Tropical geometries and multifields

Oleg Viro

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Multi-valued algebra
- Triangle addition
- Multifields
- Examples of multifields
- Tropical addition of complex numbers
- Properties of tropical addition
- Operation induced on a subset
- Tropical addition of real numbers
- Other submultifields of $\mathcal{T}\mathbb{C}$
- Multiring homomorphisms
- Weakness of ideals

Dequantizataions

Complex Tropical Geometry

Multi-valued algebra

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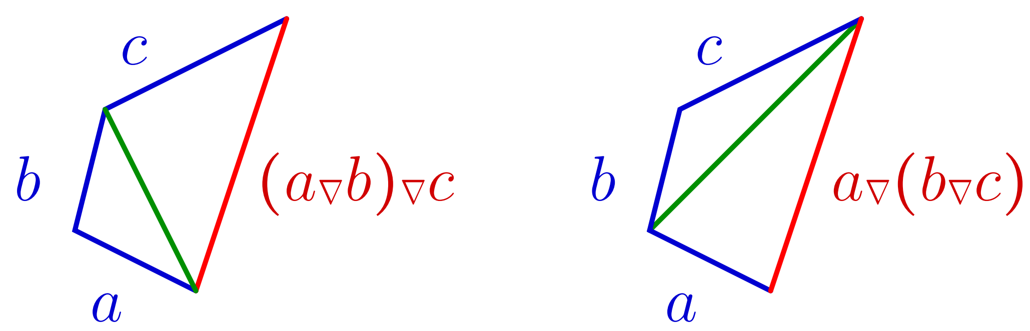

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$\mathbb{R}_{\geq 0}$ with addition $(a,b) \mapsto a \triangledown b$ and usual multiplication is a **multifield**.

Multifields

A set X with a multivalued operation

$$X \times X \to 2^X \setminus \{\varnothing\} : (a, b) \mapsto a \top b$$

and a multiplication $X \times X \to X : (a, b) \mapsto a \cdot b$ is called a **multifield**, if

- $(a, b) \mapsto a \top b$ is commutative, associative;
- $\exists 0 \in X$ such that $0 \top a = a$ for any $a \in X$;
- for $\forall a \in X$ there exists a unique $-a \in X$ such that $0 \in a \top (-a)$;
- $-(a \top b) = (-a) \top (-b)$
- $X \setminus 0$ is a commutative group under the multiplication;
- $0 \cdot a = 0$ for any $a \in X$;
- distributivity: $a(b \top c) = ab \top ac$ for any $a, b, c \in X$.

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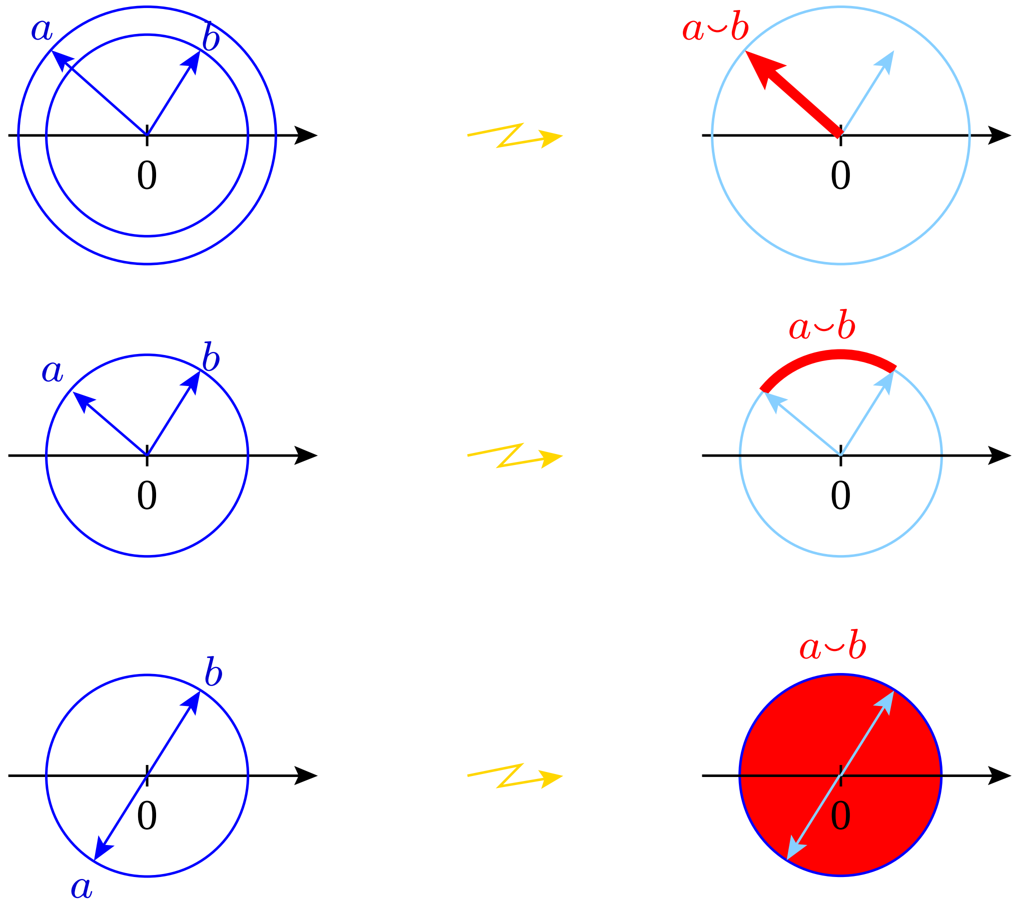
Similar to ∇, but with ultrametric triangle inequality.

Tropical addition of complex numbers



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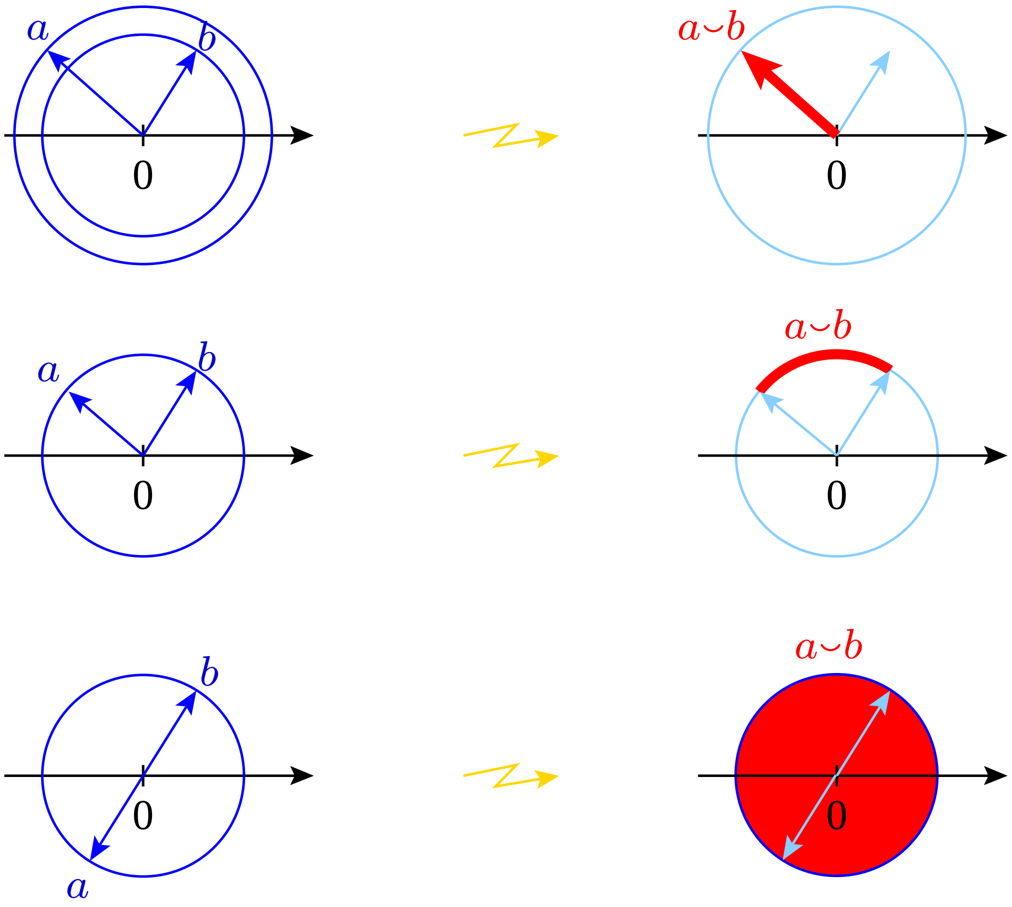

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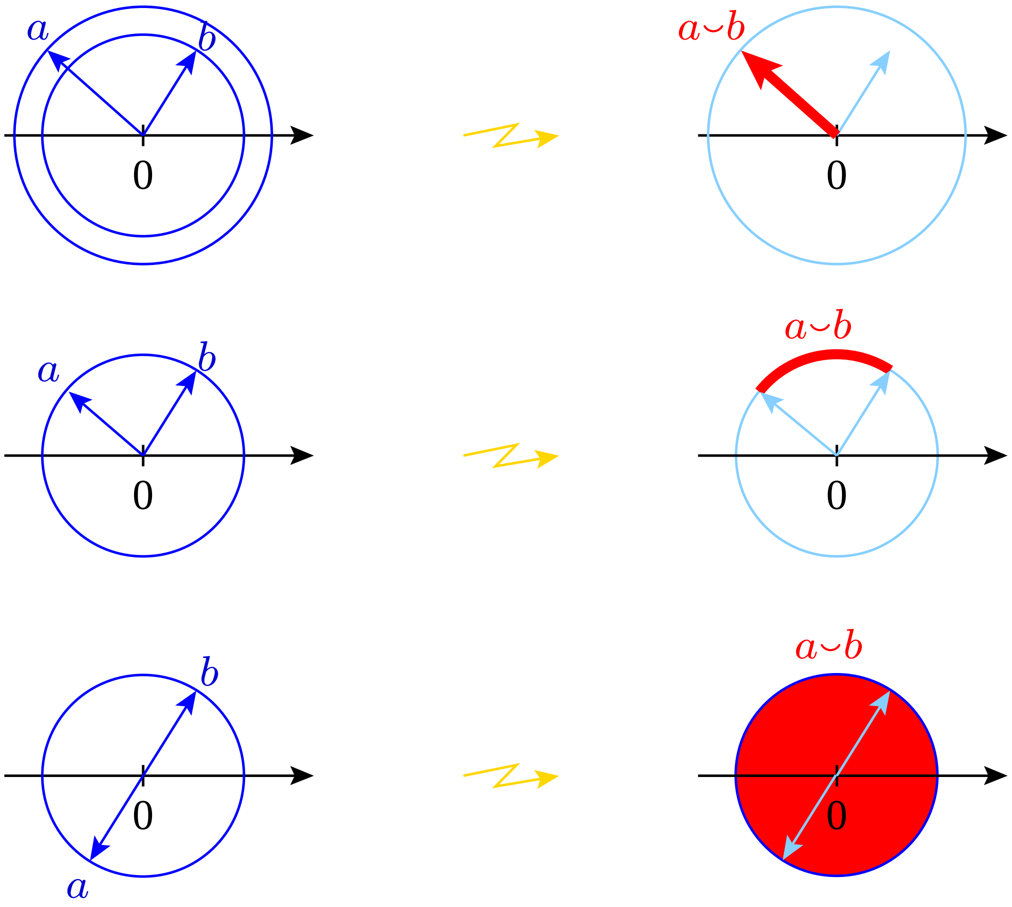

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The **complex tropical multifield** $\mathcal{T}\mathbb{C}$.

Properties of tropical addition

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How do several complex numbers with the same absolute values give zero?

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If p is a complex tropical polynomial and $X \subset \mathbb{C}$ is a closed set, then $p^{-1}(X) = \{a \mid X \subset p(a)\}$ is closed.

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(The definition of multivalued addition prohibits empty values.)

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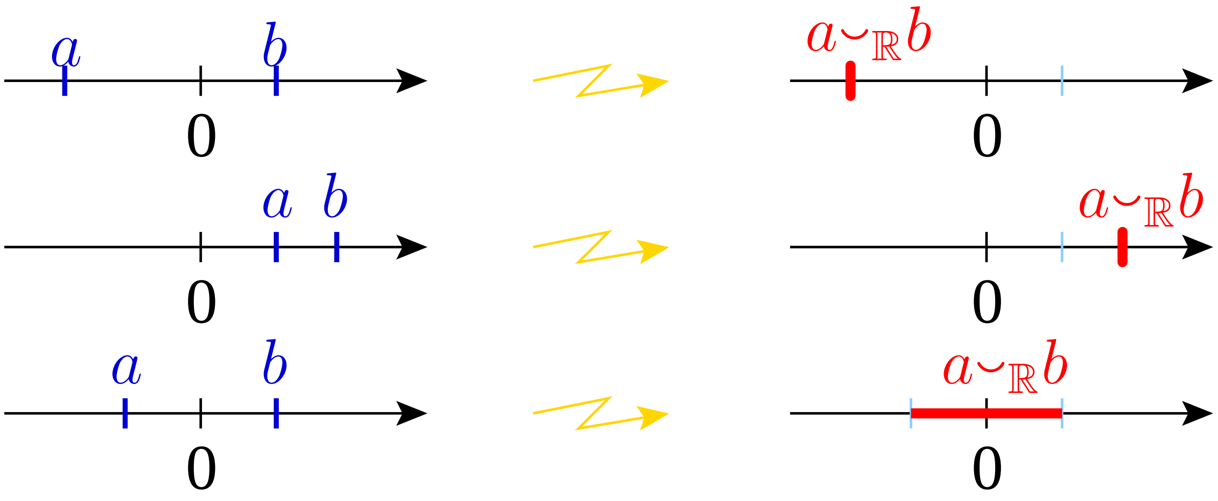

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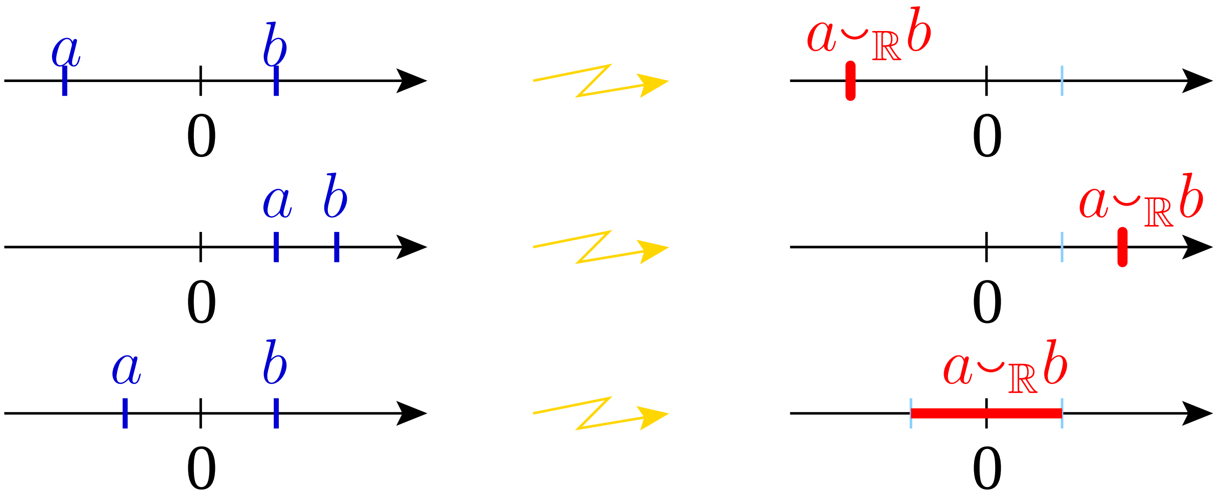

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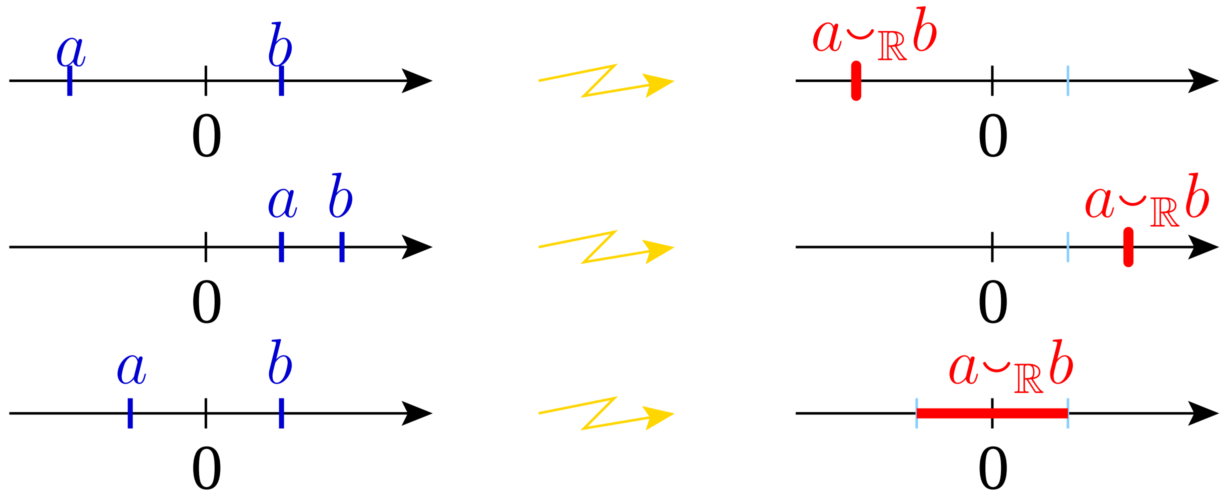

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Theorem. $\mathcal{T}\mathbb{R} = (\mathbb{R}, \smile_{\mathbb{R}}, \times)$ is a multifield.

Other submultifields of $\mathcal{T}\mathbb{C}$

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The smallest multifield $Q_1 = \{0, 1\}$ is not, because $\mathcal{T}\mathbb{C}$ is idempotent:
$a \smile a = a$ for any $a \in \mathcal{T}\mathbb{C}$, while $1 \curlyvee 1 = \{0, 1\}$ in Q_1.

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Inclusion $(\mathbb{R}_{\geq 0}, \max, \times) \to \mathcal{T}\mathbb{R}$ is a homomorphism.

Multiring homomorphisms

Multiring is a multifield without division.

A map $f : X \to Y$ is called a (multiring) homomorphism if $f(a \top b) \subset f(a) \top f(b)$ and $f(ab) = f(a)f(b)$ for any $a, b \in X$.

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A multiplicative non-archimedean norm $K \to \mathbb{R}$ is a multiring homomorphism from $K \to \mathrm{U}\nabla$.

non-archimedian = satisfies the ultra-metric triangle inequality

$$|a + b| \leq \max(a, b) \text{ for any } a, b \in K.$$

Multiring homomorphisms

A map $f : X \to Y$ is called a (multiring) homomorphism if $f(a \top b) \subset f(a) \top f(b)$ and $f(ab) = f(a)f(b)$ for any $a, b \in X$.

Example. $\mathbb{C} \to \nabla : z \mapsto |z|$ is a multiring homomorphism.

Generalization. A multiplicative norm $K \to \mathbb{R}_{\geq 0}$ in a ring K is a multiring homomorphism $K \to \nabla$.

A multiplicative non-archimedean norm $K \to \mathbb{R}$ is a multiring homomorphism from $K \to \mathsf{U}\nabla$.

Example.

$$\operatorname{sign} : \mathbb{R} \to \{0, 1, -1\} : x \mapsto \begin{cases} \frac{x}{|x|}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

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Example.

$$\operatorname{phase} : \mathbb{C} \to S^1 \cup \{0\} : x \mapsto \begin{cases} \frac{x}{|x|}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

is a multiring homomorphism $\mathbb{C} \to \Phi$ and $\mathcal{T}\mathbb{C} \to \Phi$.

Weakness of ideals

Ideal is a subset I in a multiring X such that $I \top I \subset I$ and $X I \subset I$.

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Moreover, there are non-injective multiring homomorphisms
between multifields. (e.g., $\operatorname{sign} : \mathbb{R} \to Q_2$).

Multi-valued algebra

Dequantizataions

- Litvinov-Maslov dequantization
- Dequantization $\nabla \to \mathrm{U}\nabla$
- Dequantization $\mathbb{C}$ to $\mathcal{T}\mathbb{C}$
- Dequantizations commute
- Tropical Geometry
- Graphs and curves

Complex Tropical Geometry

Dequantizataions

Litvinov-Maslov dequantization

For $h > 0$, consider a map $R_h : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$

$$x \mapsto \begin{cases} x^{\frac{1}{h}}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

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These are multiplicative homomorphisms.

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These are multiplicative homomorphisms, but they do not respect addition.

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Pull back the addition:

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Pull back the addition: $a +_h b = R_h^{-1}(R_h(a) + R_h(b))$

$$= \left(a^{1/h} + b^{1/h}\right)^h$$

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$R_h = (\mathbb{R}_{\geq 0}, +_h, \times)$ is a copy of semifield $(\mathbb{R}_{\geq 0}, +, \times)$ and $R_h : P_h \to (\mathbb{R}_{\geq 0}, +, \times)$ is an isomorphism.

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$\lim_{h\to 0}(a^{1/h} + b^{1/h})^h = \max(a, b)$.

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$\lim_{h \to 0}(a^{1/h} + b^{1/h})^h = \max(a, b)$.

P_h is a degeneration of $(\mathbb{R}_{\geq 0}, +, \times)$ to $(\mathbb{R}_{\geq 0}, \max, \times)$.

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P_h is a **dequantization** of $(\mathbb{R}_{\geq 0}, +, \times)$ to $(\mathbb{R}_{\geq 0}, \max, \times)$.

Dequantization $\nabla \to \mathsf{U}\nabla$

For $h > 0$, consider a map $R_h : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$

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These are multiplicative homomorphisms.

Dequantization $\triangledown \to \mathsf{U}\triangledown$

For $h > 0$, consider a map $R_h : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$

$$x \mapsto \begin{cases} x^{\frac{1}{h}}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

These are multiplicative homomorphisms, but they do not respect $(a, b) \mapsto a \triangledown b$.

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For $h > 0$, consider a map $R_h : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$

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Pull back the $\triangledown$-addition: $a \triangledown_h b = R_h^{-1}(R_h(a) \triangledown R_h(b))$

$$= \left\{c \in \mathbb{R}_{\geq 0} \;\middle|\; \left|a^{1/h} - b^{1/h}\right|^h \leq c \leq \left(a^{1/h} + b^{1/h}\right)^h\right\}$$

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$\nabla_h = (\mathbb{R}_{\geq 0}, \triangledown_h, \cdot)$ is a copy of ∇ and $R_h : \nabla_h \to \nabla$ is an isomorphism.

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If $a \neq b$, then

$\lim_{h\to 0} |a^{1/h} - b^{1/h}|^h = \lim_{h\to 0}(a^{1/h} + b^{1/h})^h = \max(a, b)$,

if $a = b$, then $|a^{1/h} - b^{1/h}|^h = 0$, while $\lim_{h\to 0}(a^{1/h} + b^{1/h})^h = a$.

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The endpoints of segment $a \triangledown_h b$ tend
to the endpoints of segment $a \curlyvee b$ as $h \to 0$.

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The endpoints of segment $a \triangledown_h b$ tend
to the endpoints of segment $a \curlyvee b$ as $h \to 0$.

Let $a \triangledown_0 b := a \curlyvee b$.

∇_h is a dequantization of ∇ to $\mathrm{U}\nabla$.

Dequantization $\mathbb{C}$ to $\mathcal{T}\mathbb{C}$

For $h > 0$ consider a map $S_h\colon \mathbb{C} \to \mathbb{C}$

$$z \mapsto \begin{cases} |z|^{\frac{1}{h}} \frac{z}{|z|}, & \text{if } z \neq 0; \\ 0, & \text{if } z = 0. \end{cases}$$

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These are multiplicative isomorphisms.

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These are multiplicative isomorphisms, but they do not respect the addition.

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let $\Gamma \subset \mathbb{R}_{\geq 0} \times \mathbb{C}^3$ be a graph of $+_h$ for all $h > 0$,

$$\Gamma = \{(h, a, b, c) \in \mathbb{C}^3 \mid a +_h b = c\}.$$

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$\mathbb{C}_h$ is a dequantization of $\mathbb{C}$ to $\mathcal{T}\mathbb{C}$.

Dequantizations commute

$$\mathbb{C} \equiv \mathbb{C}_h \xrightarrow{h \to 0} \mathbb{C}_0 = \mathcal{T}\mathbb{C}$$

Dequantizations commute

$$\begin{CD}
\mathbb{C} \equiv \mathbb{C}_h @>{h\to 0}>> \mathbb{C}_0 = \mathcal{T}\mathbb{C} \\
@V{x\mapsto|x|}VV @VV{x\mapsto|x|}V \\
\nabla \equiv \nabla_h @>>{h\to 0}> \nabla_0 = \mathsf{U}\nabla
\end{CD}$$

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@V{x\mapsto\log x}VV @VV{x\mapsto\log x}V \\
\mathbb{R} @>>{h\to 0}> \mathbb{Y}_0 = \mathbb{Y}
\end{CD}
$$

Dequantizations commute

Complex Algebraic Geometry

$$
\begin{array}{ccc}
\mathbb{C} \equiv \mathbb{C}_h & \xrightarrow{h \to 0} & \mathbb{C}_0 = \mathcal{T}\mathbb{C} \\
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\end{array}
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Amoebas

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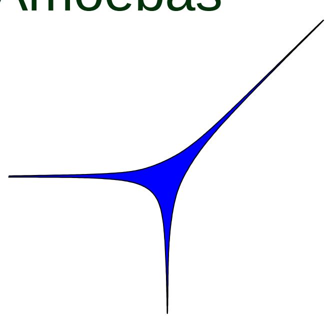
Amoebas



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\end{array}
$$

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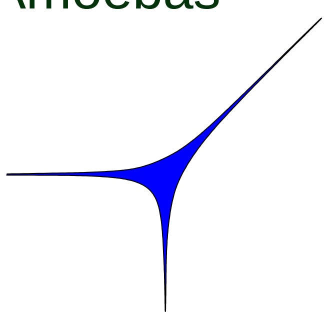

Tropical Geometry

Dequantizations commute

Complex Algebraic Geometry

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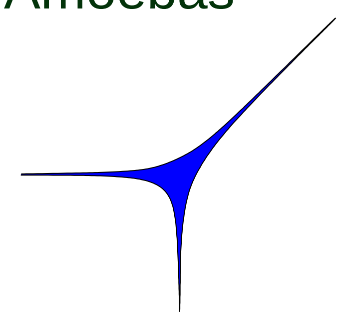

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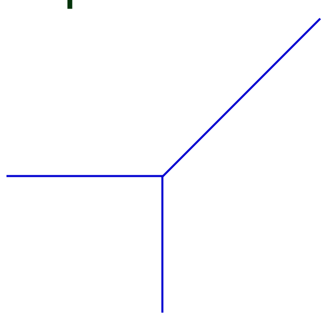

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Complex Algebraic Geometry | Complex Tropical Geometry

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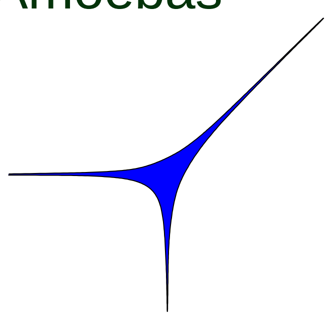

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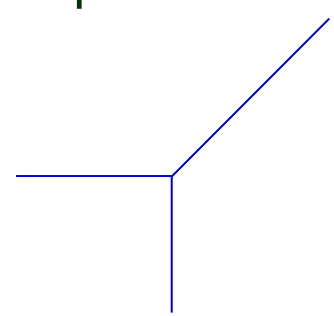

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$-\infty \in \mathsf{Y}_{k=(k_1,\dots,k_n)}(a_k + k_1 x_1 + \dots + k_n x_n)$ where the maximum $\max_{k=(k_1,\dots,k_n)}(a_k + k_1 x_1 + \dots + k_n x_n)$ is attained at least twice.

Graphs and curves

In geometry over $\mathbb{T}$

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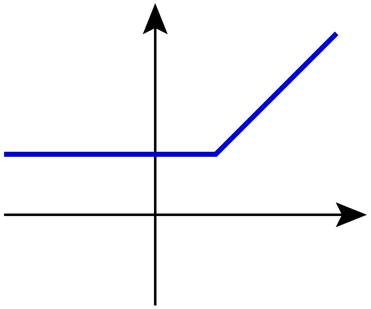

the graph of function $y = x + 1$,

Graphs and curves

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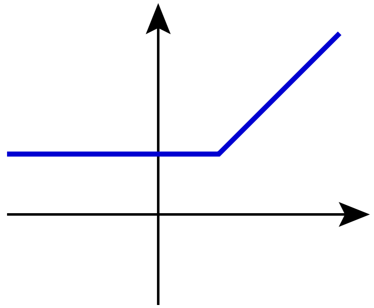

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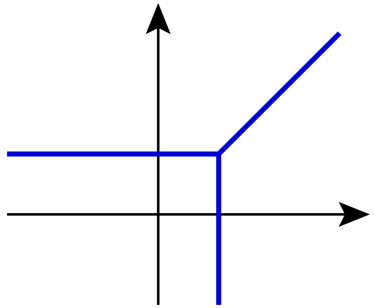
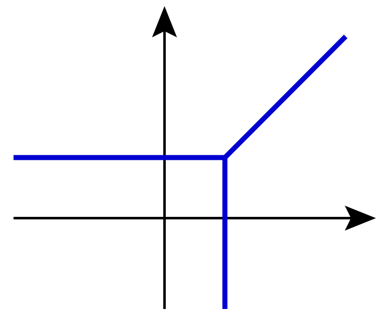


the curve defined by $x + y + 1$.

Graphs and curves

In geometry over $\mathbb{Y}$



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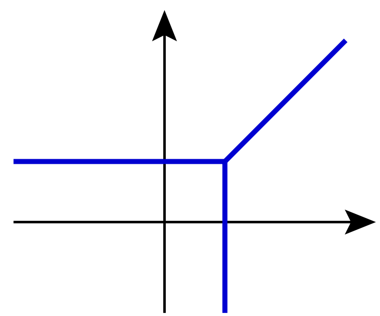

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Multi-valued algebra

Dequantizataions

Complex Tropical Geometry

- Good and bad polynomials
- Complex tropical line
- Complex tropical varieties
- Problems

Complex Tropical Geometry

Good and bad polynomials

Is $x = x \smile 1 \smile -1$?

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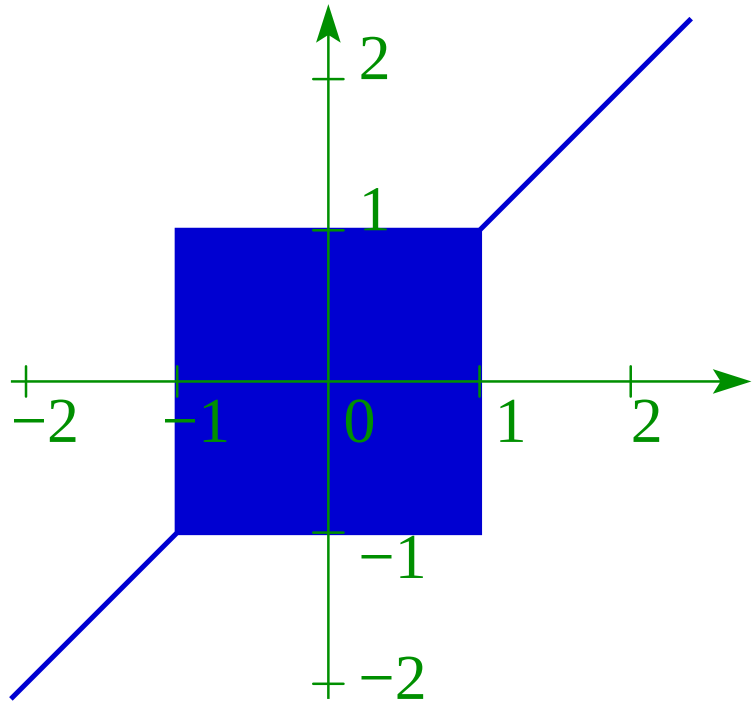

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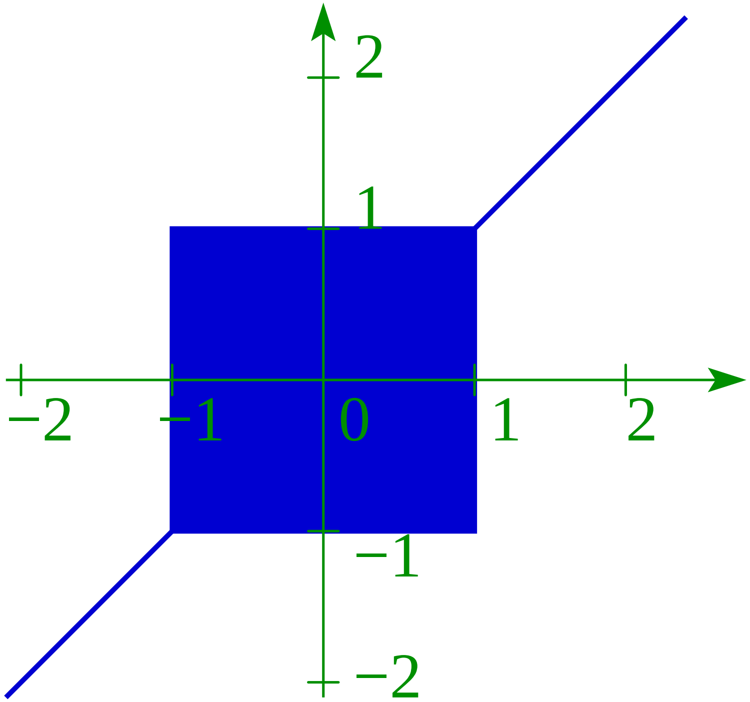

Graph of $y = x \smile 1 \smile -1$.

A polynomial is said to be **pure** if it has no two monomials with the same exponents.

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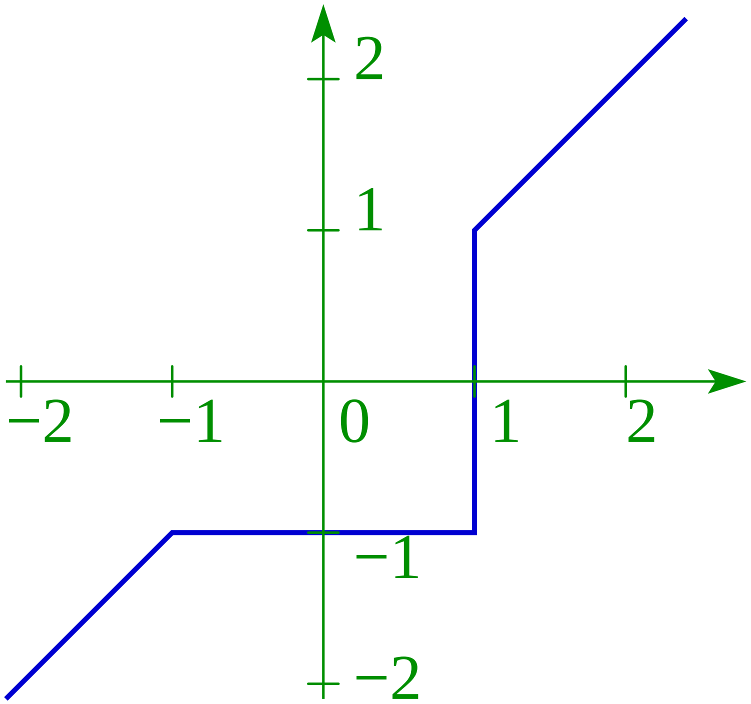

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Is $x^2 \smile -1 = (x \smile 1)(x \smile -1)$?

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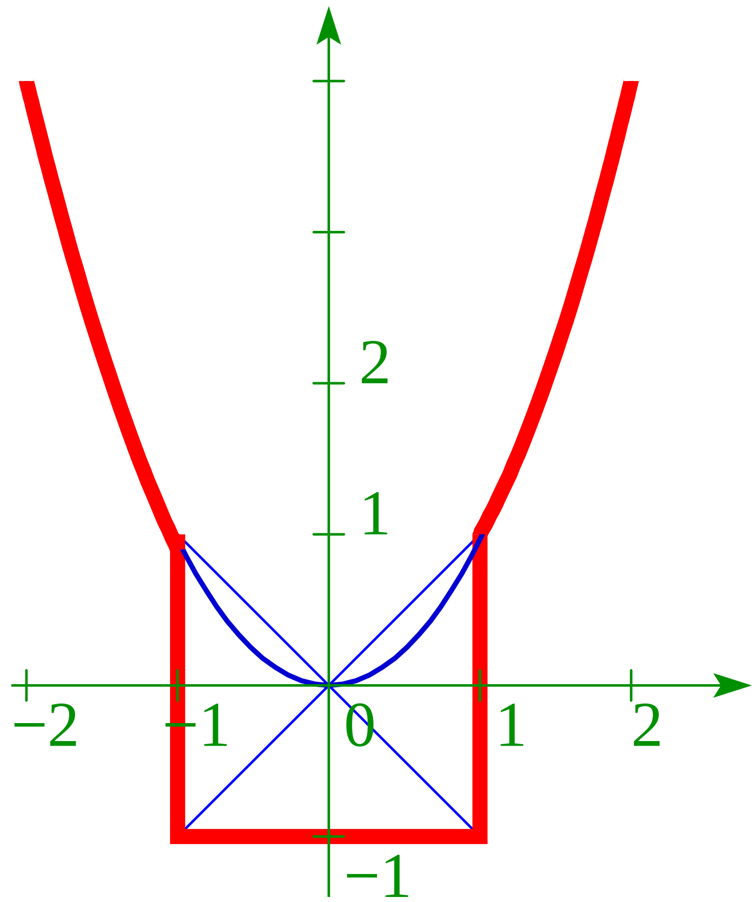
Is $x^2 \smile -1 = (x \smile 1)(x \smile -1)$? Yes, $x^2 \smile -1 = x^2 \smile x \smile -x \smile -1$.



Complex tropical line

$$\{(x, y) \in \mathbb{C}^2 \mid 0 \in x \smile y \smile 1\}$$

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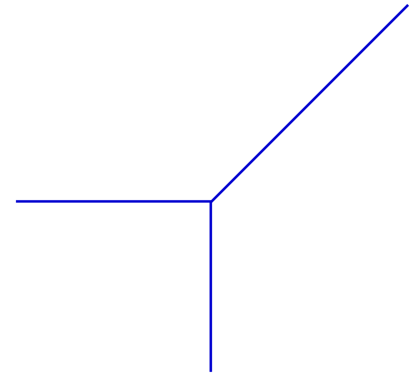

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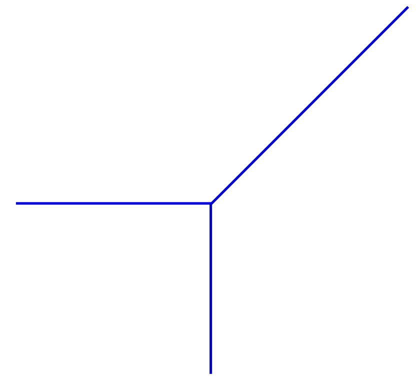

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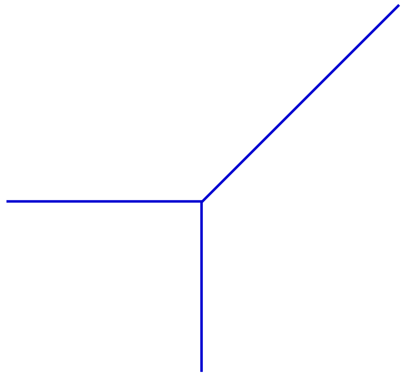

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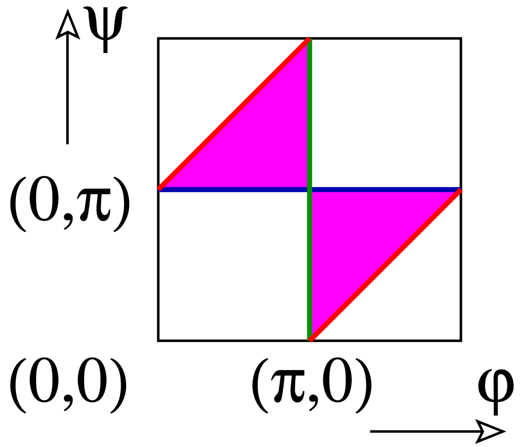

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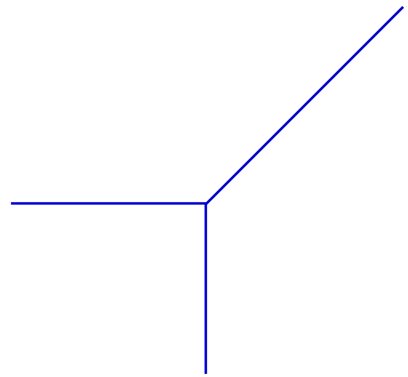

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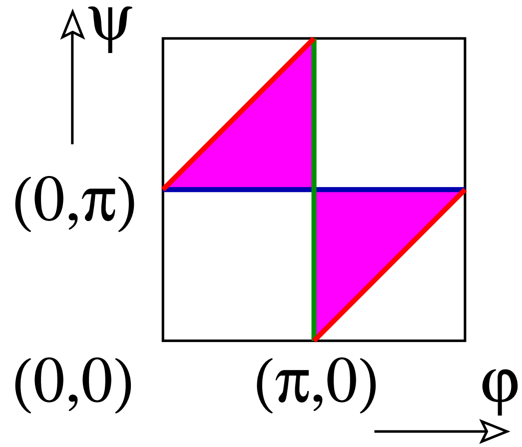

Overall a disk.

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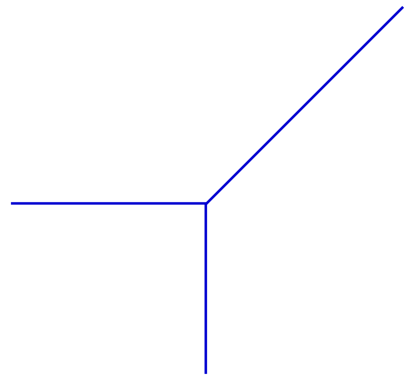

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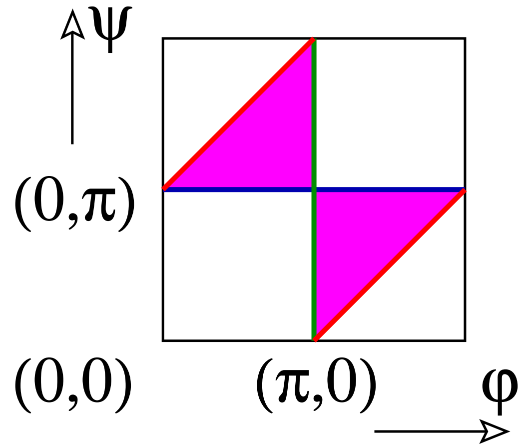

Overall a disk.   A 2-manifold!

Complex tropical varieties

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Conjecture. If under the dequantization a non-singular complex varieties tends to a non-singular complex tropical variety, then the dequantization provides an isotopy between the varieties.

Problems

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Conjecture. (Itenberg, Mikhalkin, Zharkov) Let X be a complex tropical variety, $X_q = \mathrm{Log}^{-1}(q\text{-skeleton}(\mathrm{Log}(X)))$, $H_n^q(X) = \mathrm{Im}(\mathrm{in}_* : H_n(X_q) \to H_n(X))$, $H_{p,q}(X) = H_{p+q}^q(X)/H_{p+q}^{q-1}(X)$. Then $H_{p,q}(X) \otimes \mathbb{C}$ is isomorphic to $H^{p,q}(X_h)$.

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There is a multivalued addition of p -adic numbers.

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This is a work in progress started 5 months ago.

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