DEQUANTIZATION OF REAL ALGEBRAIC GEOMETRY ON LOGARITHMIC PAPER

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Abstract. On logarithmic paper some real algebraic curves look like smoothed broken lines. Moreover, the broken lines can be obtained as limits of those curves. The corresponding deformation can be viewed as a quantization, in which the broken line is a classical object and the curves are quantum. This generalizes to a new connection between algebraic geometry and the geometry of polyhedra, which is more straightforward than the other known connections and gives a new insight into constructions used in the topology of real algebraic varieties.

1. Graphs of polynomials on logarithmic paper

1.1. How to visualize a real polynomial? If you ever tried to draw the graph for a polynomial of degree greater than, say, 4 and consisting of at least 4 monomials, you are aware of the natural difficulties. The graph is too steep. Whatever scale you choose, either some important details do not fit into the picture or are too small. The usual recipes from Calculus do not address the problem, but suggest, instead, to find roots of the first two derivatives, which does not seem to be much easier than the original problem.

1.2. Logarithmic paper. A physicist or engineer can give a more practical advice, based on their experience: *use (double) logarithmic paper.* This is a graph paper, called also *log paper*, with a non-uniform net of coordinate lines and logarithmic scales on both axes. On a log paper a point with coordinates x, y is shown at the position with the usual, Cartesian coordinates equal to $\ln x$, $\ln y$. In other words, the transition to the log paper corresponds to the change of coordinates:

$$\begin{cases} u = \ln x \\ v = \ln y. \end{cases}$$

On a log paper the first quadrant is expanded homeomorphically to the whole plane, the line $x = 1$ occupies the position of the axis of ordinates, the line $y = 1$ occupies the position of the axis of abscissas, the unit square bounded by these lines and the coordinate axes occupies the whole third quadrant.

1.3. A monomial on logarithmic paper. Let us try to follow the advice to use a log paper. Consider first the simplest special case: the graph of a monomial ax^k (i. e., the curve defined by $y = ax^k$). We are forced to consider only positive x, y and hence assume a to be positive as well. Then $v = \ln y = \ln(ax^k) = k \ln x + \ln a = ku + \ln a = ku + b$, where we denote $\ln a$ by b. Now the curve which we want to draw is defined in the coordinates u and v by the equation $v = ku + b$. Everybody knows that this is the straight line with slope k meeting the axis of ordinates (i.e., v-axis) at $(0, b)$.

1.4. Few slightly more complicated polynomials. First, consider a line $y = 1 + x$. Then

$$v = \ln y = \ln(1 + x) = \ln(1 + e^u).$$

See the left plot in Figure 1, where the graph of $v = \ln(1 + e^u)$ is shown together with lines $v = 0$ and $v = u$, which represent on the log paper the monomials 1 and x involved in our polynomial $1 + x$. The graph of $v = \ln(1 + e^u)$ looks like the broken line $v = \max\{0, u\}$ with a smoothed corner: it goes along and above of this broken line getting very close to it as $|u|$ grows. For $|u| > 4$ the difference becomes beyond the resolution.

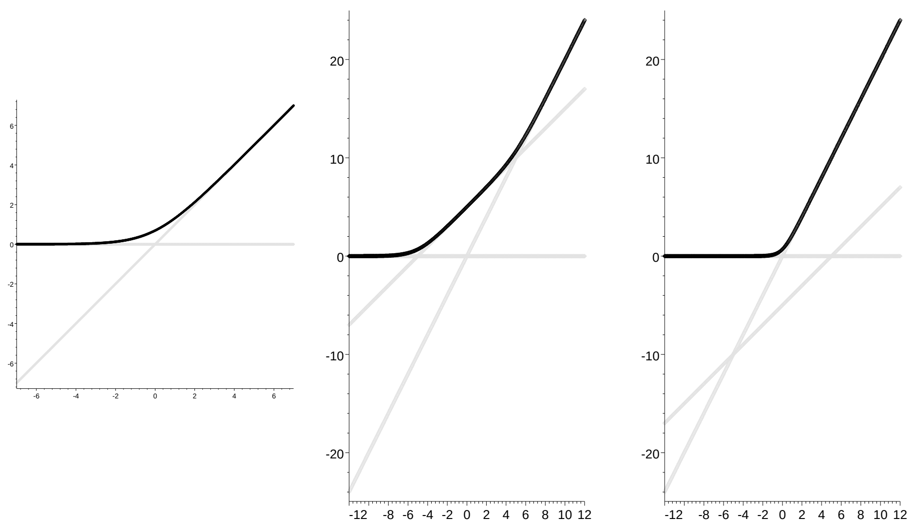

FIGURE 1. Log paper graphs of $1 + x$, $1 + e^5x + x^2$ and $1 + e^{-5}x + x^2$.

Here are two further examples: the quadratic polynomials $1 + e^{\pm 5}x + x^2$. Then

$$v = \ln(1 + e^{\pm 5}x + x^2) = \ln(1 + e^{u\pm 5} + e^{2u}).$$

See the central and right plots in Figure 1. The graph of $v = \ln(1+e^{u\pm 5}+e^{2u})$ looks like the broken line $v = \max\{0, u \pm 5, 2u\}$ with smoothed corners. It goes along and above of this broken line getting very close to it far from its corners. Notice that the lines $v = 0$, $v = u \pm 5$ and $v = 2u$ represent on the logarithmic paper the monomials 1, $e^{\pm 5}x$ and x^2, respectively.

1.5. A polynomial versus the maximum of its monomials. This suggests, for a polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ with positive real coefficients $a_i = e^{b_i}$, to compare the graphs on log paper for p and the maximum $M(p)(x) = \max\{a_n x^n, a_{n-1}x^{n-1}, \ldots, a_0\}$ of its monomials. Denote the graph on log paper of a function f by Γ_f. With respect to the usual Cartesian coordinates, Γ_p is the graph of

$$L_p(u) = \ln\left(e^{nu+b_n} + e^{(n-1)u+b_{n-1}} + \cdots + e^{b_0}\right)$$

and $\Gamma_{M(p)}$ is the graph of a piecewise linear convex function

$$M_p(u) = \max\left\{nu + b_n,\ (n-1)u + b_{n-1},\ \ldots\ ,\ b_0\right\}.$$

Obviously, $M_p(u) \le L_p(u) \le M_p(u) + \ln(n+1)$. Hence Γ_p is above $\Gamma_{M(p)}$, but below a copy of $\Gamma_{M(p)}$ shifted upwards by $\ln(n+1)$. The latter is in fact a rough estimate. It turns to equality only at u, where all linear functions, whose maximum is $M_p(u)$, are equal: $nu + b_n = (n-1)u + b_{n-1} = \cdots = b_0$.

For a generic value of u, only one of these functions is equal to $M_p(u)$. Say $M_p(u) = ku + b_k$, while $M_p(u) > d + lu + b_l$ for some positive d and each $l \ne k$. Then

$$L_p(u) < M_p(u) + \ln(1 + ne^{-d}) < M_p(u) + e^{-d}n.$$

If for some value of u the values of all of the functions $ku + b_k$ except two are smaller than $M_p(u) - d$, then

$$L_p(u) < M_p(u) + \ln(2 + (n-1)e^{-d}) < M_p(u) + \ln 2 + e^{-d}(n-1)/2.$$

Thus, on a logarithmic paper the graph of a generic polynomial with positive coefficients lies in a narrow strip along the brocken line which is the graph of the maximum of its monomials. The width of the strip is estimated by characteristics of the mutual position of the lines which are the graphs of the monomials. The less congested the configuration of these lines, the norrower this strip.

1.6. Rescalings pushing the graph of a polynomial to a PL-graph. A natural way to make a configuration of lines less congested without changing its topology is to apply a dilation $(u, v) \mapsto (Cu, Cv)$ with a large $C > 0$. In what follows it is more convenient to use instead of C a parameter h related to C by $h = 1/C$. In terms of h the dilation acts by $(u, v) \mapsto (u/h, v/h)$. It maps the graph of $v = ku + b$ to the graph of $v = ku + b/h$. The parallel operation on monomials replaces ax^k by $a^{1/h}x^k$.

Consider the corresponding family of polynomials: $p_h(x) = \sum_k a_k^{1/h} x^k$. On log paper, the graphs of its monomials are obtained by dilation with ratio $1/h$ from the graphs of the corresponding monomials of p. Hence $\Gamma_{M(p_h)}$ is the image of $\Gamma_{M(p)}$ under the same dilation. However, Γ_{p_h} is not the image of Γ_p. It still lies in a strip along $\Gamma_{M(p_h)}$ and the strip is getting narrower as h decreases, but at the corners of $\Gamma_{M(p_h)}$ the width of the strip cannot become smaller than $\ln 2$.

To keep the picture of our expanding configuration of lines (the graphs of monomials) independent on h, let us make an additional calibration of coordinates: set $u_h = hu = h\ln x$, $v_h = hv = h\ln y$. Denote by Γ^h_f the graph of a function $y = f(x)$ in the plane with coordinates u_h, v_h.

Then $\Gamma^h_{M(p_h)}$ does not depend on h. The additional scaling reduces the width of the strip along $\Gamma^h_{M(p_h)}$, where $\Gamma^h_{p_h}$ lies, forcing the width to tend to 0 as $h \to 0$. Thus $\Gamma^h_{p_h}$ tends to $\Gamma^h_{M(p_h)}$ (in the C^0 sense) as $h \to 0$.

2. Quantization

2.1. Maslov dequantization of positive real numbers. The rescaling formulas $u_h = h \ln x$, $v_h = h \ln y$ bring to mind formulas related to the Maslov dequantization of real numbers, see e.g. [4], [5]. The core of the Maslov dequantization is a family of semirings $\{S_h\}_{h\in[0,\infty)}$ (recall that a semiring is a sort of ring, but without subtraction). As a set, each of S_h is $\mathbb{R}$. The semiring operations $\oplus_h$ and $\odot_h$ in S_h are defined as follows:

$$a \oplus_h b = \begin{cases} h\ln(e^{a/h} + e^{b/h}), & \text{if } h > 0 \\ \max\{a, b\}, & \text{if } h = 0 \end{cases} \tag{1}$$

$$a \odot_h b = a + b \tag{2}$$

These operations depend continuously on h. For each $h > 0$ the map

$$D_h : \mathbb{R}_+ \smallsetminus \{0\} \to S_h : x \mapsto h \ln x$$

is a semiring isomorphism of $\{\mathbb{R}_+ \smallsetminus \{0\}, +, \cdot\}$ onto $\{S_h, \oplus_h, \odot_h\}$, that is

$$D_h(a + b) = D_h(a) \oplus_h D_h(b), \qquad D_h(ab) = D_h(a) \odot_h D_h(b).$$

Thus S_h with $h > 0$ can be considered as a copy of $\mathbb{R}_+ \smallsetminus \{0\}$ with the usual operations of addition and multiplication. On the other hand, S_0 is a copy of $\mathbb{R}$ where the operation of taking maximum is considered as an addition, and the usual addition, as a multiplication.

Applying the terminology of quantization to this deformation, we must call S_0 a classical object, and S_h with $h \neq 0$, quantum ones. The analogy with Quantum Mechanics motivated the following *correspondence principle* formulated by Litvinov and Maslov [4] as follows:

"There exists a (heuristic) correspondence, in the spirit of the correspondence principle in Quantum Mechanics, between important, useful and interesting constructions and results over the field of real (or complex) numbers (or the semiring of all nonnegative numbers) and similar constructions and results over idempotent semirings."

This principle proved to be very fruitful in a number of situations, see [4], [5]. According to the correspondence principle, the idempotent counterpart of a polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_0$ is a convex PL-function $M_p(u) = \max\{nu + b_n,\ (n-1)u + b_{n-1},\ \dots\ ,\ b_0\}$. As we have seen above, p and M_p are related not only on an heuristic level. In Section 1.6 we connected the graph Γ_p of p on logarithmic paper and the graph $\Gamma_{M(p)}$ of M_p by a continuous family of graphs $\{\Gamma^h_{p_h}\}_{h\in(0,1)}$.

2.2. Logarithmic paper as a graphical device for the Maslov dequantization. As we saw in Section 1.5, the graph of a polynomial $p(x) = \sum_k a_k x^k$ with positive real coefficients $a_k = e^{b_k}$ on log paper is the graph of function $\mathbb{R} \to \mathbb{R}$ defined by $v = \ln\left(\sum_k e^{ku+b_k}\right)$. Observe that $\ln\left(\sum_k e^{ku+b_k}\right)$ is the value in S_1 of the polynomial $\sum_k b_k x^k$ at $x = u$. Therefore we can identify the graph Γ_p of $p(x) = \sum_k a_k x^k$ on log paper with the (Cartesian) graph of the polynomial $\sum_k b_k x^k$ on S_1^2.

Furthermore, $\Gamma_{p_h}^h$ is the graph of the function $\mathbb{R} \to \mathbb{R}$ defined by

$$v = h \ln \left(\sum_k a_k^{1/h} e^{(ku)/h} \right) = h \ln \left(\sum_k e^{(ku+b_k)/h} \right).$$

Observe, that the right hand side is the value in S_h of the same polynomial $\sum_k b_k x^k$ at u. Therefore we can identify the graph $\Gamma_{p_h}^h$ of $p_h(x) = \sum_k a_k^{1/h} x^k$ on log paper with the (Cartesian) graph of $\sum_k b_k x^k$ on S_h^2.

At last, the graph of $\sum_k b_k x^k$ on S_0^2 is the the graph of M_p.

We see that the whole job of deforming Γ_p to the graph of a piecewise linear convex function can be done by the Maslov dequantization: the deformation consists of the graphs of the same polynomial $\sum_k b_k x^k$ on S_h^2 for $h \in [0, 1]$. The coefficients b_k of this polynomial are logarithms of the coefficients of the original polynomial: $b_k = \ln a_k$. Since the map $x \mapsto \ln x : \mathbb{R}_+ \smallsetminus 0 \to S_1$ was denoted above by D_1, we denote by $D_1 F$ the polynomial obtained from a polynomial F with positive coefficients by replacing its coefficients with their logarithms. Thus $\sum_k b_k x^k = D_1 p(x)$. Since D_1 is a semiring homomorphism, the graph Γ_p of p on log paper is the graph of $D_1 p$ on S_1^2. The other graphs involved into the deformation are the graphs of the same polynomial $D_1 p$ on S_h^2. They coincide with the graphs on log paper of the preimages p_h of $D_1 p$ under D_h. Indeed, $p_h(x) = \sum_k a_k^{1/h} x^k$ and $D_h^{-1}(b_k) = D_h^{-1} D_1(a_k) = e^{D_1(a_k)/h} = e^{(\ln a_k)/h} = a_k^{1/h}$.

For a real polynomial $p(x) = \sum_k a_k x^k$ with positive coefficients, we shall call $p_h(x) = \sum_k a_k^{1/h} x^k$ with $h > 0$ the *dequantizing family* of polynomials.

2.3. Real algebraic geometry as quantized PL-geometry. The notion of polynomial is central in algebraic geometry. (I believe the subject of algebraic geometry would be better described by the name of *polynomial geometry*.) Since a polynomial over $\mathbb{R}$ is presented so explicitly as a quantization of a piecewise linear convex function, one may expect to find along this line explicit relations between other objects and phenomena of algebraic geometry over $\mathbb{R}$ and piecewise linear geometry. Indeed, in piecewise linear geometry the notion of piecewise linear convex function plays almost the same rôle as the notion of polynomial in algebraic geometry.

A representation of real algebraic geometry as a quantized PL-geometry may be rewarding in many ways. For example, in any quantization there are *classical objects*, i.e., objects which do not change much under the quantization. Objects of PL-geometry are easier to construct. If we knew conditions under which a PL object gives rise to a real algebraic object, which is classical with respect to the Maslov quantization, then we would have a simple way to construct real algebraic objects with controlled properties.

3. Algebraic geometry on logarithmic paper

3.1. Speak of real algebraic geometry only positively. Above in Section 1 we discussed the drawing of graphs on a logarithmic paper only for a polynomial with positive coefficients. The graphs allowed us to see the behavior of the polynomials only at positive values of the argument. This was for a good reason: we used logarithms of coordinates. The Maslov dequantization deals only with positive numbers. Therefore each fragment of algebraic geometry that we want to dequantize must be reformulated first only in terms of positive numbers. This seems to be possible for everything belonging to real algebraic geometry.

3.2. Visualizing roots of a polynomial on logarithmic paper. Above we could not encounter roots of polynomials, for a polynomial with *positive* coefficients has no *positive* roots. However if we really want to do algebraic geometry on log paper, we must figure out how to use graphs on log paper for visualizing roots (well, only *positive* roots) of an arbitrary real polynomial.

Any real polynomial $p(x)$ is a difference $p^+(x) - p^-(x)$ of polynomials with positive coefficients. We can reformulate the problem of finding the positive roots of p as the problem of finding positive values of x at which $p^+(x) = p^-(x)$. The graphs of p^+ and p^- can be drawn on a log paper, where they are localized in the strips along broken lines, see Section 1.5 above. For some polynomials this picture gives a decent information on the number and position of the positive roots.

The negative roots of $p(x)$ can be treated in the same way, since their absolute values are the positive roots of $p(-x)$.

3.3. Plane algebraic curves on logarithmic paper. Now consider a real polynomial $p(x,y) = \sum_{k,l} a_{k,l} x^k y^l$ in two variables. Similarly to the case of polynomials in one variable, in the *logarithmic space* the graph of a monomial $ax^k y^l$ with $a > 0$ is a plane $w = ku + lv + \ln a$, and the graph Γ_p of a polynomial $p(x,y)$ with positive coefficients lies in a neighborhood of a convex piecewise linear surface, which is the graph $\Gamma_{M(p)}$ of the maximum $M(p)$ of the monomials. Furthermore, p is included into a dequantizing family p_h defined as $\sum_{k,l} a_{k,l}^{1/h} x^k y^l$ for $h > 0$, cf. Section 2.2. The graph $\Gamma^h_{p_h}$ of p_h in the logarithmic space with scaled coordinates $u_h = h \ln x$, $v_h = h \ln y$, $w_h = h \ln z$ coincides with the graph of the polynomial $D_1 p(x,y) = \sum_{k,l} (\ln a_{k,l})\, x^k y^l$ in S^3_h. These graphs with $h \in (0,1]$ constitute a continuous deformation of $\Gamma_p = \Gamma^1_{p_1}$ to $\Gamma_{M(p)}$.

For a polynomial p in two variables with arbitrary real coefficients, denote by p^+ the sum of its monomials with positive coefficients, and put $p^- = p^+ - p$. Thus p is canonically presented as a difference $p^+ - p^-$ of two polynomials with positive coefficients. To obtain the curve defined on logarithmic paper by the equation $p(x,y) = 0$, one can construct the graphs Γ_{p^+} and Γ_{p^-} for p^+ and p^- in the logarithmic space, which are the surfaces defined in the usual Cartesian coordinates by $w = \ln(p^\pm(e^u, e^v))$, and project the intersection $\Gamma_{p^+} \cap \Gamma_{p^-}$ to the plane of arguments.

For the first approximation of this curve, one may take the broken line, which is the projection of the intersection of the piecewise linear surfaces $\Gamma_{M(p^+)}$ and $\Gamma_{M(p^-)}$ corresponding to p^+ and p^-.

Of course, it may well happen that the broken line does not even resemble the curve. This happens to first approximations. However, it is very appealing to figure out circumstances under which the broken line is a good approximation, for a broken line seems to be much easier to deal with than an algebraic curve.

3.4. Constructing algebraic curves, which are classical from our quantum point of view. Recall that in the logarithmic space the graph of ax^ky^l is a plane $w = ku + lv + \ln a$. It has a normal vector $(k, l, -1)$ and intersects the vertical axis at $(0, 0, \ln a)$. Thus if we want to construct a curve of a given degree m, we have to arrange planes whose normals are fixed: they are $(k, l, -1)$ with integers k, l, satisfying inequalities $0 \le k, l, k + l \le m$. The only freedom is in moving them up and down.

Consider the pieces of these planes which do not lie under the others. They form a convex piecewise linear surface U, the graph of the maximum of the linear forms defining our planes. The combinatorial structure of faces in U depends on the arrangement. Assume that at each vertex of U exactly three of the planes meet. This is a genericity condition, which can be satisfied by small shifts of the planes.

Divide now the faces of U arbitrarily into two classes. Denote the union of one of them by U^+, the union of the other by U^-. By genericity of the configuration, the common boundary of U^+ and U^- is union of disjoint polygonal simple closed curves. It can be easily realized as the intersection of PL-surfaces $\Gamma_{M(p^+)}$ and $\Gamma_{M(p^-)}$ as above: take for p^ε with $\varepsilon = \pm$ the sum of monomials corresponding to the planes of faces forming U^ε and put $p = p^+ - p^-$.

Consider now for $1 \ge h \ge 0$ the curve $C_h \subset S^3_h$ which is the intersection of the graphs in S^3_h of the polynomials D_1p^+ and D_1p^-. At $h = 0$ this is the intersection of the convex PL-surfaces $\Gamma_{M(p^+)}$, $\Gamma_{M(p^-)}$. Due to the genericity condition above, this intersection is as transversal as one could wish: at all but a finite number of points the interior part of a face of one of them meets the interior part of a face of the other one, and at the rest of the points an edge of one of the surfaces intersects transversaly the interior of a face of the other surface.

When h gets positive, the graphs are smoothed, their corners are rounded off. The same happens to their intersection curve. While the graphs are transveral, the intersection curve is deformed isotopically.

Take the curve corresponding to a value of h such that the transversality is preserved between 0 and this value. The projection to (u, v)-plane of C_h represents an algebraic curve of degree m on the scaled logarithmic paper and it can be obtained by a small isotopy of the projection of ∂U^+ to the (u, v)-plane.

3.5. Is this a patchworking? A construction, which looks similar, has been known in the topology of real algebraic varieties for about 20 years as *patchworking,* or *Viro's method*. It has been used to construct real algebraic varieties with controlled topology and helped to solve a number of problems. For example, to classify up to isotopy non-singular real plane projective curves of degree 7 [7], [8] and disprove the Ragsdale conjecture [2] on the topology of plane curves formulated [6] as early as in 1906. To the best of my knowledge, the patchworking has never been related to the Maslov quantization.

4. Patchworking real algebraic curves

4.1. The simplest patchworking. Here is a description of a simplified version of patchworking. The simplifications are of the following 3 kinds:

- we restrict to the case of nonsingular planar curves,
- we assume that all patches are trinomials, and
- we consider only the part of the curve contained in the first quadrant (what happens in other quadrants is described soon after).

Initial data. Let m be a positive integer (it will be the degree of the curve under construction) and Δ be the triangle in $\mathbb{R}^2$ with vertices $(0,0)$, $(m,0)$, $(0,m)$. Let τ be a *convex* triangulation of Δ with vertices having integer coordinates. The convexity of τ means that there exists a convex piecewise linear function $\nu : \Delta \longrightarrow \mathbb{R}_+$ which is linear on each triangle of τ and is not linear on the union of any two triangles of τ. Let the vertices of τ be equipped with signs. The sign (plus or minus) at the vertex with coordinates (k,l) is denoted by $\sigma_{k,l}$.

Construction of the piecewise linear curve. If a triangle of the triangulation τ has vertices of different signs, draw a midline separating pluses from minuses. Denote by L the union of these midlines. It is a collection of polygonal lines contained in Δ. The pair (Δ, L) is called the *result of combinatorial patchworking.*

Construction of polynomials. Given initial data m, Δ, τ and $\sigma_{k,l}$ as above and a positive convex function ν certifying, as above, that the triangulation τ is convex. Consider a one-parameter family of polynomials

$$b_t(x,y) = \sum_{\substack{(k,l) \text{ runs over} \\ \text{vertices of } \tau}} \sigma_{k,l} t^{\nu(k,l)} x^k y^l. \tag{3}$$

The polynomials b_t are called the results of *polynomial patchworking.*

Patchwork Theorem. *Let m, Δ, τ, $\sigma_{k,l}$ and ν be initial data as above. Denote by b_t the polynomials obtained by the polynomial patchworking of these initial data, and by L the PL-curve in Δ obtained from the same initial data by combinatorial patchworking.*

Then for all sufficiently small $t > 0$ the polynomial b_t defines in the first quadrant $\mathbb{R}^2_{++} = \{(x,y) \in \mathbb{R}^2 \mid x, y > 0\}$ a curve a_t such that the pair $(\mathbb{R}^2_{++},\ a_t)$ is homeomorphic to the pair $(\operatorname{Int}\Delta,\ L \cap \operatorname{Int}\Delta)$.

4.2. Patchwork in other quadrants. The Patchwork Theorem applied to $b_t(-x,y)$, $b_t(x,-y)$ and $b_t(-x,-y)$ gives a similar topological description of the curve defined in the other quadrants by b_t with sufficiently small $t > 0$. The results can be collected in the following natural combinatorial construction.

Construction of the PL-curve. Take copies $\Delta_x = s_x(\Delta)$, $\Delta_y = s_y(\Delta)$, $\Delta_{xy} = s(\Delta)$ of Δ, where $s_x,\ s_y$ are reflections with respect to the coordinate axes and $s = s_x \circ s_y$. Denote by $A\Delta$ the square $\Delta \cup \Delta_x \cup \Delta_y \cup \Delta_{xy}$. Extend the triangulation τ to a symmetric triangulation of $A\Delta$, and the distribution of signs $\sigma_{i,j}$ to a distribution at the vertices of the extended triangulation by the following rule: $\sigma_{i,j}\sigma_{\varepsilon i,\delta j}\varepsilon^i\delta^j = 1$, where $\varepsilon, \delta = \pm 1$. In other words,

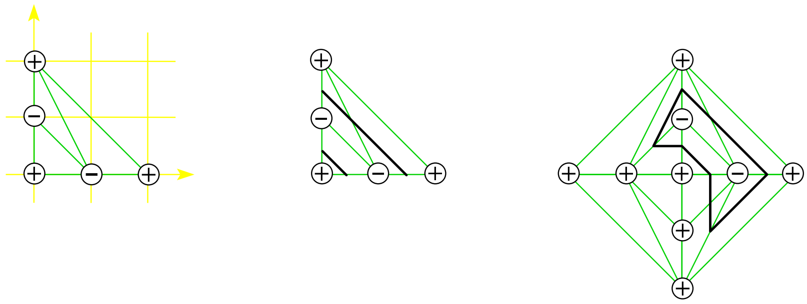

FIGURE 2. Patchworking: initial data, construction of the PL-curve in the first quadrant and on the whole plane. The corresponding algebraic curves are ellipses meeting the coordinate axes in their positive halves.

passing from a vertex to its mirror image with respect to an axis we preserve its sign if the distance from the vertex to the axis is even, and change the sign if the distance is odd.

If a triangle of the triangulation of $A\Delta$ has vertices of different signs, select (as above) a midline separating pluses from minuses. Denote by AL the union of the selected midlines. It is a collection of polygonal lines contained in $A\Delta$. The pair $(A\Delta, AL)$ is called the *result of affine combinatorial patchworking.* Glue by s the sides of $A\Delta$. The resulting space $P\Delta$ is homeomorphic to the real projective plane $\mathbb{R}P^2$. Denote by PL the image of AL in $P\Delta$ and call the pair $(P\Delta, PL)$ the *result of projective combinatorial patchworking.*

Addendum to the Patchwork Theorem. *Under the assumptions of Patchwork Theorem above, for all sufficiently small $t > 0$ there exist a homeomorphism $A\Delta \to \mathbb{R}^2$ mapping AL onto the the affine curve defined by b_t and a homeomorphism $P\Delta \to \mathbb{R}P^2$ mapping PL onto the projective closure of this affine curve.*

4.3. The Simplest Patchworking Coincides With Construction of Section 3.4.

The polynomial b_t defined by (3) is presented as $b_t^+ - b_t^-$, where

$$b_t^{\varepsilon}(x,y) = \sum_{\substack{(k,l) \text{ runs over the vertices} \\ \text{of } \tau, \text{ at which } \sigma_{k,l}=\varepsilon}} t^{\nu(k,l)} x^k y^l$$

Observe that polynomials $b_t^{\pm}$ comprise dequantizing families. Indeed, if we take $p(x,y) = \sum_{k,l} a_{k,l} x^k y^l$ with $a_{k,l} = e^{-\nu(k,l)}$, then for $h = -1/\ln t$ we obtain

$$p_h(x,y) = \sum_{k,l} a_{k,l}^{1/h} x^k y^l = \sum_{k,l} e^{-\nu(k,l)/h} x^k y^l =$$

$$\sum_{k,l} e^{\nu(k,l)\ln t} x^k y^l = \sum_{k,l} t^{\nu(k,l)} x^k y^l.$$

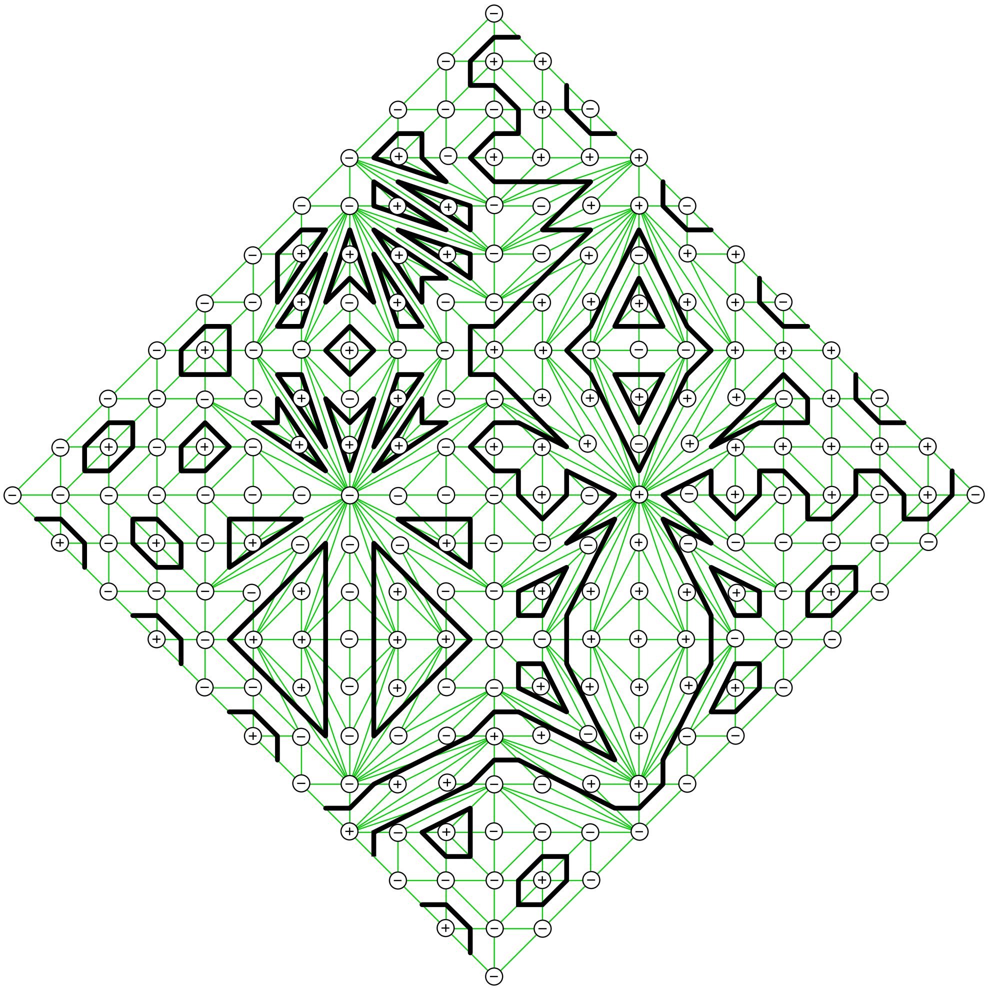

FIGURE 3. Patchworking of a counter-example to the Ragsdale Conjecture. A curve of degree 10 with 32 odd ovals constructed by Itenberg [2].

Patchwork Theorem deals with sufficiently small positive t, while h in a dequantizing family of polynomials was a small positive number approaching 0. This is consistent with our setting $h = -1/\ln t$.

A monomial $a_{k,l}x^k y^l = e^{-\nu(k,l)}x^k y^l$ is presented in the logarithmic space by the graph of $w = ku + lv - \nu(k,l)$. Hence the graph of the maximum of linear forms corresponding to all monomials of p^+ and p^- is defined by

$$w = \max\{ku + lv - \nu(k,l) \mid (k,l) \text{ runs over vertices of } \tau\}. \tag{4}$$

In (4) we recognize the convex function conjugate to ν. The graph of (4) is a convex PL surface, whose natural stratification is dual to the triangulation τ of Δ: the face which lies on the plane $w = ku + lv - \nu(k,l)$ corresponds to the vertex (k,l) of τ, two such faces meet at an edge in the graph of (4) iff the corresponding vertices are connected with an edge of τ, three faces meet at a vertex iff the corresponding vertices of τ belong to a triangle of τ. In particular, we see that the configuration of planes satisfies the genericity condition of Section 3.4 and planes $w = ku + lv - \nu(k,l)$ corresponding to all monomials of $b_t^{\pm}$ show up in the graph of (4) as its faces.

Some of these faces correspond to monomials of b_t^{+}, the others to monomials of b_t^{-}. The edges which separate the faces of these two kinds constitute a broken line as in Section 3.4. These edges are dual to the edges of τ which intersect the result L of the combinatorial patchworking.

Therefore the topology of the projection to the (u,v)-plane of the broken line coincides with the topology of L in Δ. □

Conclusion. We see that the quantum point of view (or its graphical log paper equivalent) gives a natural explanation to the simplest patchwork construction. The proofs become more conceptual and straight-forward. Of course, similar but slightly more involved quantum explanations can be given to all versions of patchwork.

Let me shortly mention other problems which can be attacked using similar arguments.

First of all, this is the Fewnomial Problem. Although A. G. Khovansky [3] proved that basically all topological characteristics of a real algebraic variety can be estimated in terms of the number of monomials in the equations, the known estimates seem to be far weaker than conjectures. For varieties classical from the quantum point of view a strong estimates are obvious. It is very compelling to estimate how much the topology can be complicated by the quantizing deformations.

There seem to be deep relations between the dequantization of algebraic geometry considered above and the results of I. M. Gelfand, M. M. Kapranov and A. V. Zelevinsky on discriminants [1]. In particular, some monomials in a discriminant are related to intersections of hyperplanes in the dequantized polynomial.

Complex algebraic geometry also deserves a dequantization. Especially relevant may be amoebas introduced in [1].

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