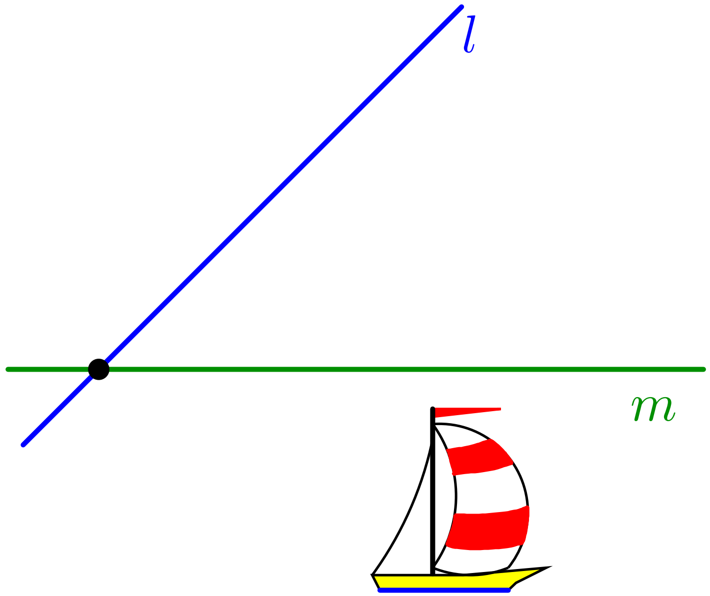

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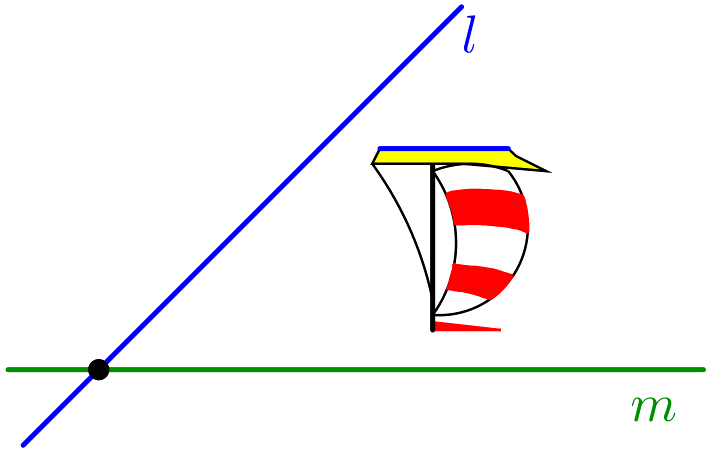

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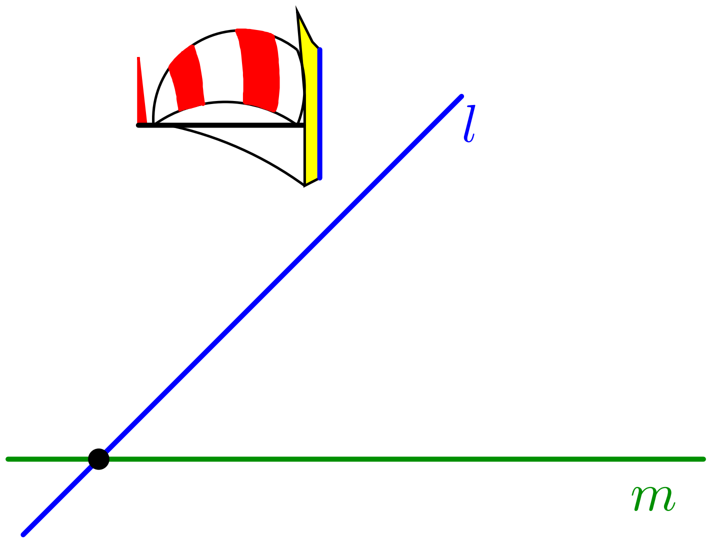

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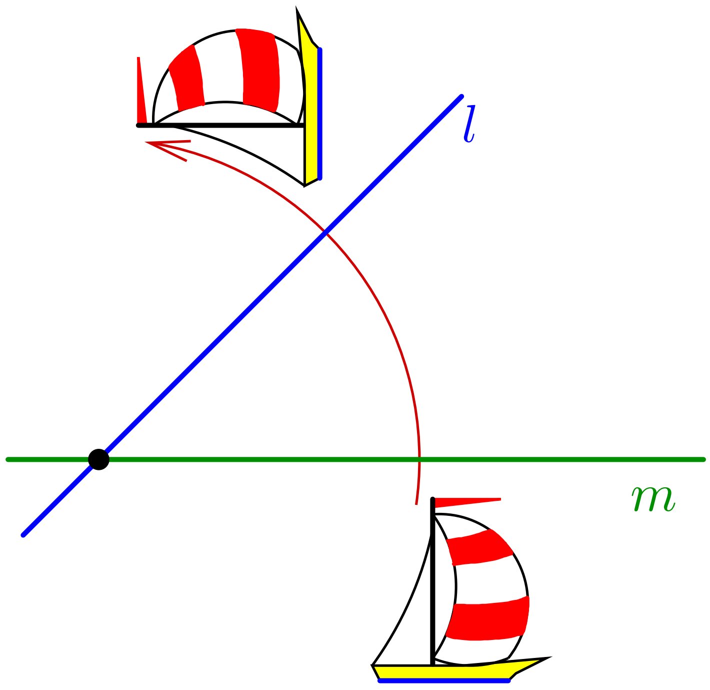

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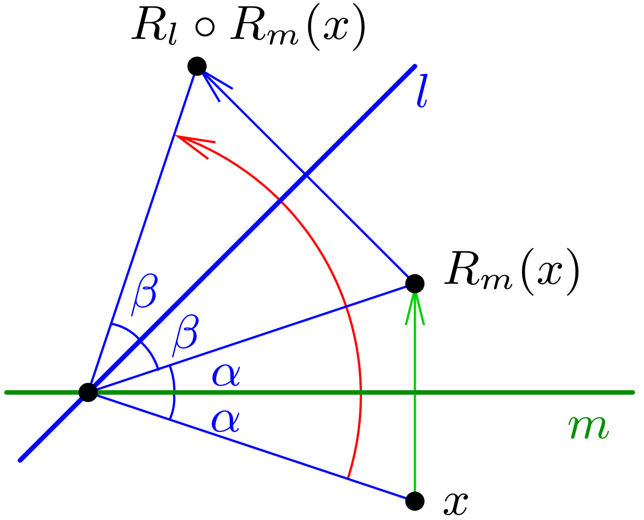

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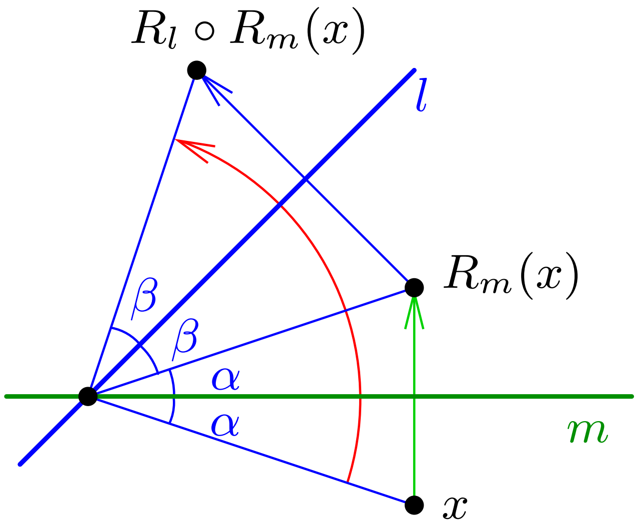

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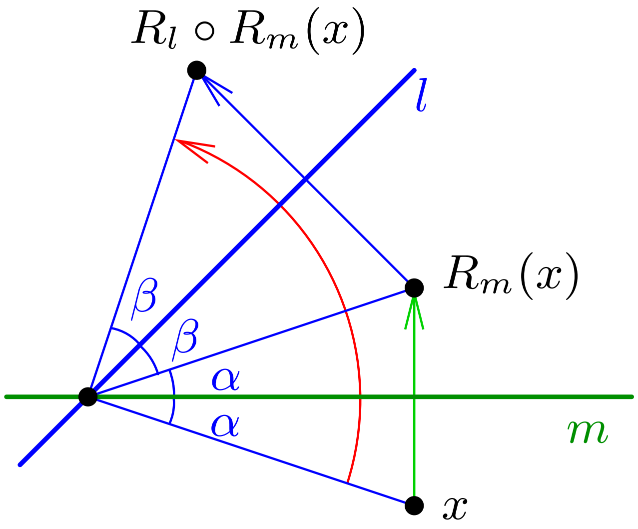

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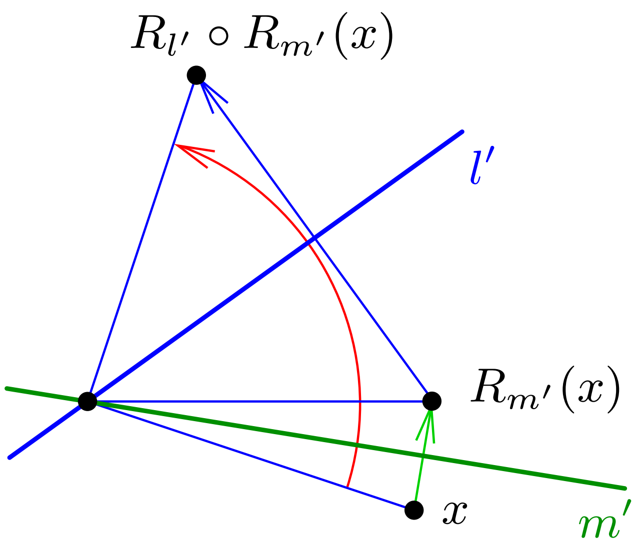

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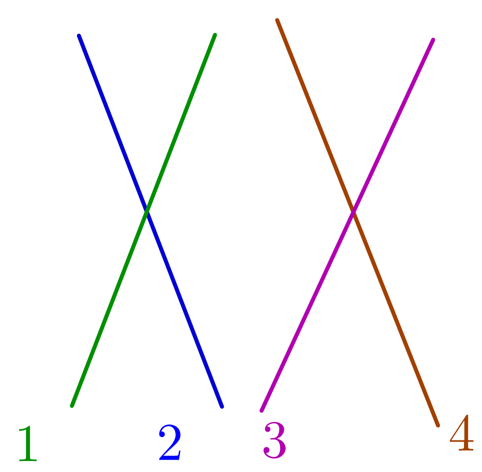

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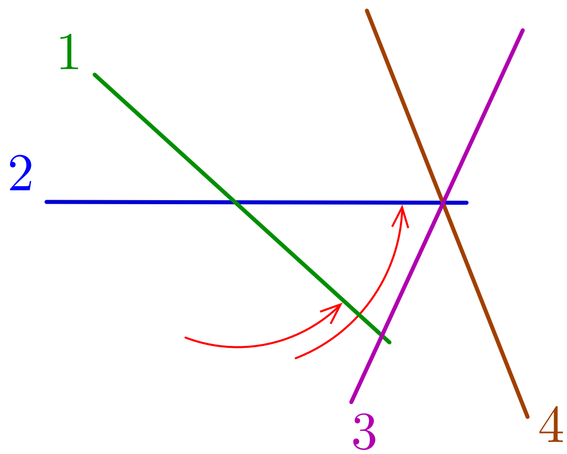

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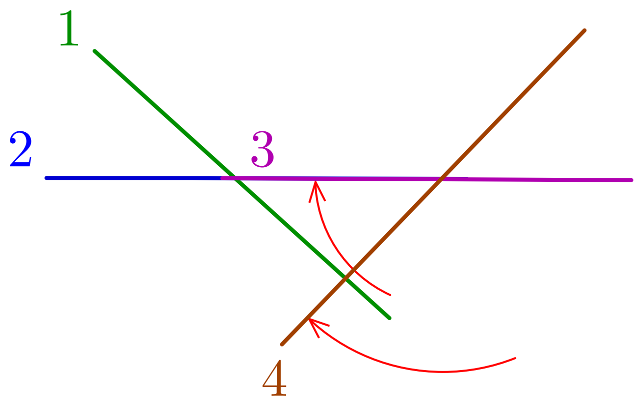

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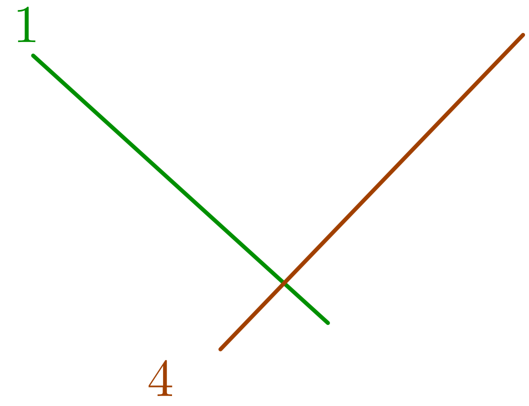

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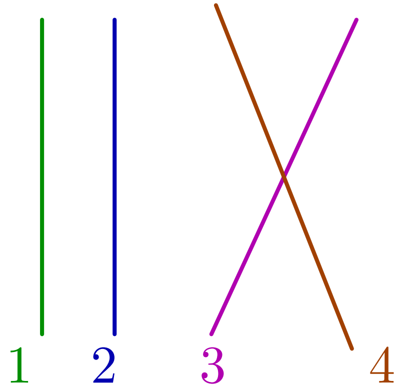

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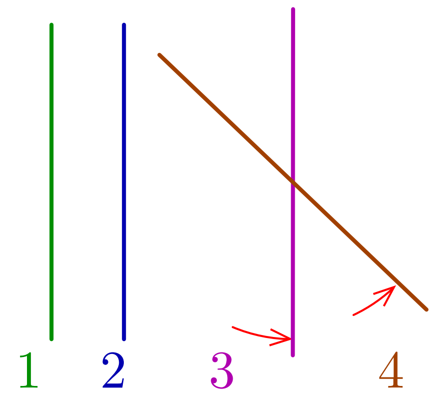

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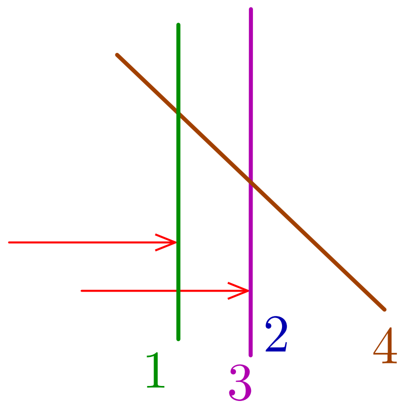

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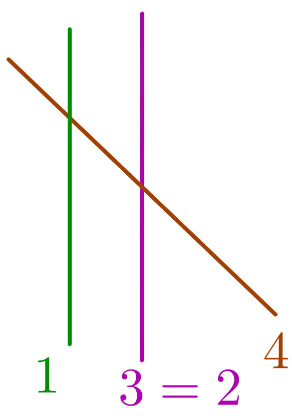

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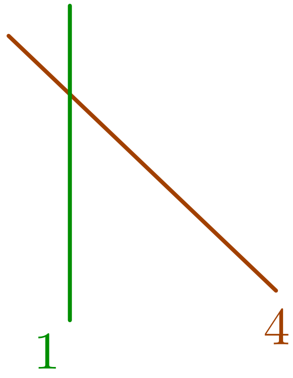

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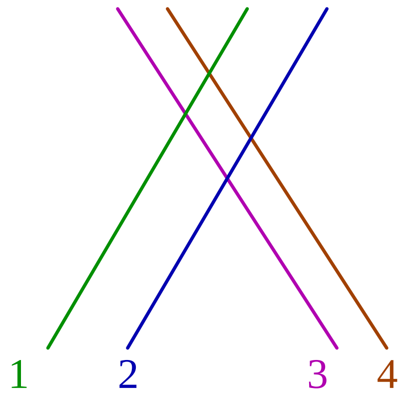

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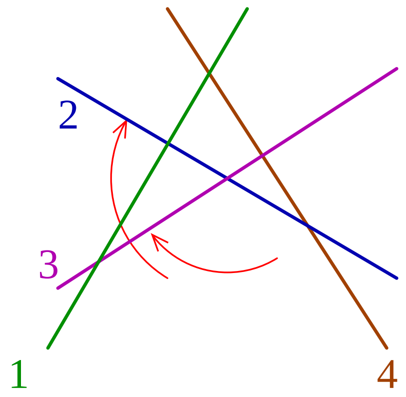

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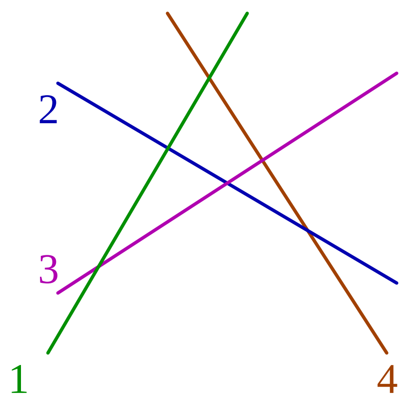

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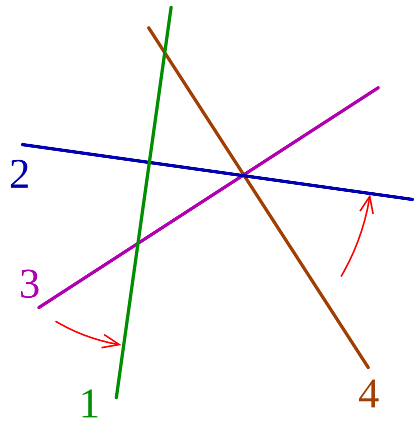

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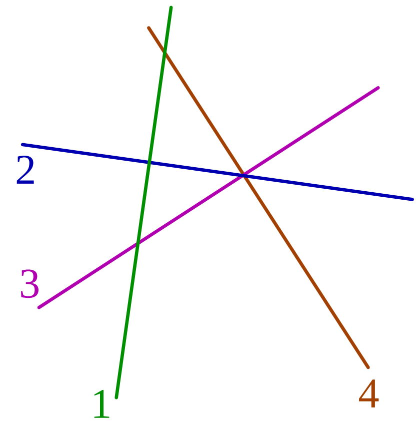

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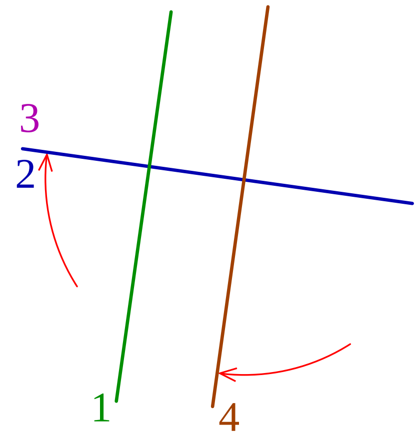

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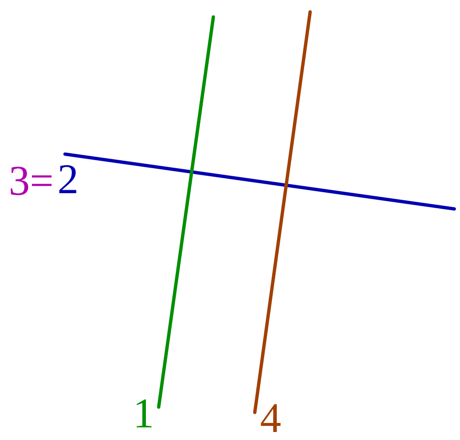

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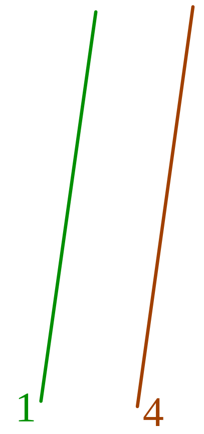

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A composition of odd number of reflections reverses orientation and cannot be id.
A composition of two different reflections is not identity. $\square$

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Generalization of Lemma. In $\mathbb{R}^n$,
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Generalization of Theorem. Any relation among reflections in hyperplanes of $\mathbb{R}^n$ follow
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Key example: $\mathbb{R} \to \mathbb{R} : x \mapsto 2a - x$, the reflection of $\mathbb{R}$ in a point a.

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Generalize reflections!

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Correspondence Flipper $S \longleftrightarrow$ Flip in S is the shortest connection between simple static geometric objects - flippers - and isometries.

Symmetry about a point

is a flip.

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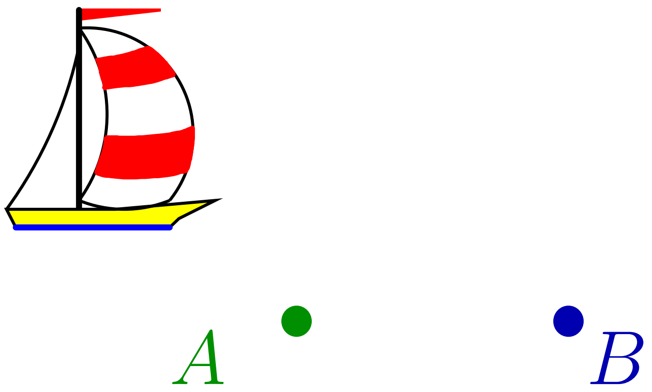

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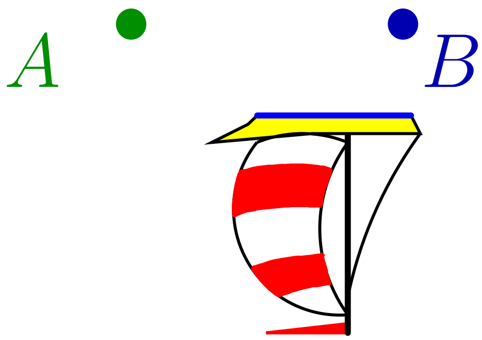

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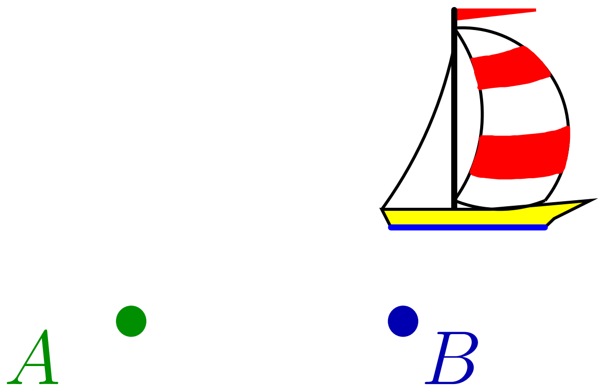

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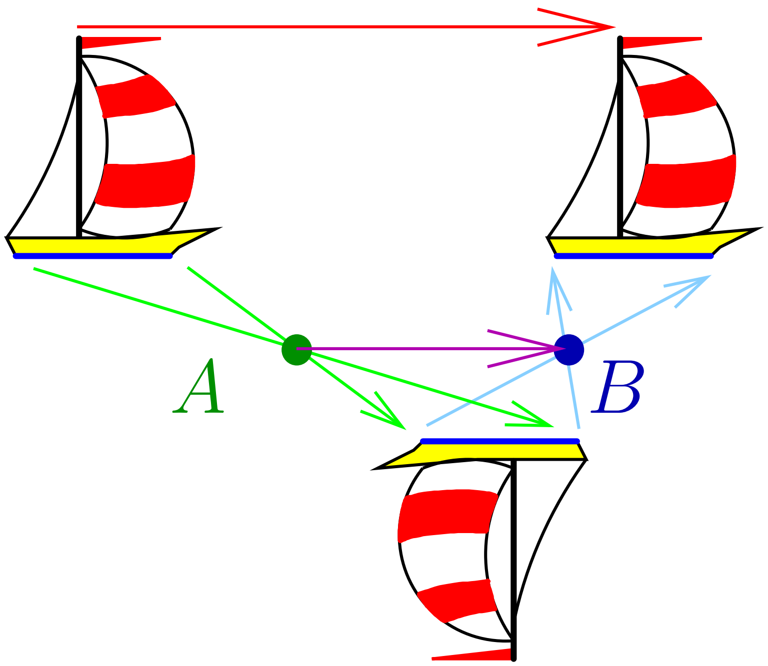

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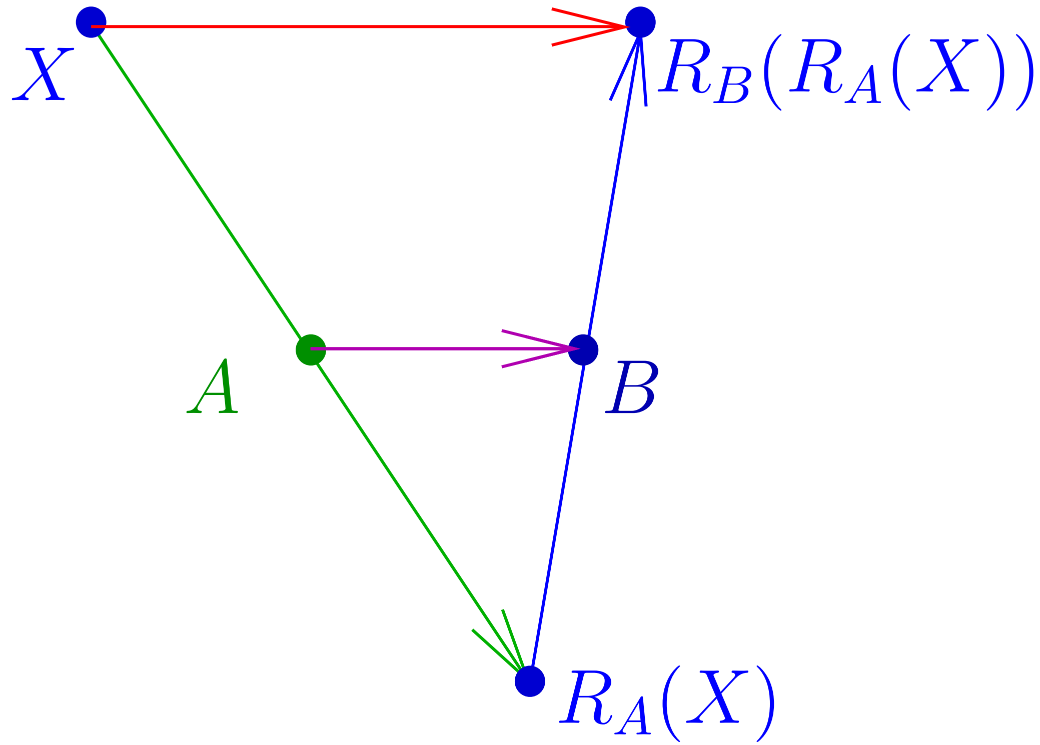

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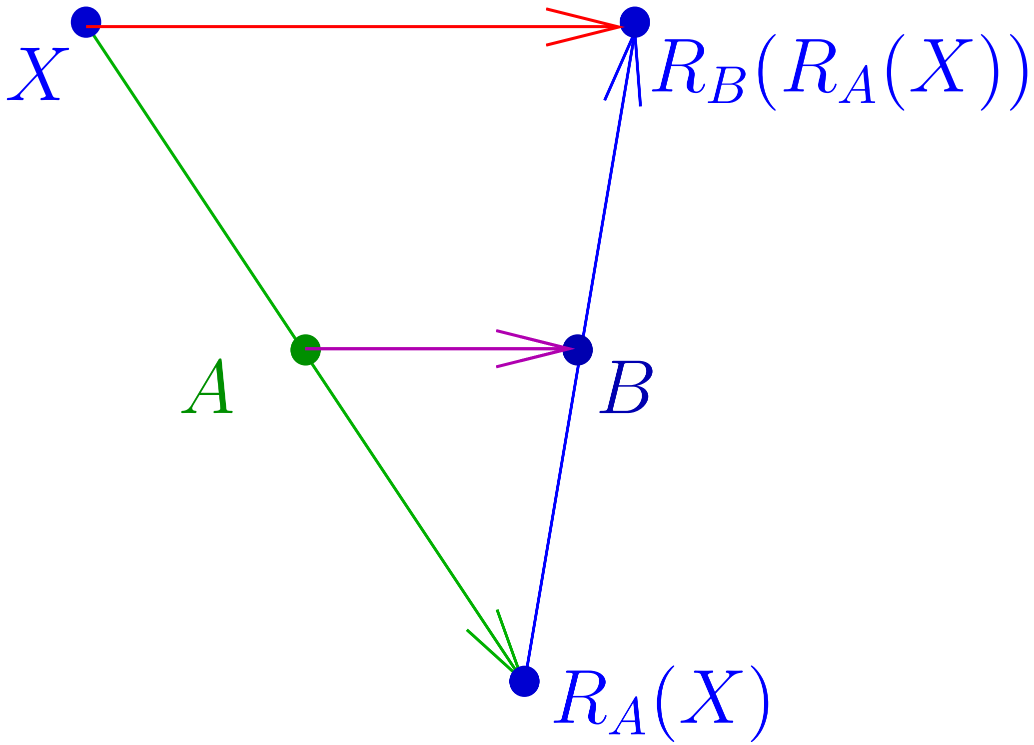

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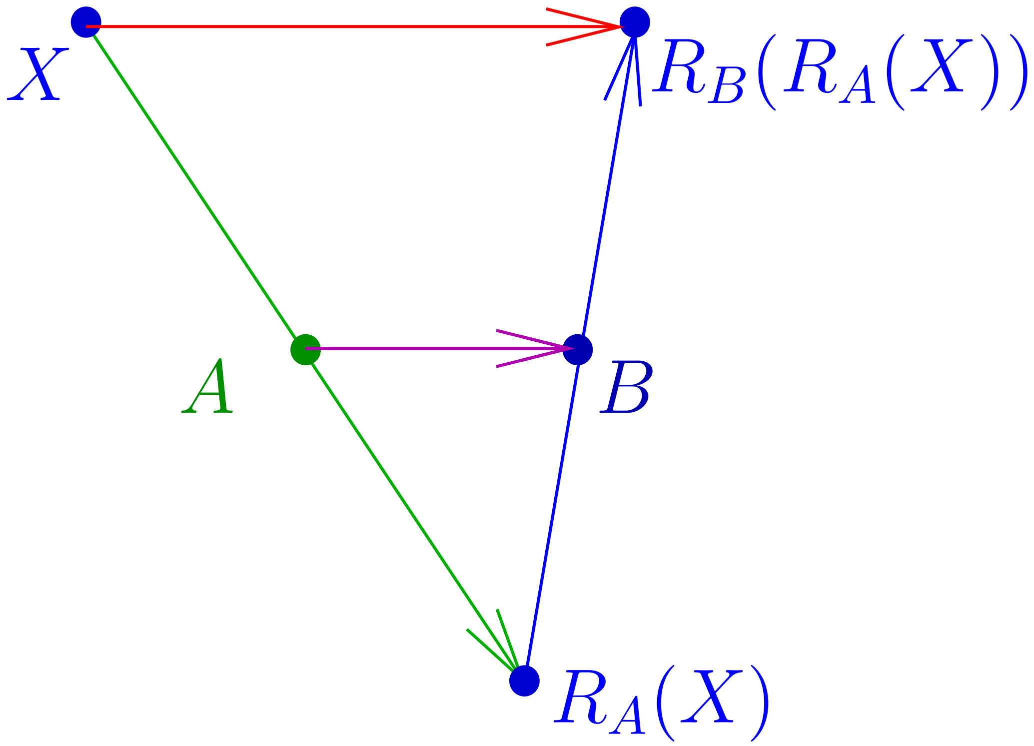

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$\overrightarrow{AB}$ is half the arrow representing $R_B \circ R_A$.

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Compare to the head to tail addition

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}.$$

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Corollary. *Any isometry of a hyperbolic space, sphere, projective space, etc. is a composition of two flips.*

A flip-flop decomposition.

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Because this is so in the symmetric group.

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$f \in G$ is strongly reversible iff $\exists$ an involution $\alpha \in G$ such that $f\alpha$ is an involution.

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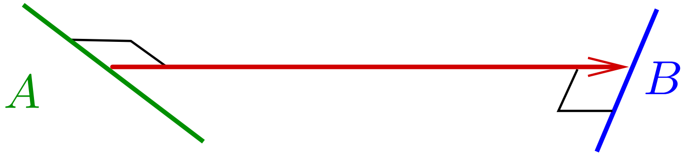

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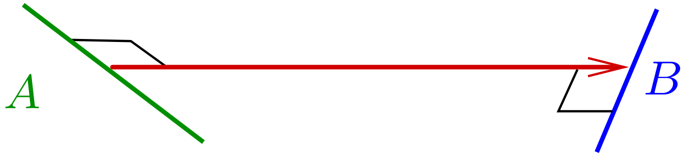

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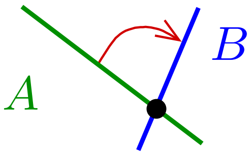

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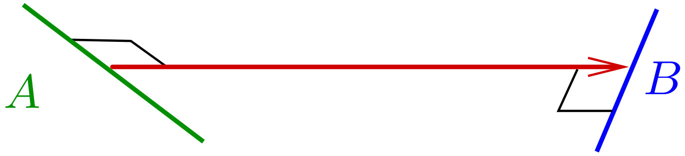

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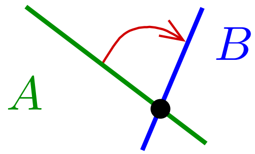

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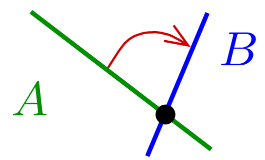

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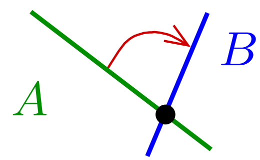

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Problem. Find an explicit description for the equivalence.

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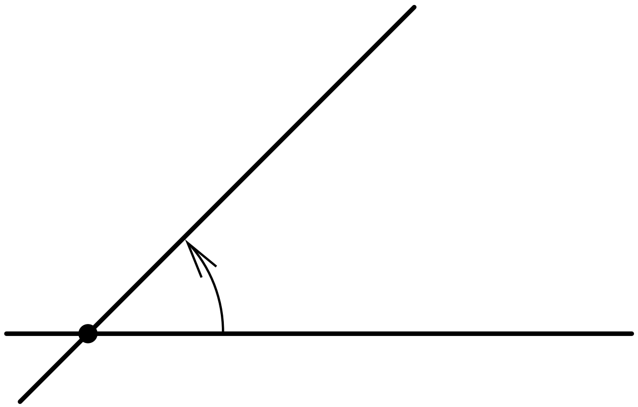

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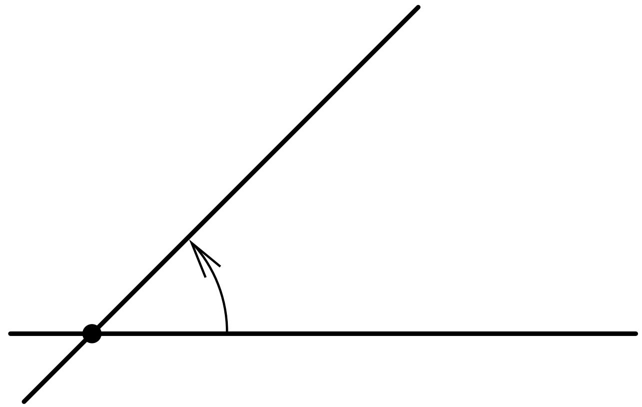

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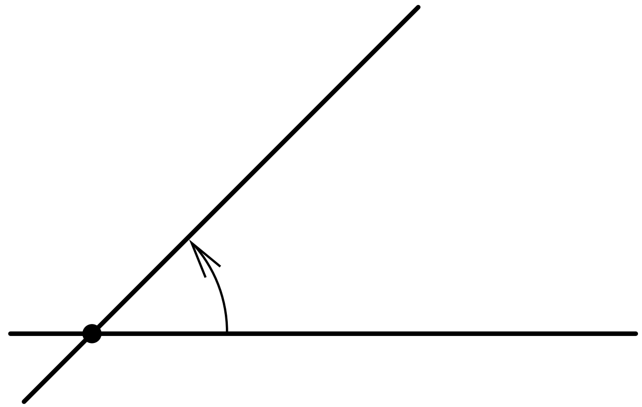

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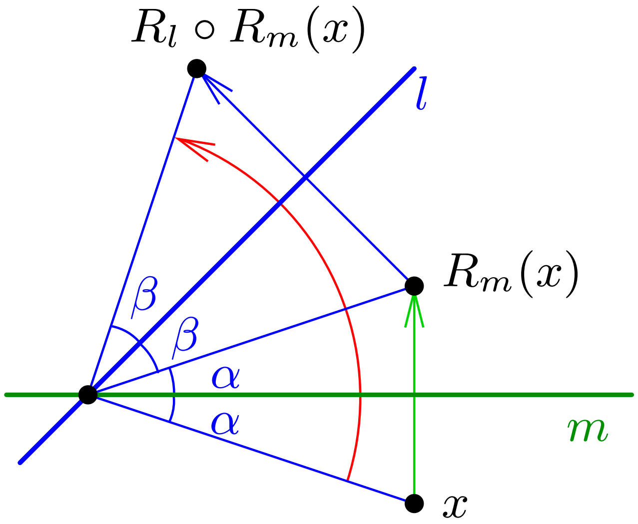

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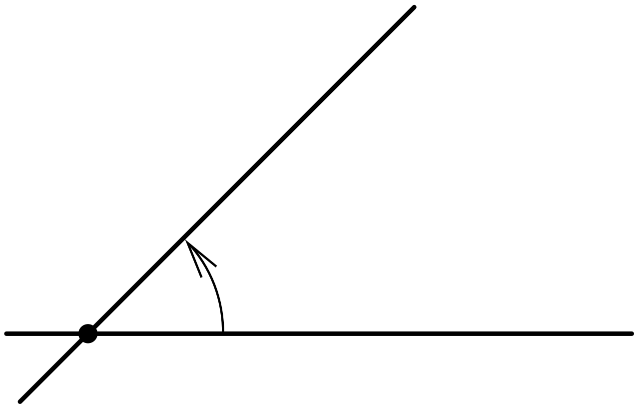

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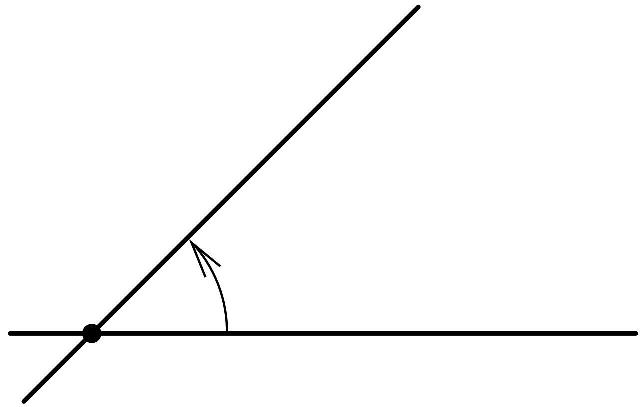

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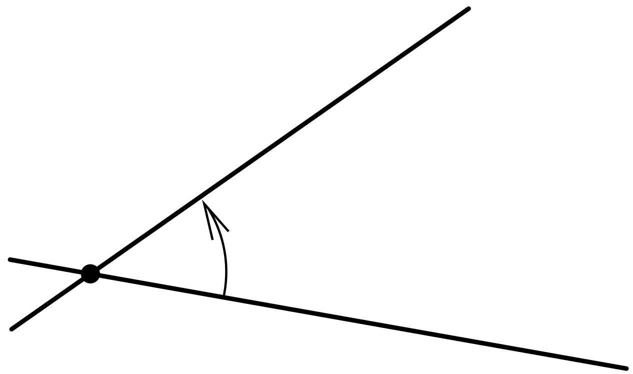

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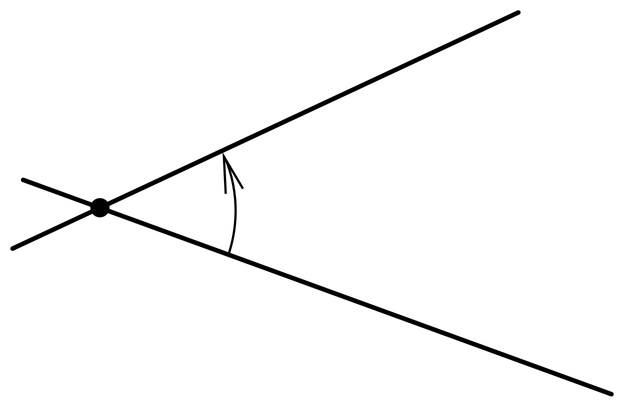

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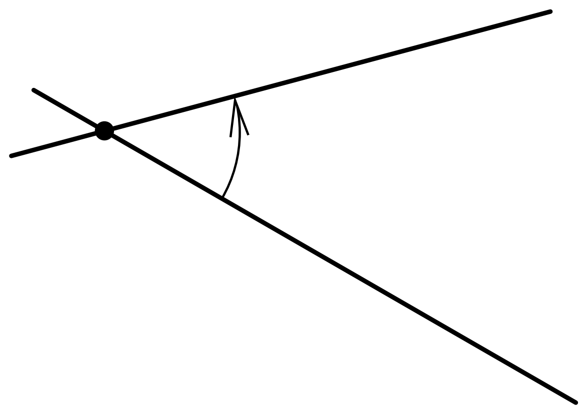

Head to tail for rotations

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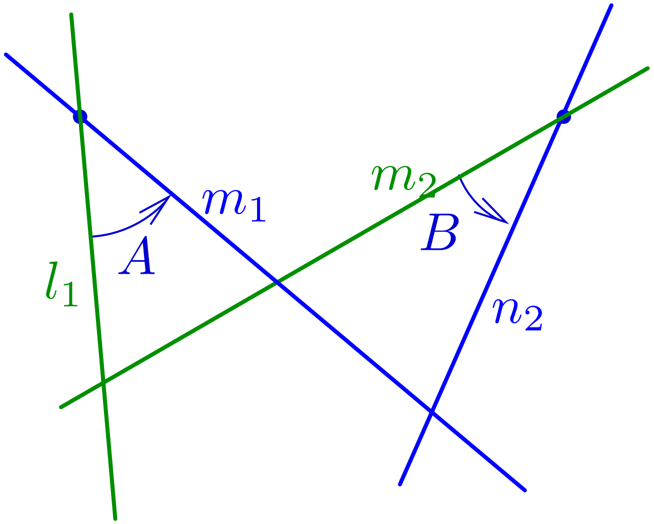
Given two rotations, present them by biflippers.



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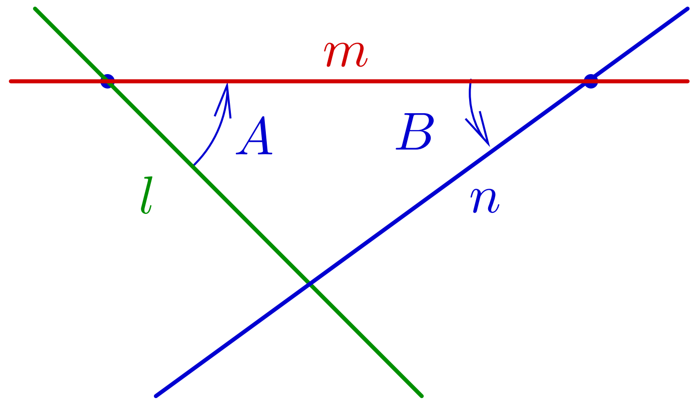

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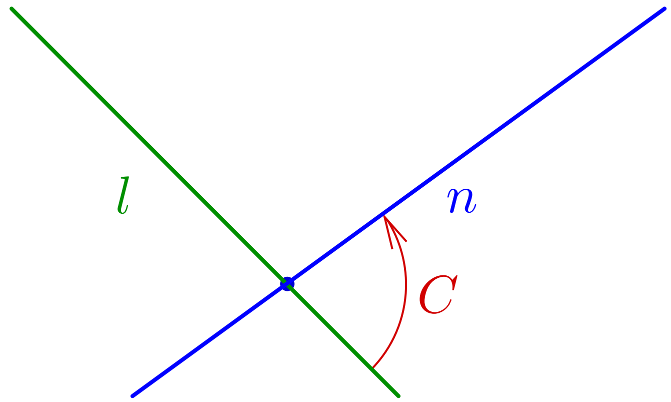

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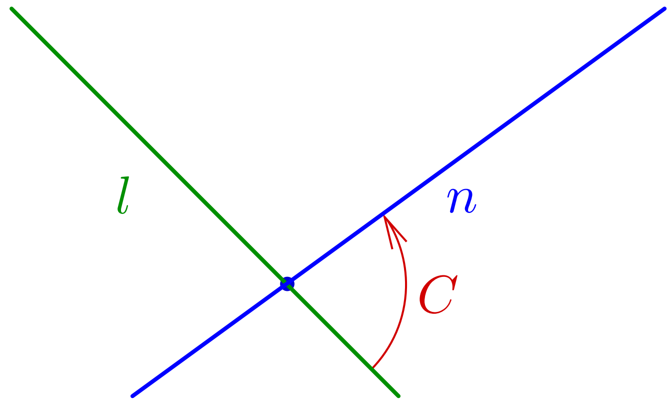

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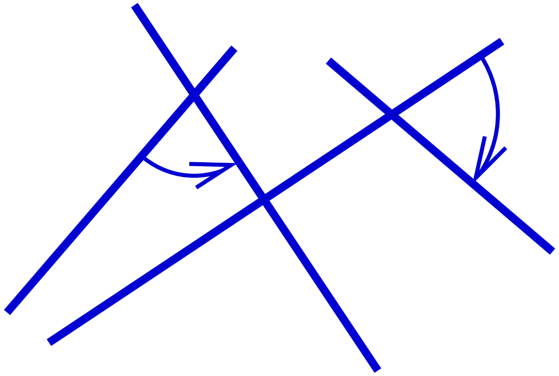

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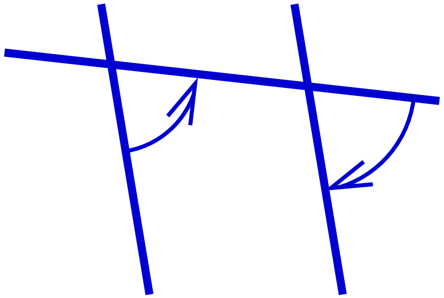

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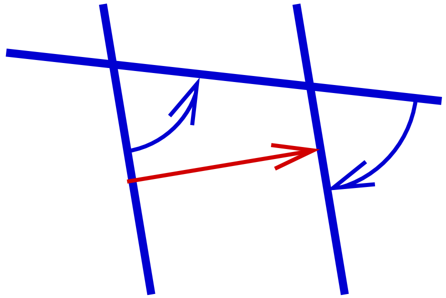

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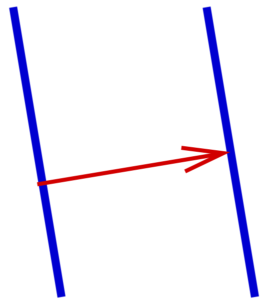

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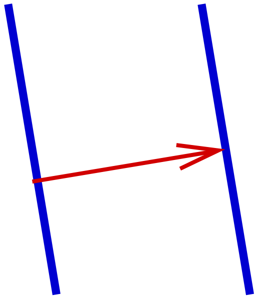

This is a translation.

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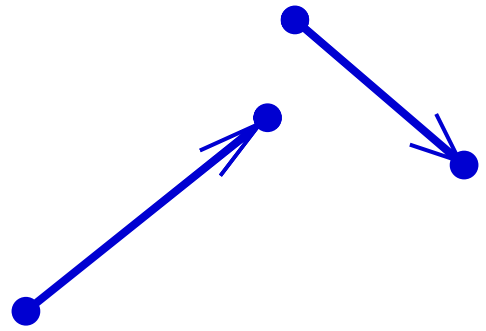

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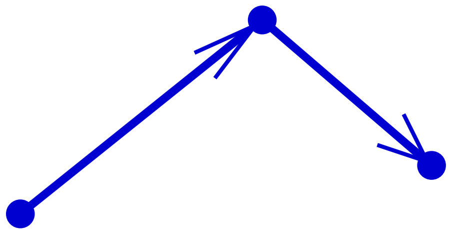

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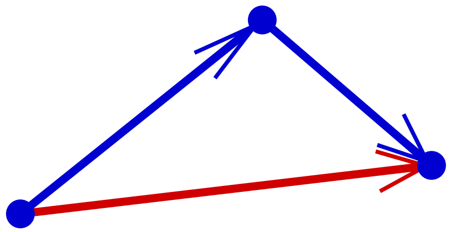

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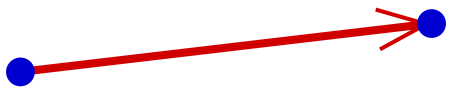
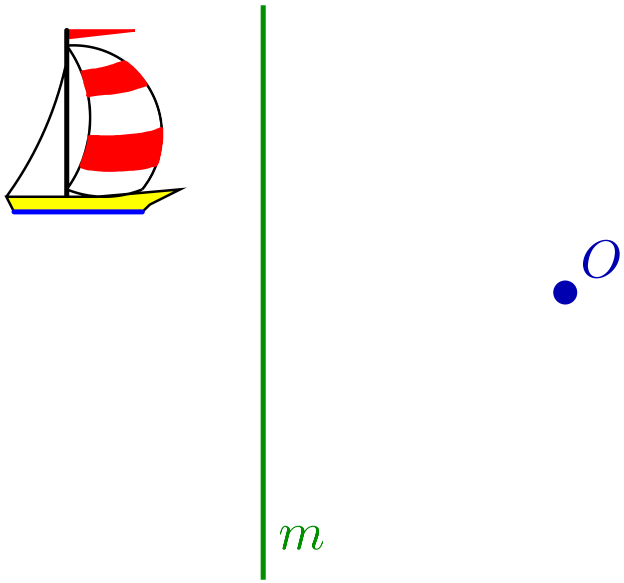
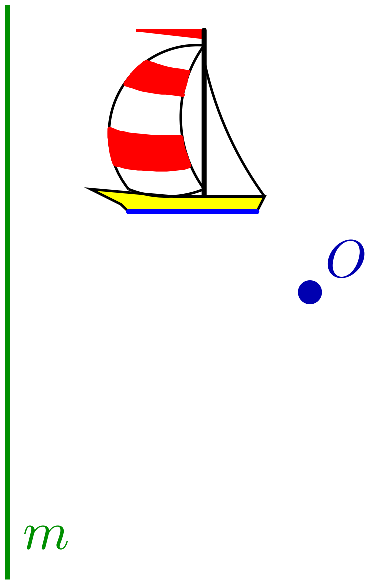


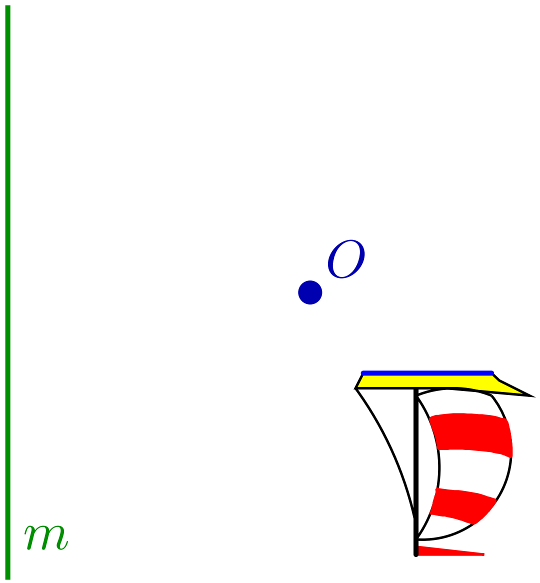
Composing reflections in line and point



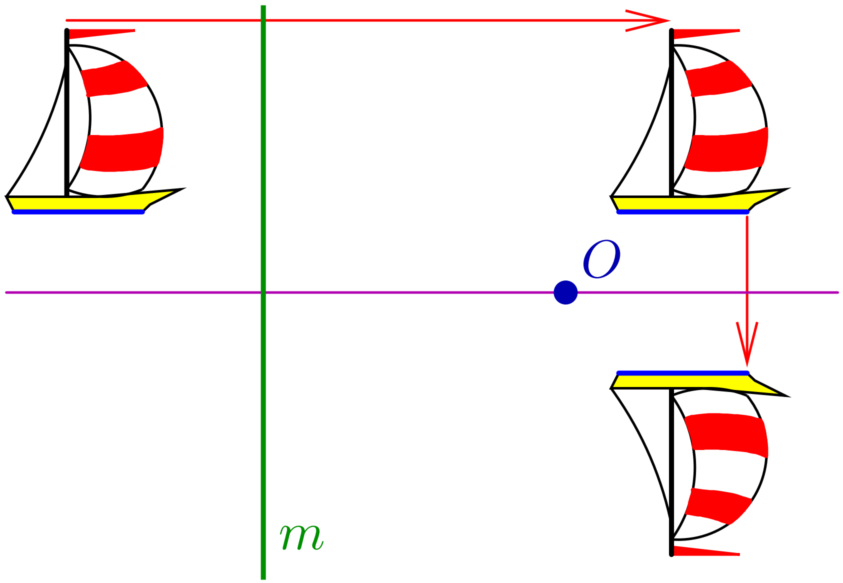
Composing reflections in line and point



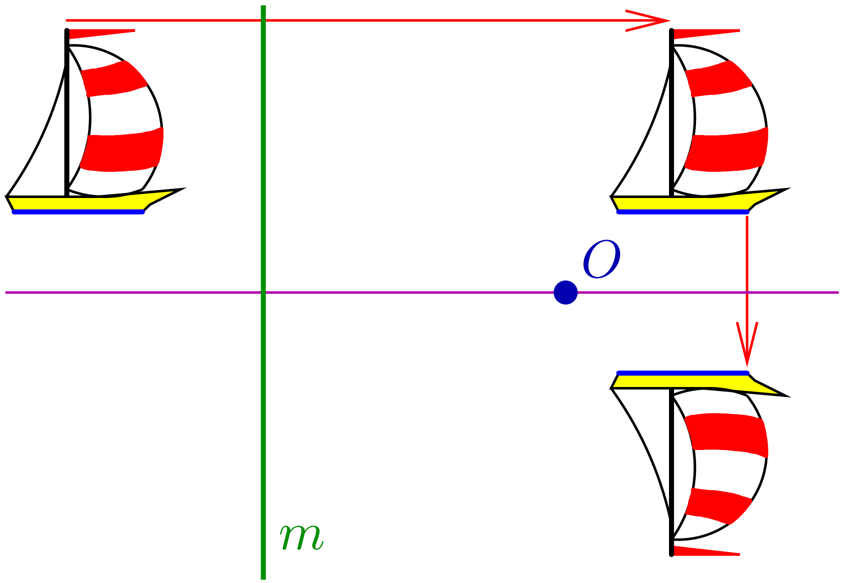
Composing reflections in line and point



Composing reflections in line and point



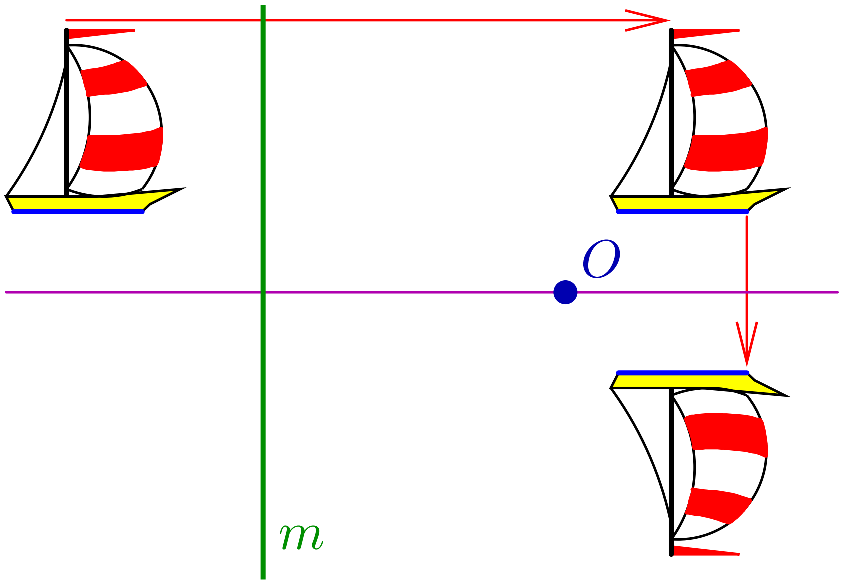
Composing reflections in line and point



This is a glide reflection!

Composing reflections in line and point

Indeed!



Composing reflections in line and point

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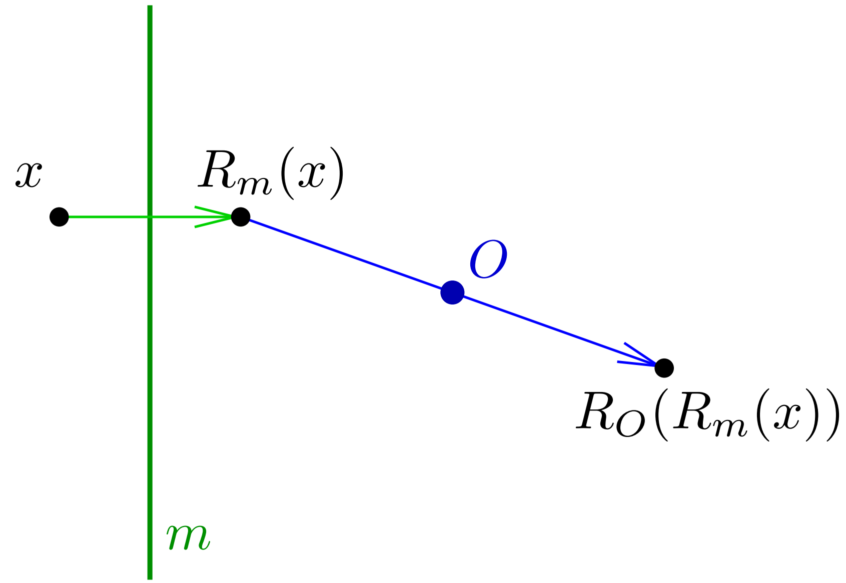

Composing reflections in line and point

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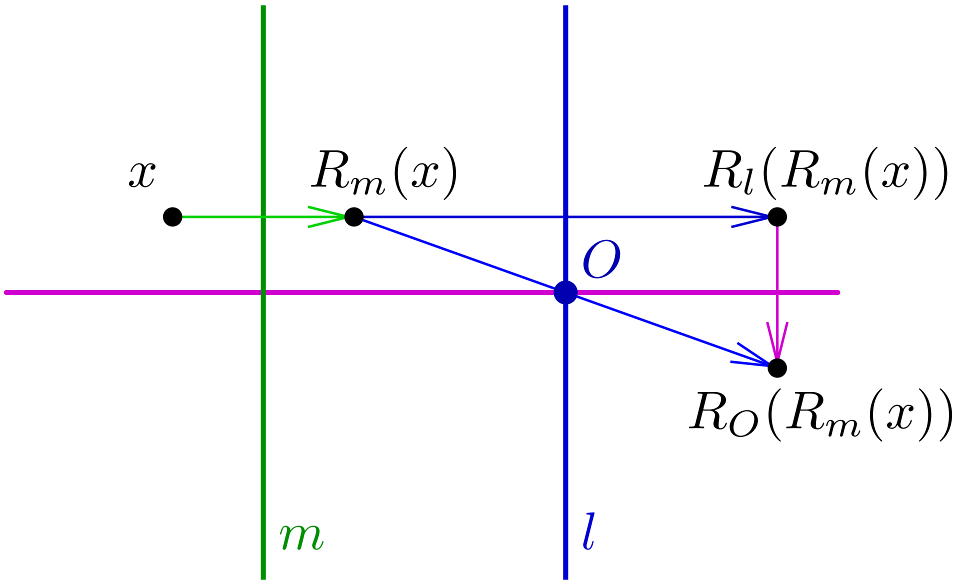
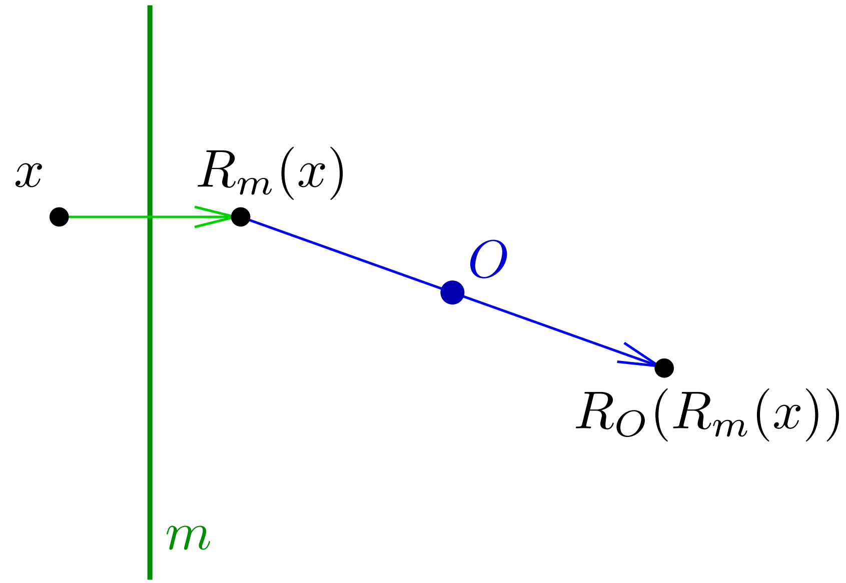


Biflippers for a glide reflection



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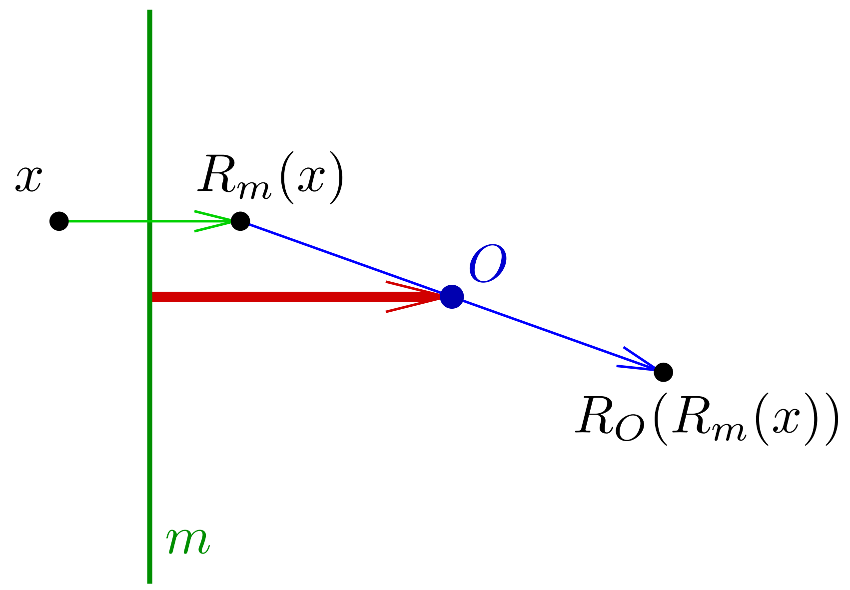

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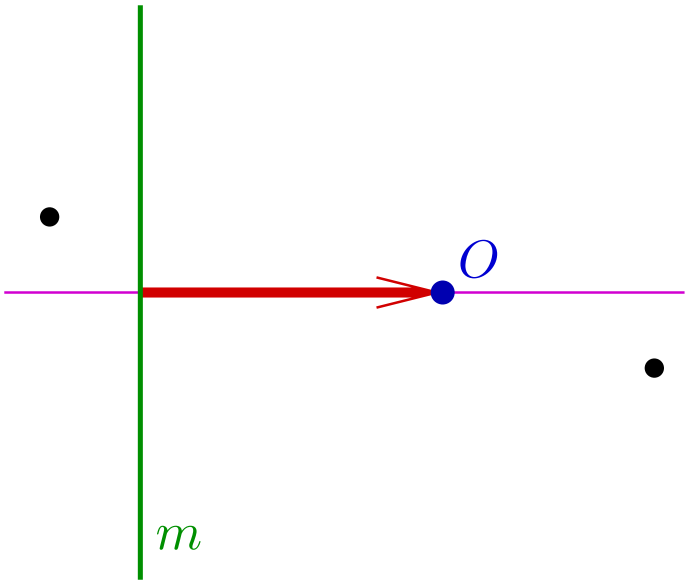

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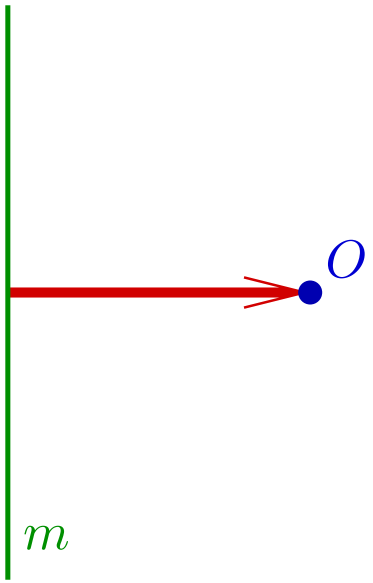
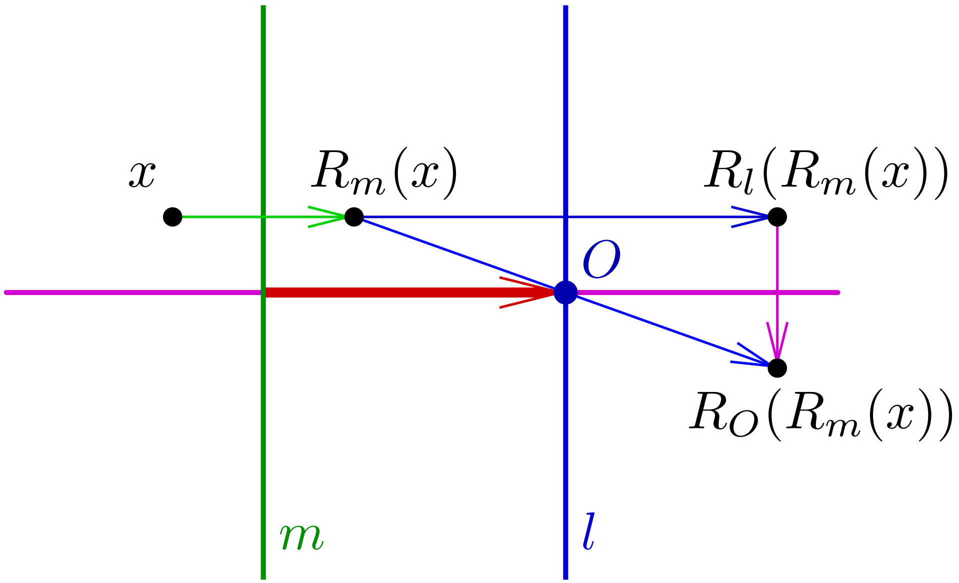
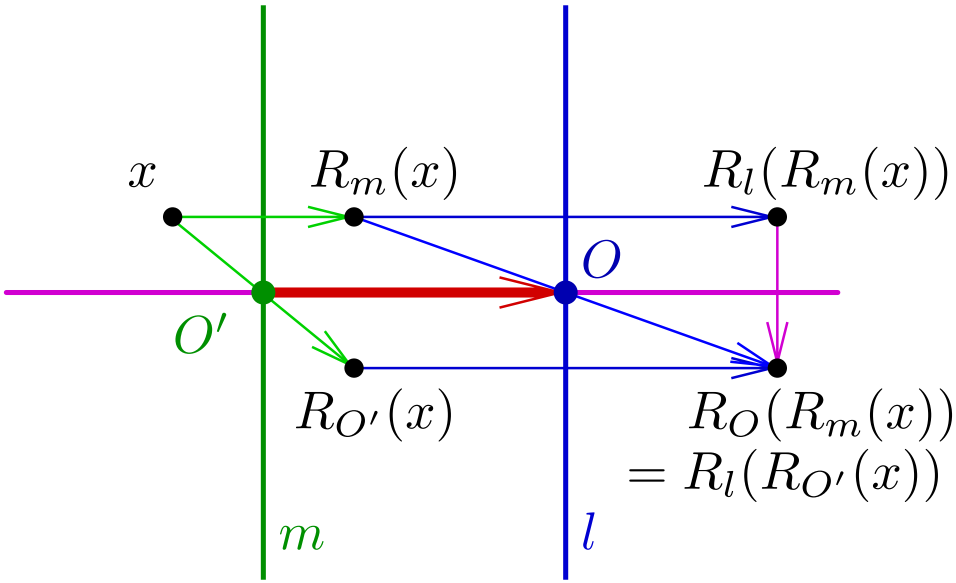


Biflippers for a glide reflection



Biflippers for a glide reflection



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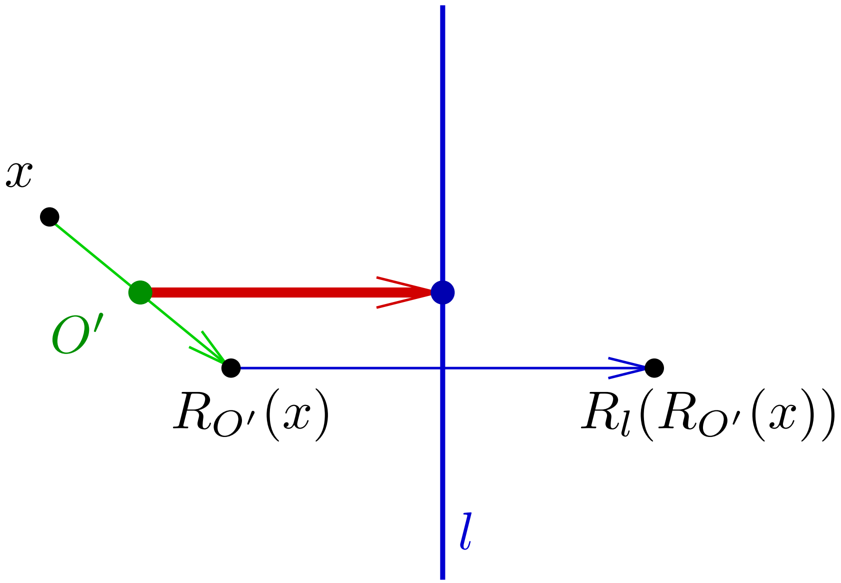
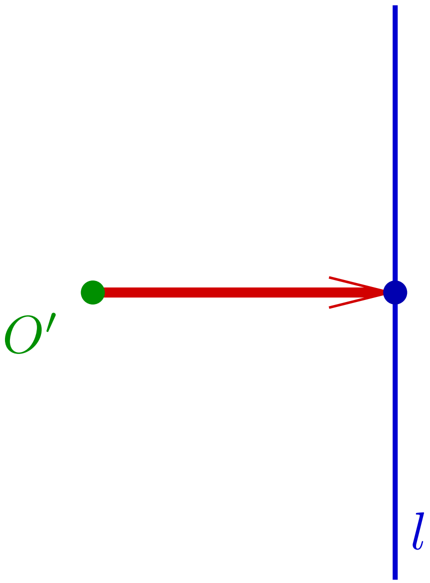
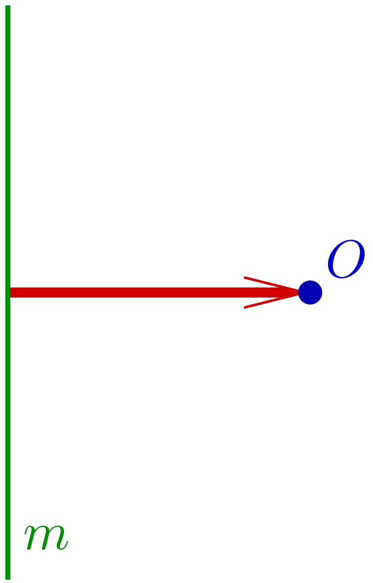


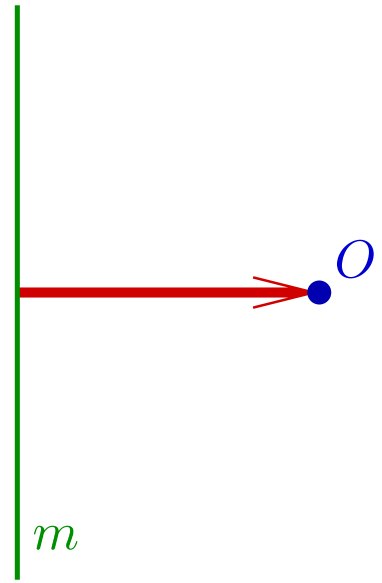
Biflippers for a glide reflection



Biflippers for a glide reflection

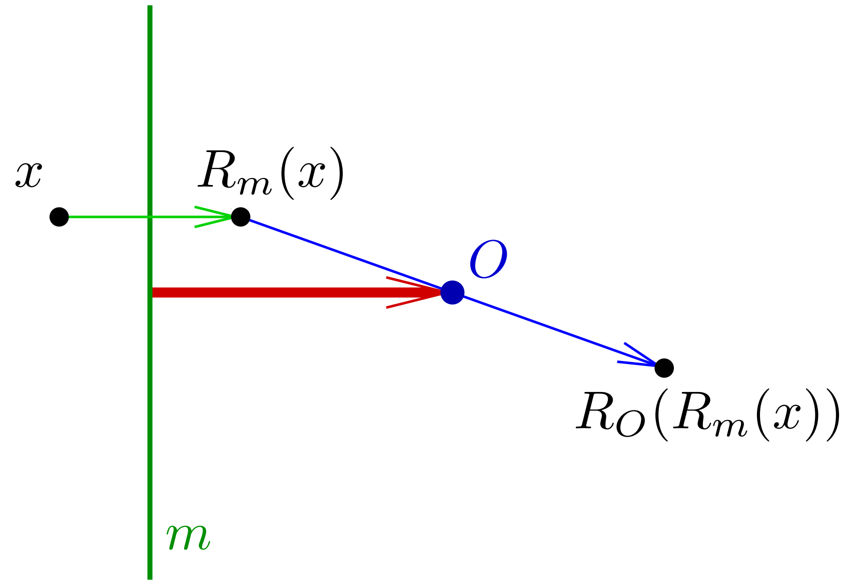


Biflippers for a glide reflection



A biflipper for a glide reflection may glide along itself.

Biflippers for a glide reflection



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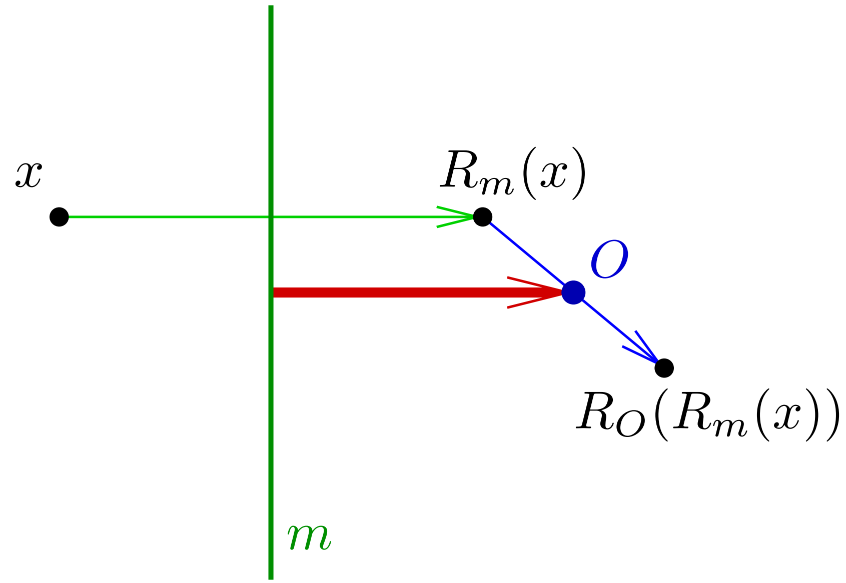

A biflipper for a glide reflection may glide along itself.

Head to tail for glide reflections

Head to tail for glide reflections

Given two glide reflections, present them by biflippers.

Head to tail for glide reflections

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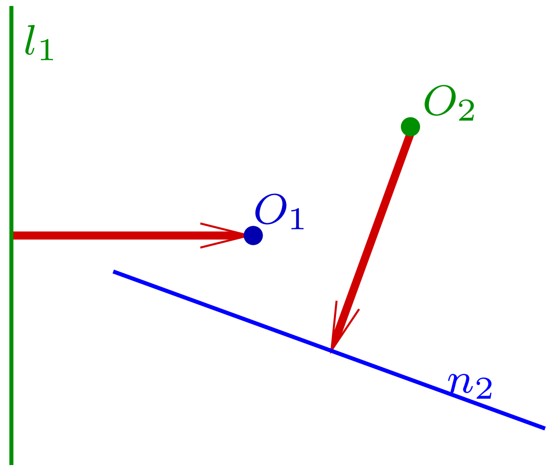

Head to tail for glide reflections

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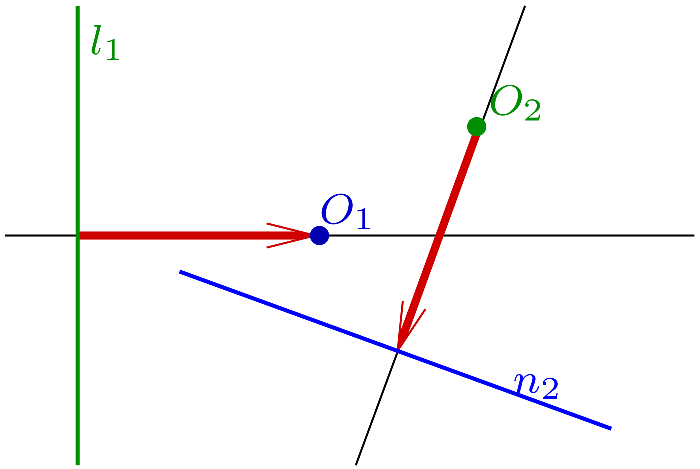

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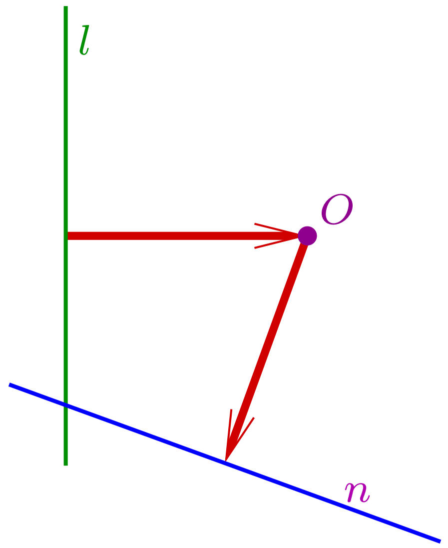

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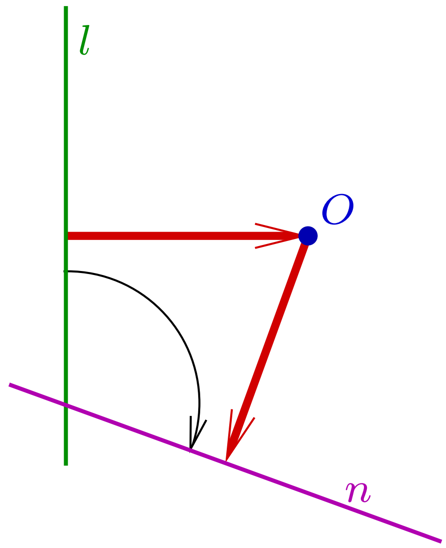

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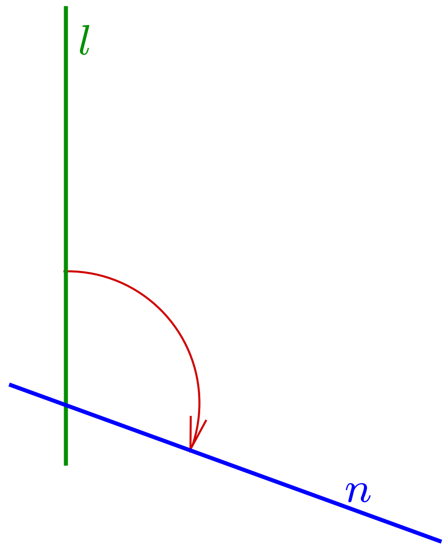

Head to tail for glide reflections

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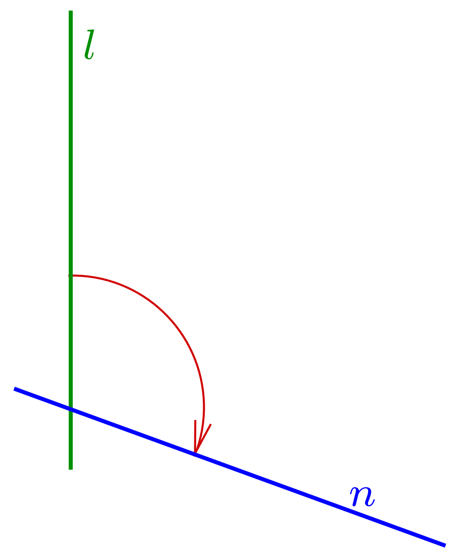
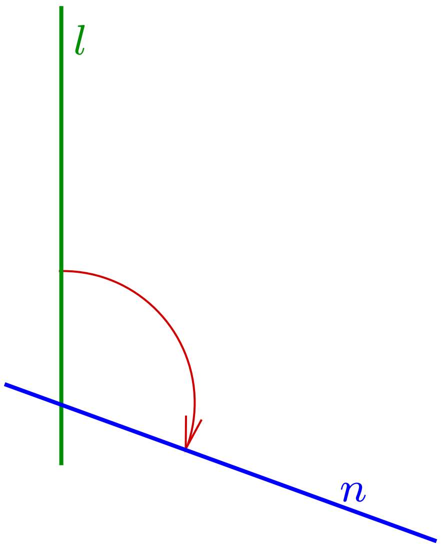


This is a rotation!

Head to tail for glide reflections

Given two glide reflections, present them by biflippers.



Exercise. *Find head to tail rules for rotation ∘ glide reflection.*

Direct product of biflippers

Direct product of biflippers

For any isometries $S : \mathbb{R}^p \to \mathbb{R}^p$ and $T : \mathbb{R}^q \to \mathbb{R}^q$, the direct product $S \times T : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^p \times \mathbb{R}^q : (x, y) \mapsto (S(x), T(y))$ is an isometry of $\mathbb{R}^{p+q} = \mathbb{R}^p \times \mathbb{R}^q$.

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If S and T are flips in flippers A and B, then $S \times T$ is the flip in the flipper $A \times B \subset \mathbb{R}^{p+q}$.

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If S and T are flips in flippers A and B,
then $S \times T$ is the flip in the flipper $A \times B \subset \mathbb{R}^{p+q}$.

If S and T are isometries, defined by biflippers (A, A') and (B, B'),
then $S \times T$ is defined by biflipper $(A \times B, A' \times B')$.

Direct product of biflippers

For any isometries $S:\mathbb{R}^p\to\mathbb{R}^p$ and $T:\mathbb{R}^q\to\mathbb{R}^q$, the direct product $S\times T:\mathbb{R}^p\times\mathbb{R}^q\to\mathbb{R}^p\times\mathbb{R}^q:(x,y)\mapsto(S(x),T(y))$ is an isometry of $\mathbb{R}^{p+q}=\mathbb{R}^p\times\mathbb{R}^q$.

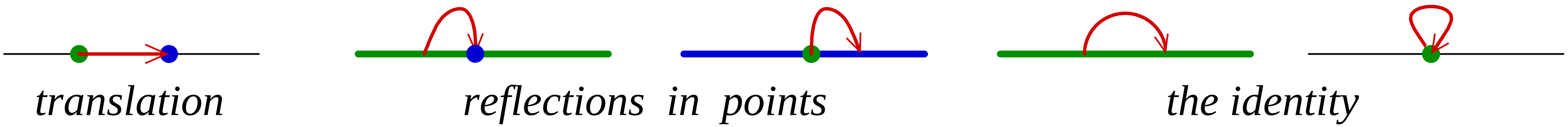
If S and T are flips in flippers A and B, then $S\times T$ is the flip in the flipper $A\times B\subset\mathbb{R}^{p+q}$.

If S and T are isometries, defined by biflippers (A,A') and (B,B'), then $S\times T$ is defined by biflipper $(A\times B, A'\times B')$.

Any isometry of $\mathbb{R}^n$ is a direct product of isometries of $\mathbb{R}^2$ and $\mathbb{R}^1$.

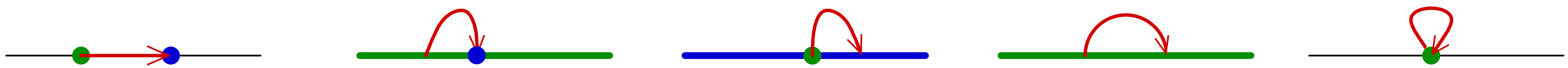
Biflippers on line and plane

On line:



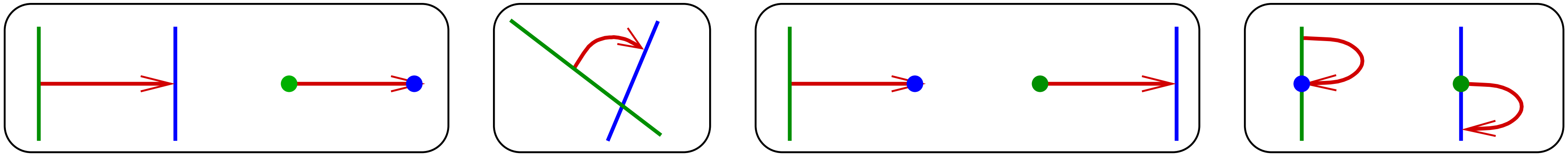
Biflippers on line and plane

On line:



translation *reflections in points* *the identity*

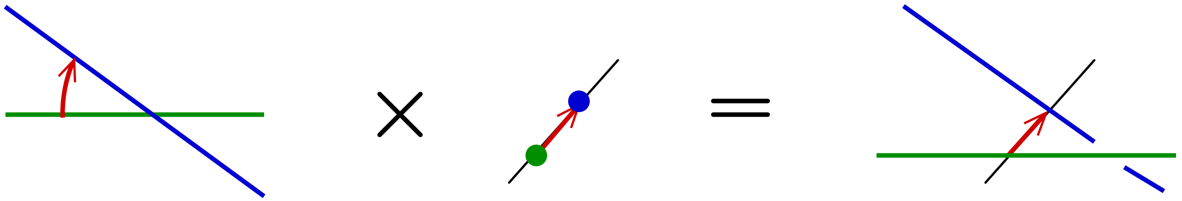
On the plane:



translations *rotation* *glide reflections* *reflections*

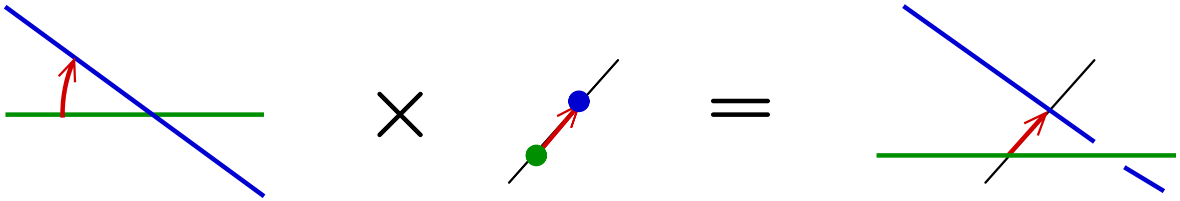
In 3D

In 3D

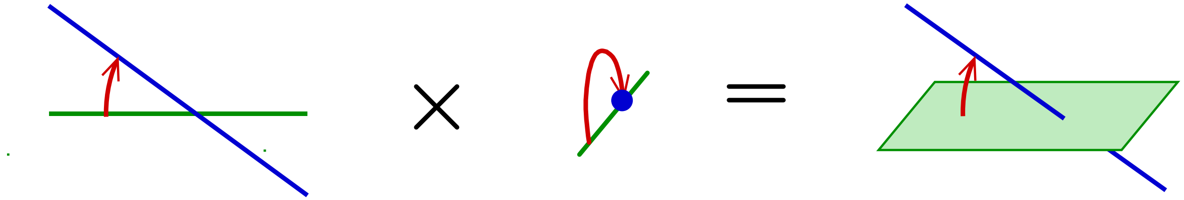
Screw motion: 

In 3D

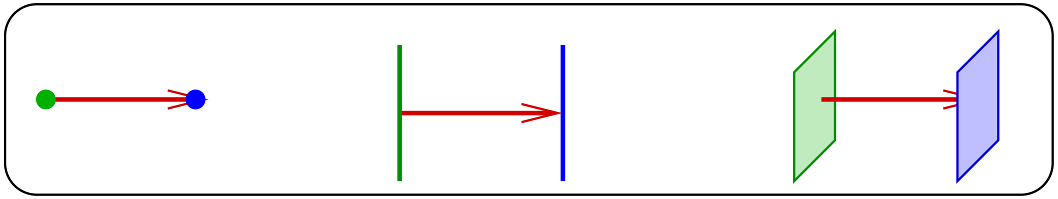
Screw motion:



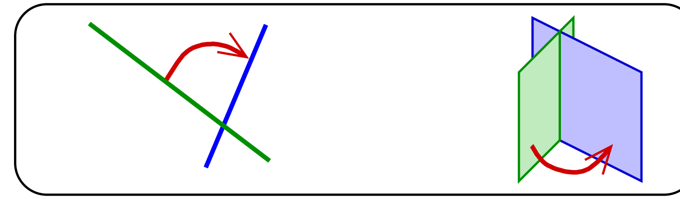
Rotary reflection:



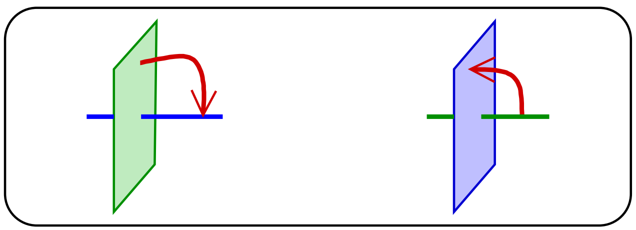
In 3D



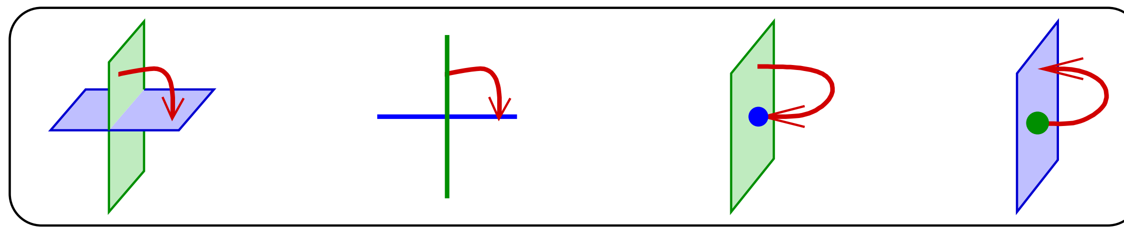
translations



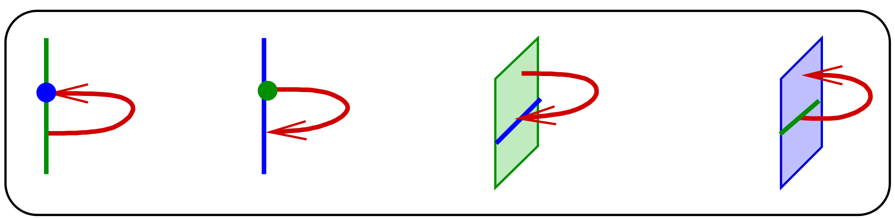
rotations



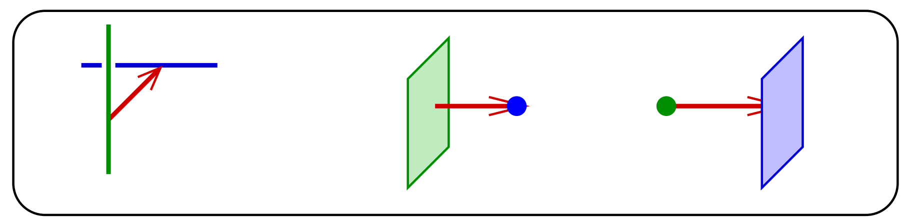
central symmetries



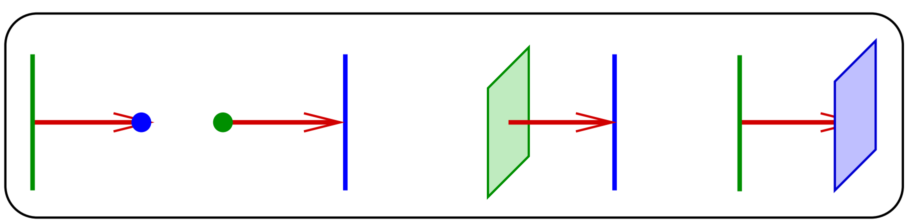
symmetries about a line (half−turns)



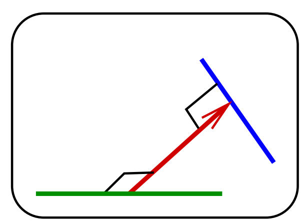
reflections



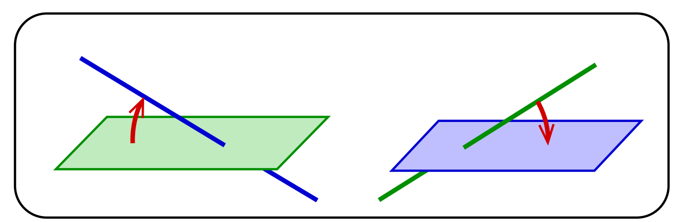
glide symmetries about a line



glide reflections



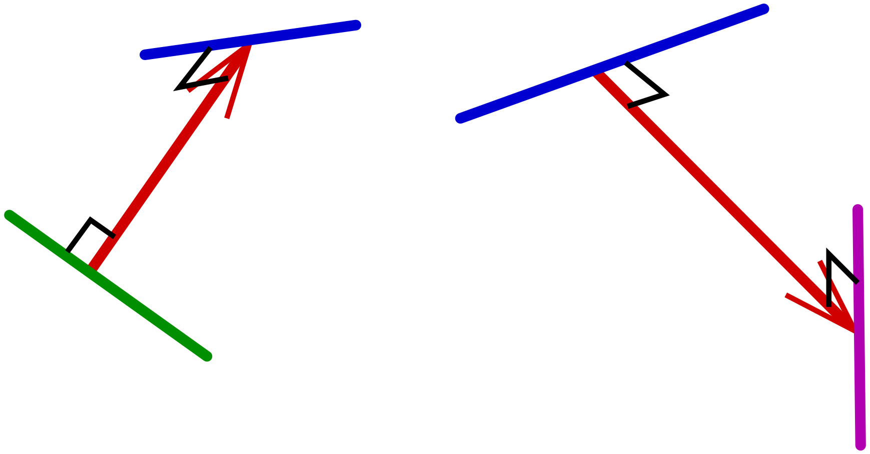
screw motion



rotary reflections

Head to tail for screws

Head to tail for screws



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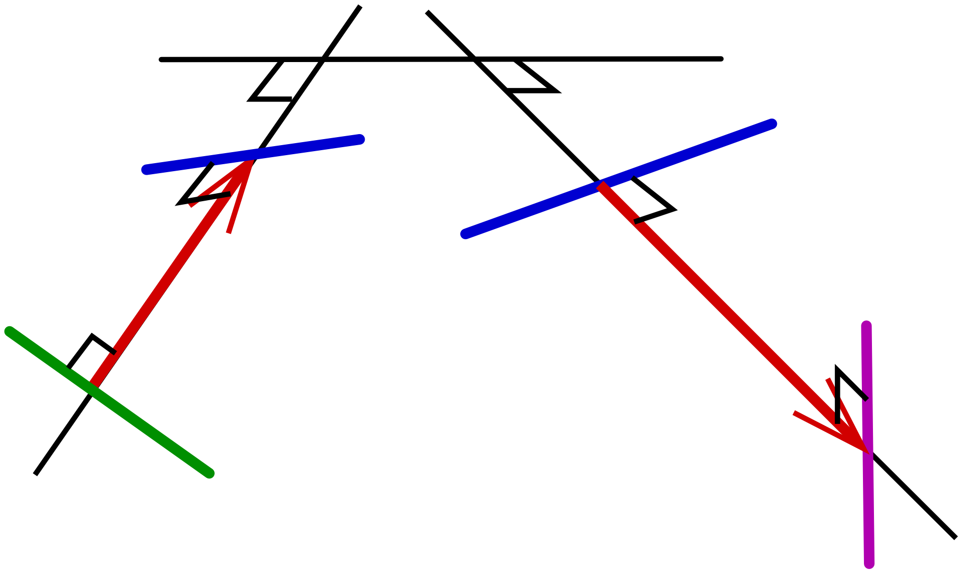

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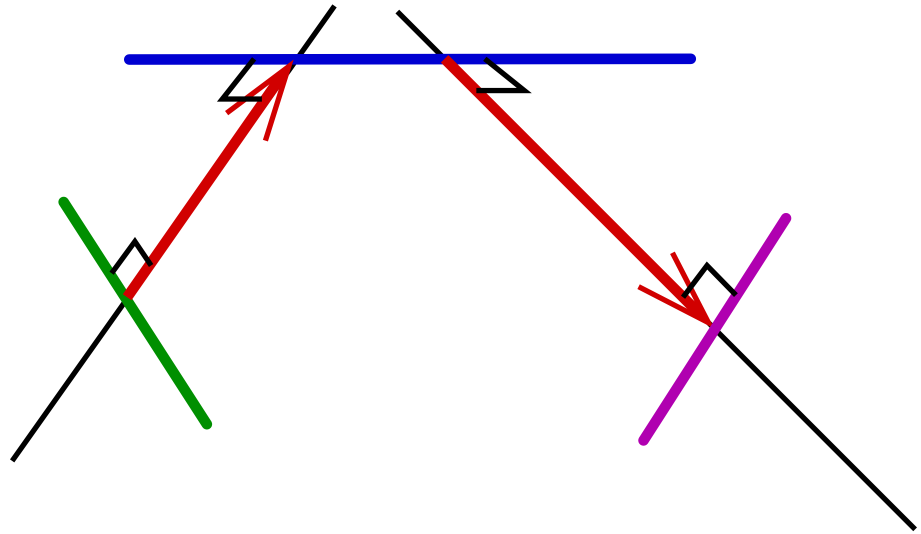

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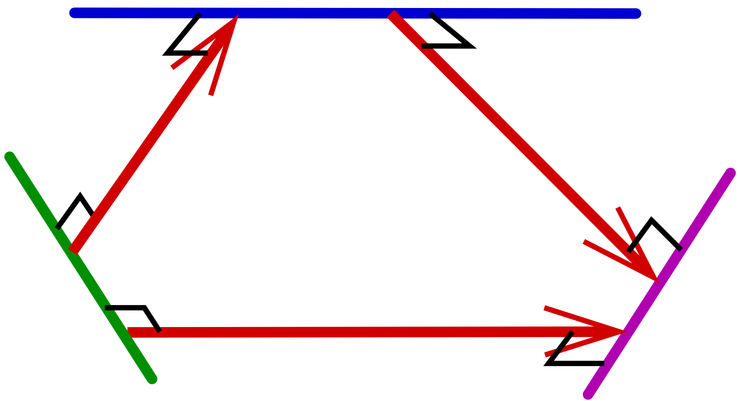
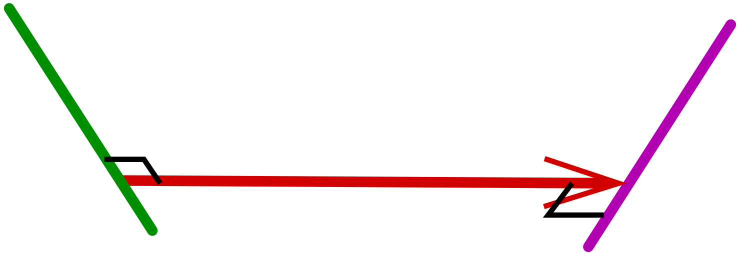
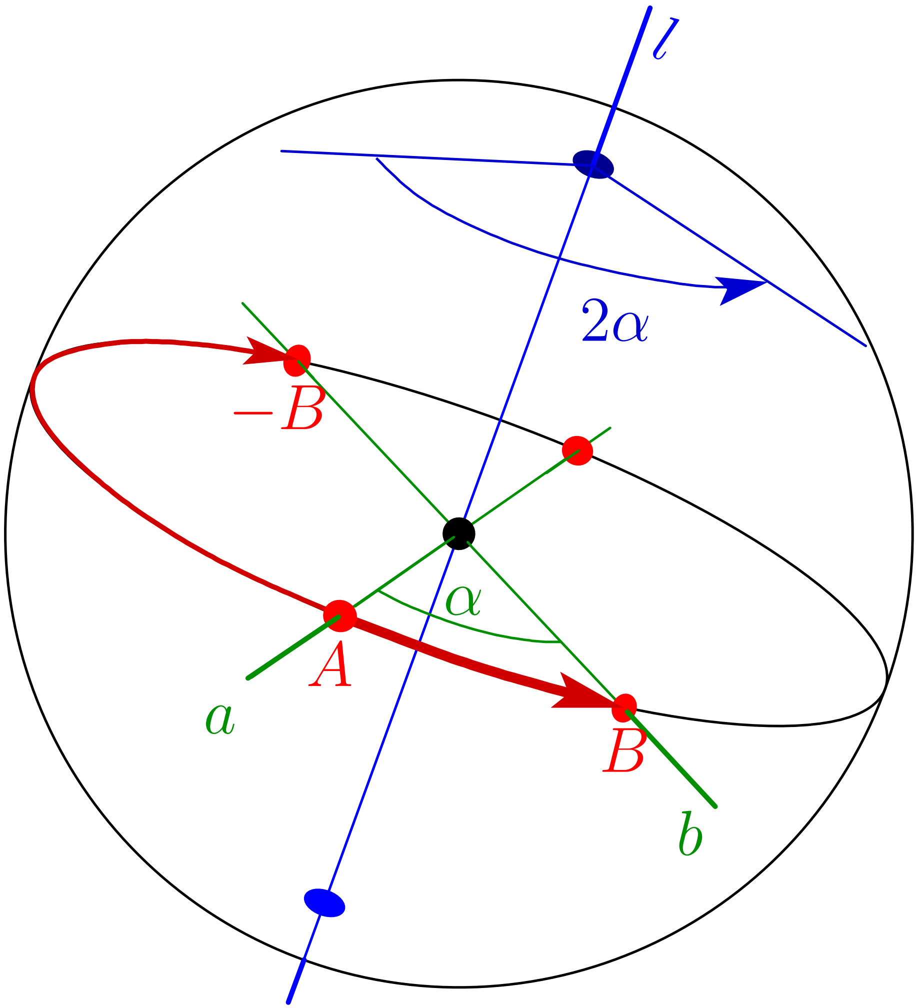


Head to tail for screws

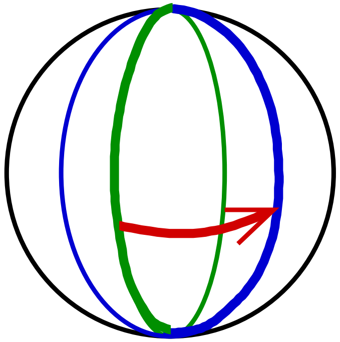


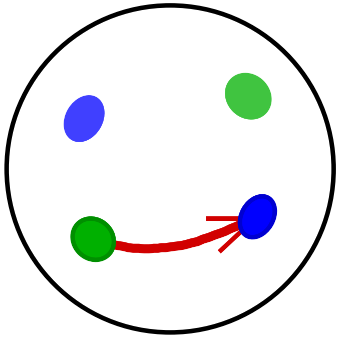
Rotations of 2-sphere



Rotations of 2-sphere

Biflippers:





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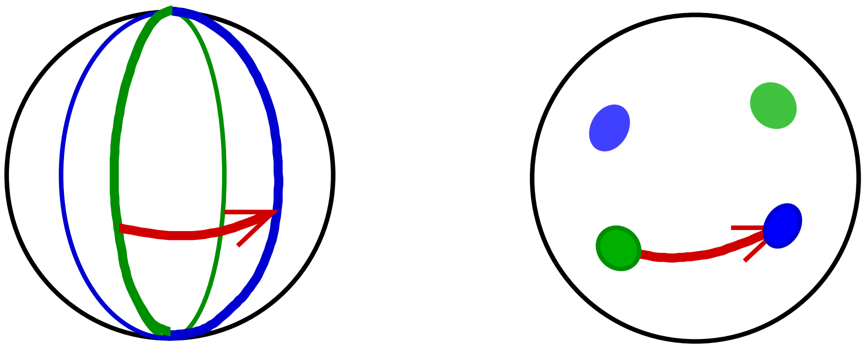
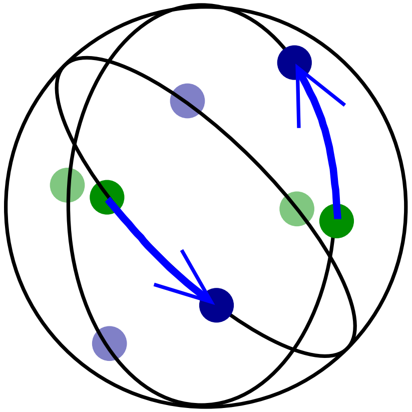


Head to tail for rotations:



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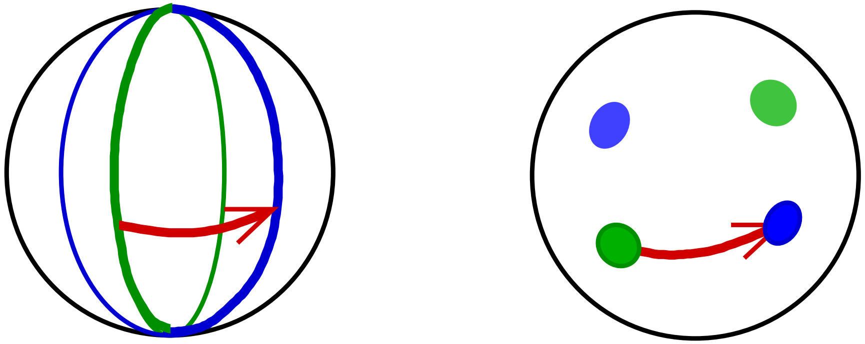
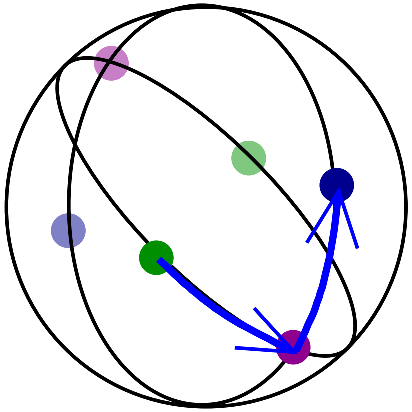


Head to tail for rotations:



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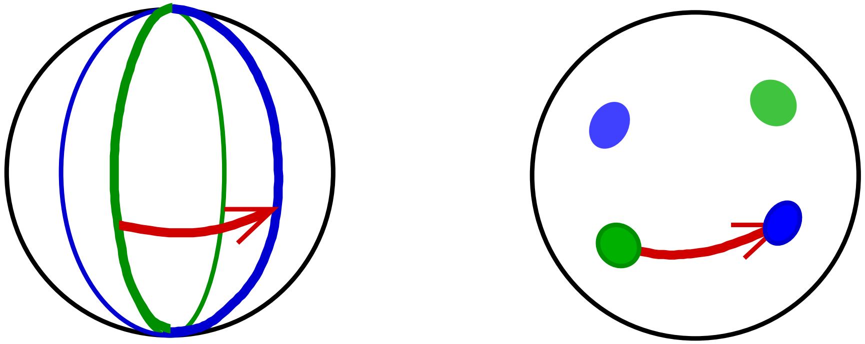
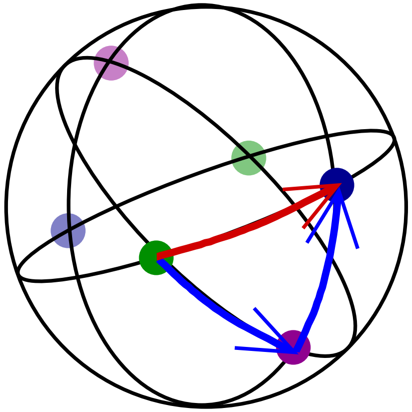


Head to tail for rotations:



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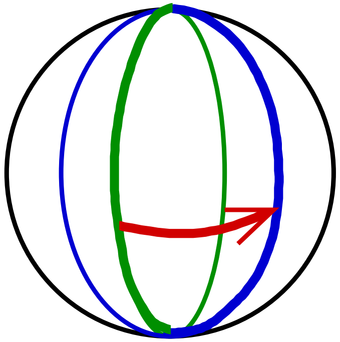
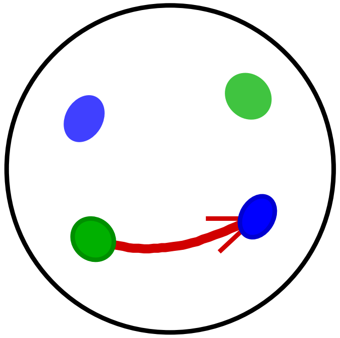
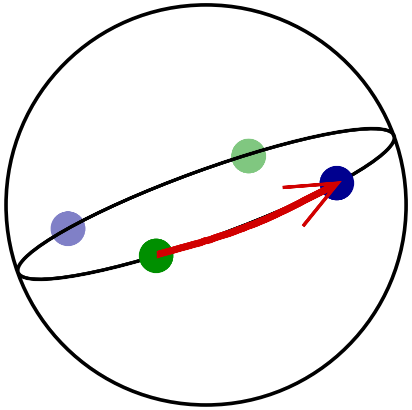




Head to tail for rotations:



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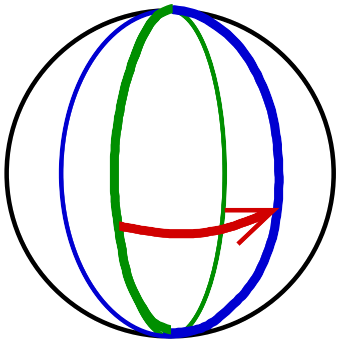
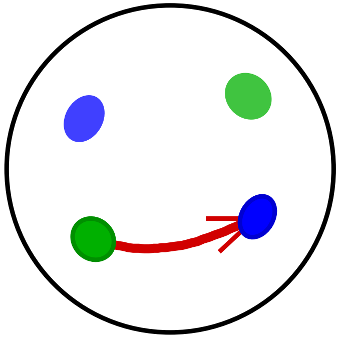
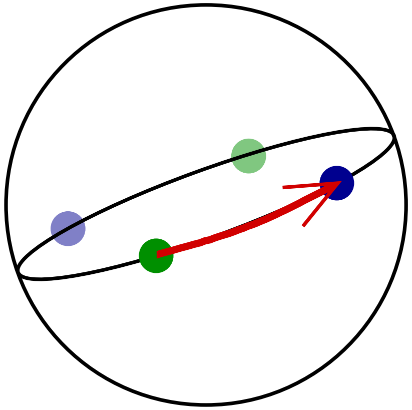




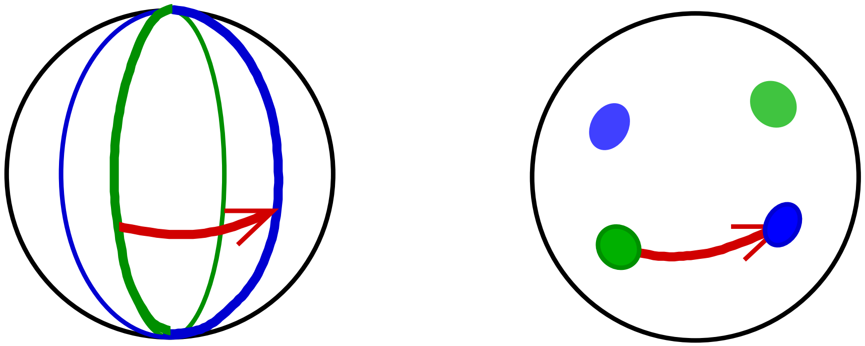
Head to tail for rotations:



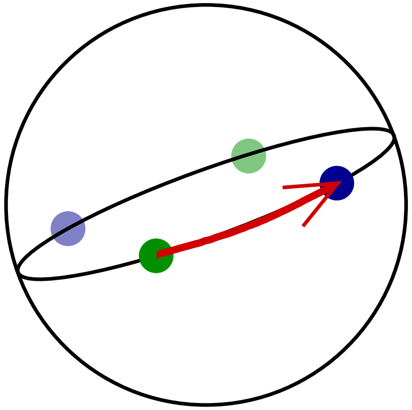
Biflipper vs. angular displacement vector vs. unit quaternion.

Rotations of 2-sphere

Biflippers:



Head to tail for rotations:

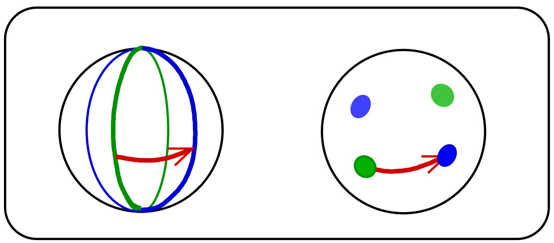


Biflipper vs. angular displacement vector vs. unit quaternion.

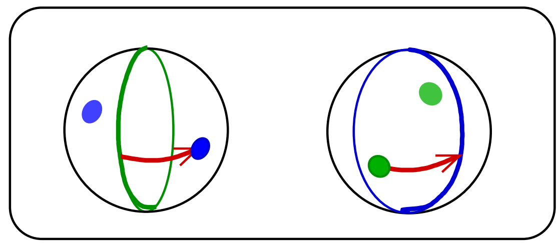
The rotation encoded by bilipper $\overrightarrow{wv}$ is defined by quaternion $vw = v \times w - v \cdot w$.

All biflippers on 2-sphere

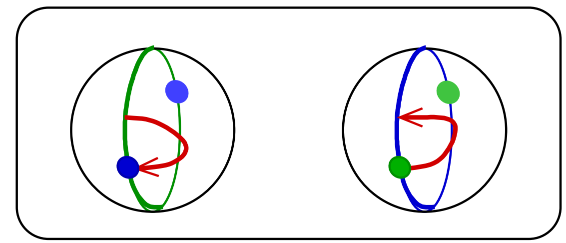
All biflippers on 2-sphere



rotations



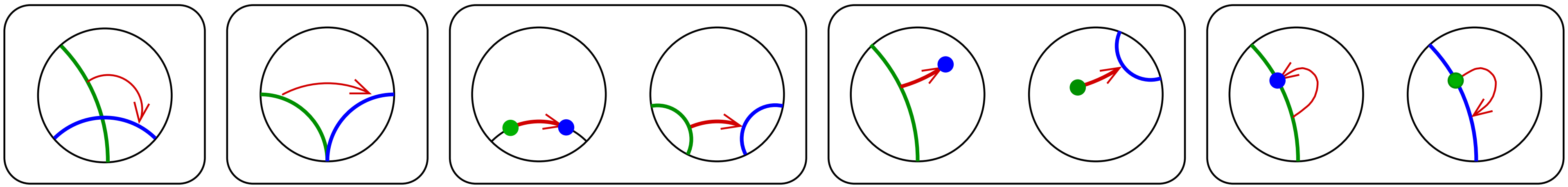
rotary reflections



reflections

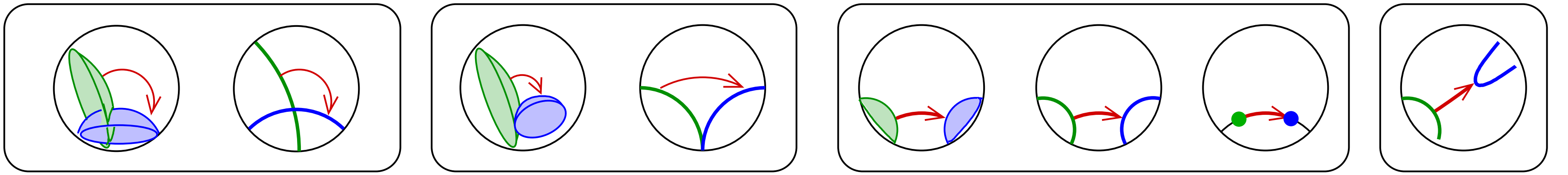
On the hyperbolic plane

On the hyperbolic plane

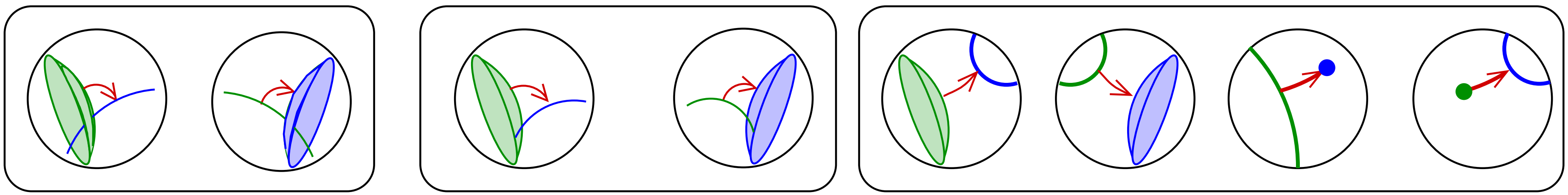


rotation *parallel motion* *translation* *glide reflections* *reflections*

In hyperbolic 3-space



rotation *parallel motion* *translation* *screw motion*



rotary reflections *parallel reflections* *glide reflections*

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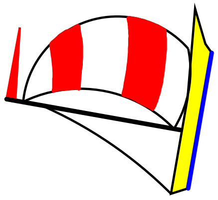

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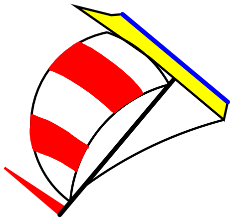

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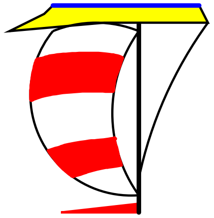

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