Patchworking Algebraic Curves Disproves the Ragsdale Conjecture

Ilia Itenberg and Oleg Viro

Real algebraic curves seem to be quite distant from combinatorial geometry. In this article we intend to demonstrate how to build algebraic curves in a combinatorial fashion: to patchwork them from pieces which essentially are lines. One can trace related constructions back to Newton's consideration of branches at a singular point of a curve. Nonetheless, an explicit formulation is not familiar to many mathematicians.

This technique was developed by the second author in the beginning of the eighties. Using it, the first author has recently found counterexamples to the oldest and most famous conjecture on the topology of real algebraic curves. The conjecture was formulated as early as 1906 by V. Ragsdale [14] on the basis of experimental material provided by A. Harnack's and D. Hilbert's constructions [5,6].

A Combinatorial Look at Patchworking

Initial Data. Let m be a positive integer (it will be the degree of the curve under construction) and T be the triangle in $\mathbb{R}^2$ with vertices $(0, 0)$, $(m, 0)$, $(0, m)$. Let τ be a triangulation of T with vertices having integer coordinates and equipped with signs. The sign (plus or minus) at the vertex with coordinates (i, j) is denoted by $\sigma_{i,j}$.

Construction of a Piecewise Linear Curve. Take copies

$$T_x = s_x(T), \qquad T_y = s_y(T), \qquad T_{xy} = s(T)$$

of T, where $s = s_x \circ s_y$ and s_x, s_y are reflections with respect to the coordinate axes. Denote by T_* the square $T \cup T_x \cup T_y \cup T_{xy}$. Extend the triangulation τ to a symmetric triangulation of T_*, and the distribution of signs $\sigma_{i,j}$ to a distribution at the vertices of the extended tri-

Ilia Itenberg

Oleg Viro

Ilia Itenberg graduated from Leningrad State University in 1988. He received his Ph.D. in Leningrad in 1991; his adviser was O. Viro. His research area is the topology of real algebraic varieties. He obtained a classification of real algebraic curves of degree 6 in $\mathbb{R}P^2$ with one nondegenerate double point up to isotopy and up to rigid isotopy. In 1993 he constructed counterexamples to the Ragsdale conjecture. Itenberg now works at the Institut de Recherche Mathématique de Rennes, France.

Oleg Viro received his Candidate degree (Ph.D.) under the direction of V. A. Rokhlin in 1974. His first research areas were knot theory and low-dimensional topology. In the late seventies the focus of his research shifted to the topology of real algebraic varieties. In particular, he completed the topological classification of nonsingular plane projective curves of degree 7. In 1983 he defended his second degree (Doctor), then he returned to low-dimensional topology, constructing exotic (homeomorphic but nondiffeomorphic) knottings of a surface in 4-space (jointly with M. Kreck and S. Finashin) and a new topological quantum field theory (with V. Turaev). After 16 years teaching at Leningrad State University, he is now at the University of California Riverside and Uppsala University, Sweden, keeping his research position in the St. Petersburg branch of the Steklov Institute.

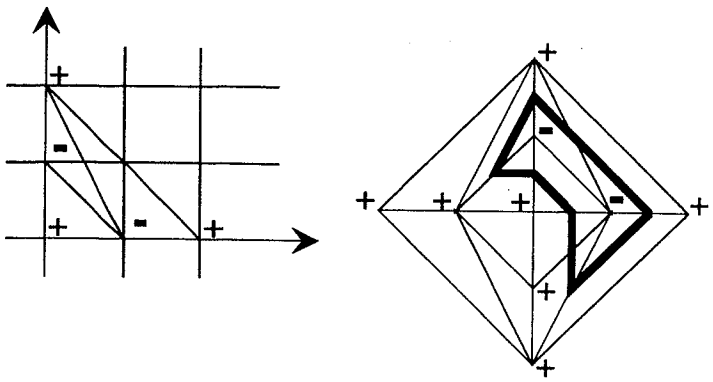

Figure 1. Initial data and the result of combinatorial patchworking of it.

angulation by the following rule: $\sigma_{i,j}\sigma_{\epsilon i,\delta j}\epsilon^i\delta^j = 1$, where $\epsilon, \delta = \pm 1$. In other words, passing from a vertex to its mirror image with respect to an axis we preserve its sign if the distance from the vertex to the axis is even, and change the sign if the distance is odd.

If a triangle of the triangulation of T_* has vertices of different signs, select a midline separating pluses from minuses. Denote by L the union of the selected midlines. It is a collection of polygonal lines contained in T_*. The pair (T_*, L) is called the *result of affine combinatorial patchworking*. Glue the sides of T_* by s. The resulting space $\mathfrak{T}$ is homeomorphic to the real projective plane $\mathbb{R}P^2$. Denote by $\mathfrak{L}$ the image of L in $\mathfrak{T}$, and call the pair $(\mathfrak{T}, \mathfrak{L})$ the *result of projective combinatorial patchworking*. (See, for example, Figures 1–3).

Let us introduce an additional assumption: the triangulation τ of T is *convex*. This means that there exists a convex piecewise-linear function $T \to \mathbb{R}$ which is linear on each triangle of τ and not linear on the union of any two triangles of τ.

PATCHWORK THEOREM: *Under the assumptions above on the triangulation τ of T, there exist a nonsingular real algebraic plane affine curve of degree m and a homeomorphism of the plane $\mathbb{R}^2$ onto the interior of the square T_* mapping the set of real points of this curve onto L. Furthermore, there exists a homeomorphism $\mathbb{R}P^2 \to \mathfrak{T}$ mapping the set of real points of the corresponding projective curve onto $\mathfrak{L}$.*

Real Plane Algebraic Curves

The word *curve* is known to be one of the most ambiguous in mathematics. Thus, we had better specify the

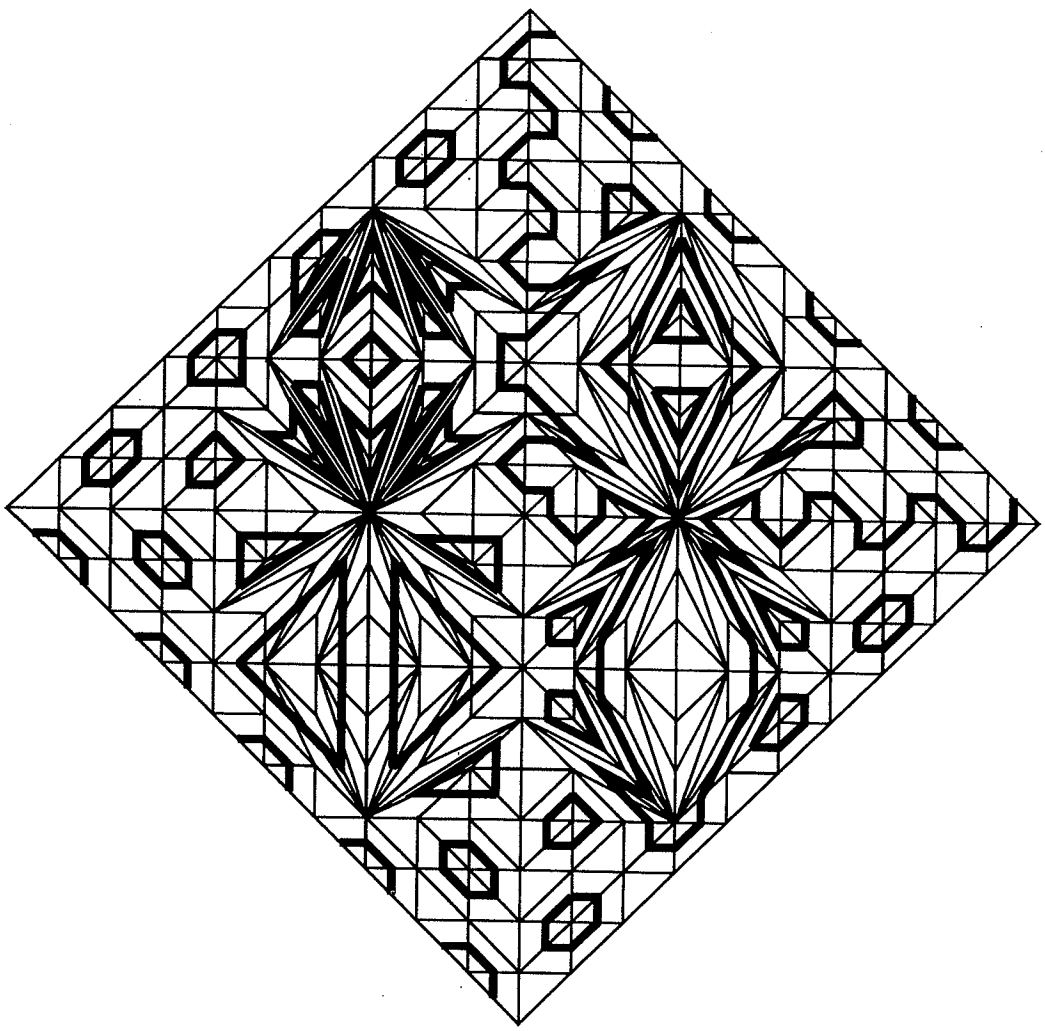

Figure 2. Patchwork of a counterexample to the Ragsdale Conjecture with degree 10 and $p = 32$.

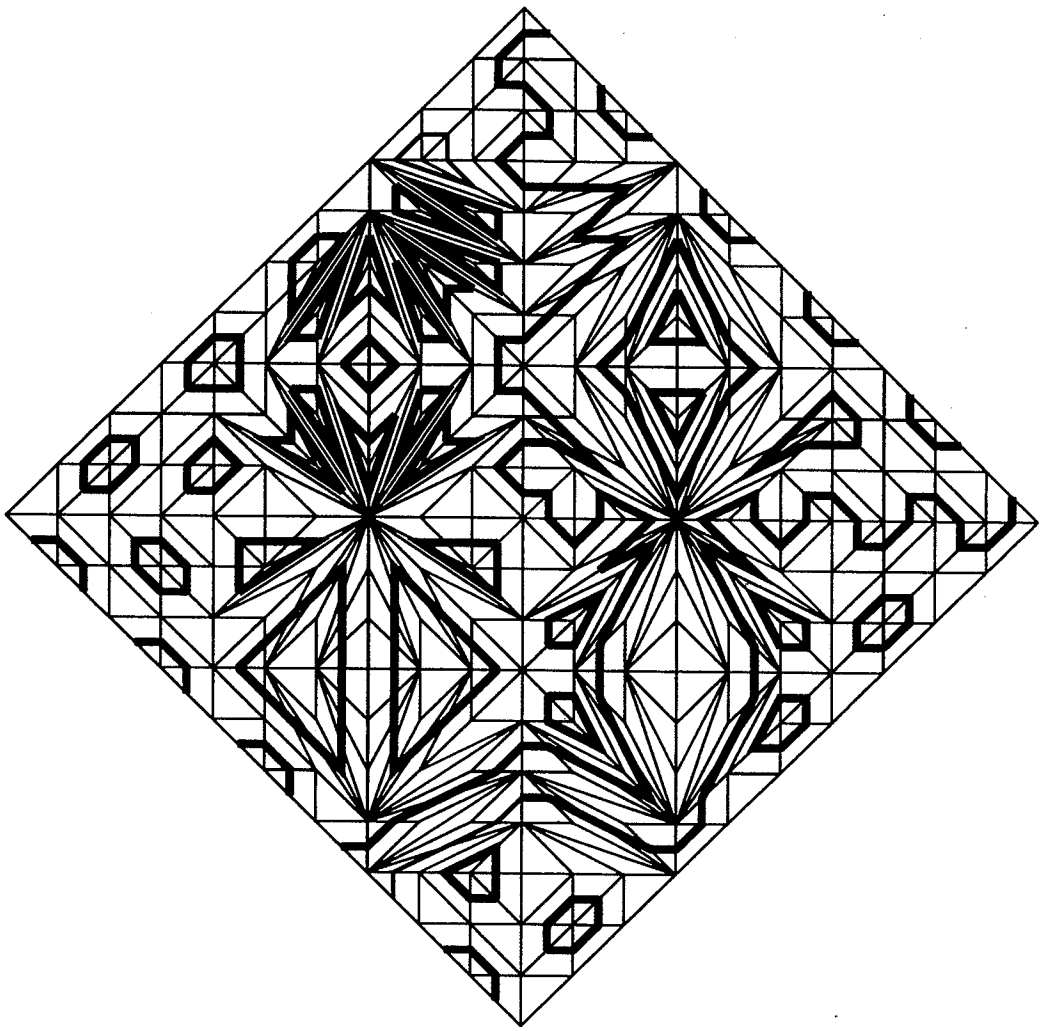

Figure 3. Patchwork of a counterexample to the Ragsdale Conjecture with degree 10 and $n = 32$.

type of curves to be considered. They are real algebraic plane curves, i.e., plane curves defined by equations $f = 0$, where f is a polynomial over the field $\mathbb{R}$ of real numbers. The constructions below give us real polynomials $f(x, y)$ of a given degree such that the curves $f(x, y) = 0$ are positioned in a complicated way (for this degree) in the plane $\mathbb{R}^2$.

For many reasons we prefer projective curves. To a reader who is not comfortable with the projective plane, we offer the following motivations and definitions.

It was probably Isaac Newton [10] who first observed that a curve $f(x, y) = 0$ in the plane $\mathbb{R}^2$ is a more complicated object (e.g., to classify) than the cone generated by it in $\mathbb{R}^3$. If m is the degree of f, then the cone is defined by the equation $z^m f(x/z, y/z) = 0$. Newton [10] found 99 classes of curves of degree 3 on $\mathbb{R}^2$, but at the end of his text he noted that curves of all 99 classes can be obtained as plane sections of only 5 cubic cones.

In the 19th century this observation and similar ones led to the notion of the projective plane and the idea that it is simpler to study curves in the projective plane than in the affine plane.

The real projective plane $\mathbb{R}P^2$ can be defined as the set of lines in $\mathbb{R}^3$ passing through the origin (0, 0, 0). The line passing through (0, 0, 0) and (x_0, x_1, x_2) is denoted by $(x_0 : x_1 : x_2)$; the numbers x_0, x_1, and x_2 are called *homogeneous coordinates* of $(x_0 : x_1 : x_2)$.

A cone in $\mathbb{R}^3$ with vertex (0, 0, 0) can be thought of as a collection of lines passing through (0,0,0). This is a curve in the projective plane provided that the collection is one-parameter. An equation $F(x, y, z) = 0$, where F is a homogeneous real polynomial, defines a cone in $\mathbb{R}^3$ with vertex (0, 0, 0) and hence a curve in the projective plane $\mathbb{R}P^2$.

Take a curve on $\mathbb{R}^2$ defined by an equation $f(x, y) = 0$ of degree m, shift it with its plane to the plane $z = 1$ in $\mathbb{R}^3$, and consider lines passing through it and the origin (0, 0, 0). These lines lie on the cone $z^m f(x/z, y/z) = 0$ and fill it together with its intersection with the plane $z = 0$. The corresponding curve on $\mathbb{R}P^2$ is called the *projective completion* of the affine curve $f(x, y) = 0$. The study of real algebraic curves in the affine plane $\mathbb{R}^2$ splits naturally into study of their projective completions and investigation of the position of the completions with respect to the line at infinity $\mathbb{R}P^2 \setminus \mathbb{R}^2$.

A curve (at least an algebraic curve) is something more than just the set of points which belong to it. It is only slightly less than its equation: equations differing

by a constant factor define the same curve. Modern algebraic geometry provides a lot of ways to define algebraic curve. We adopt the following definition, which at first glance seems to be overly algebraic.

By a *real projective algebraic plane curve of degree m* we mean a homogeneous real polynomial of degree m in three variables, considered up to a constant factor. (Similarly, by a *real affine algebraic plane curve of degree m* we mean a real polynomial of degree m in two variables, considered up to a constant factor.) If F is such a polynomial, then the equation $F(x_0, x_1, x_2) = 0$ defines the *set of real points of the curve* in the real projective plane $\mathbb{R}P^2$. Let $\mathbb{R}A$ denote the set of real points of the curve A. Following tradition, we also call this set a curve, avoiding this terminology only in cases where confusion could result.

A point $(x_0 : x_1 : x_2) \in \mathbb{R}P^2$ is called a (real) *singular point* of the curve defined by a polynomial F if the first partial derivatives of F vanish at (x_0, x_1, x_2) (vanishing of the derivatives implies vanishing of the homogeneous polynomial: by the Euler formula $\deg(F)\cdot F(x_0, x_1, x_2) = \sum_i x_i(\partial F/\partial x_i)\ (x_0, x_1, x_2)$). A curve is said to be (real) *nonsingular* if it has no real singular points. The set $\mathbb{R}A$ of real points of a nonsingular real projective plane curve A is a smooth closed one-dimensional submanifold of the projective plane. Then $\mathbb{R}A$ is a union of disjoint circles smoothly embedded in $\mathbb{R}P^2$. A circle can be positioned in $\mathbb{R}P^2$ either one-sidedly, like a projective line, or two-sidedly, like a conic. A two-sided circle is called an *oval*. An oval divides $\mathbb{R}P^2$ into two parts. The part homeomorphic to a disk is called the *interior* of the oval. Two ovals can be situated in two topologically distinct ways: each may lie outside the other one—i.e., each is in the outside component of the complement of the other—or else one of them is in the inside component of the complement of the other—in that case, we say that the first is the *inner* oval of the pair and the second is the *outer* oval. In the latter case we also say that the outer oval of the pair *envelops* the inner oval. The topological type of the pair $(\mathbb{R}P^2, \mathbb{R}A)$ is defined by the scheme of disposition of the ovals of $\mathbb{R}A$. This scheme is called *the real scheme of curve A*.

In 1900 D. Hilbert [7] included the following question in the 16th problem of his famous list: *what real schemes can be realized by curves of a given degree?* The complete answer is known now only for curves of degree not greater than 7.

T-Curves

Now let us come back to the Patchwork Theorem. It states that for any convex triangulation τ of T with integer vertices and a distribution of signs at vertices of τ there exists a nonsingular real algebraic plane projective curve A of degree m such that the pair $(\mathbb{R}P^2, \mathbb{R}A)$ is homeomorphic to the pair $(\mathfrak{T}, \mathfrak{L})$ constructed as in the first section, i.e., the result of projective combinatorial patchworking.

In fact, a polynomial defining the curve can be presented quite explicitly.

Construction of Polynomials. Given initial data m, T, τ, and $\sigma_{i,j}$ as in the first section, and a convex function ν certifying that the triangulation τ is convex, consider the one-parameter family of polynomials

$$b_t(x, y) = \sum_{\substack{(i,j) \text{ runs over} \\ \text{vertices of } \tau}} \sigma_{i,j} x^i y^j t^{\nu(i,j)}.$$

Denote by B_t the corresponding homogeneous polynomials:

$$B_t(x_0, x_1, x_2) = x_0^m b_t(x_1/x_0, x_2/x_0).$$

Polynomials b_t and B_t are called the results of *affine and projective polynomial patchworking*.

DETAILED PATCHWORK THEOREM: *Let m, T, τ, $\sigma_{i,j}$, and ν be initial data as above. Denote by b_t and B_t the nonhomogeneous and homogeneous polynomials obtained by the polynomial patchworking of these initial data, and by L and $\mathfrak{L}$ the piecewise-linear curves in the square T_* and its quotient space $\mathfrak{T}$, respectively, obtained from the same initial data by the combinatorial patchworking.*

Then there exists $t_0 > 0$ such that for any $t \in (0, t_0]$

(1) *b_t defines an affine curve a_t such that the pair $(\mathbb{R}^2, \mathbb{R}a_t)$ is homeomorphic to the pair (T_*, L);*
(2) *B_t defines a projective curve A_t such that the pair $(\mathbb{R}P^2, \mathbb{R}A_t)$ is homeomorphic to the pair $(\mathfrak{T}, \mathfrak{L})$.*

A curve obtained by this construction is called a *T-curve*.

All real schemes of curves of degree ≤ 6 and almost all real schemes of curves of degree 7 have been realized by the patchwork construction described above. On the other hand, there exist real schemes realizable by algebraic curves of some (high) degree, but not realizable by T-curves of the same degree. Probably such a scheme can be found even for degree 7 or 8.

The construction of T-curves is a special case of more general patchwork construction; see [17] and [13]. In this generalization the patches are more complicated: they may be algebraic curves of any genus with arbitrary Newton polygon. Therefore, the patches demand more care than above. This is why we restrict ourselves here to T-curves. However, even when constructing T-curves, it is useful to think in terms of blocks more complicated than a single triangle (made of several triangles).

The rest of the article is devoted to applications of the patchwork construction.

The Ragsdale Conjecture

The year 1876 is often considered as the beginning of the topological study of real algebraic curves. Prior to that, topological properties were not separated from other geo-

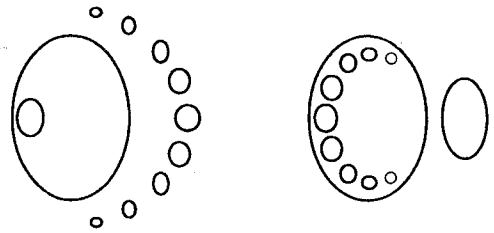

Figure 4. Real schemes of Harnack's and Hilbert's curves of degree 6.

metric properties, which are more subtle and could keep geometers busy with curves of a few lower degrees.

In 1876, A. Harnack published a paper [5] where he found an exact upper bound for the number of components for a curve of a given degree. Harnack proved that the number of components of a real plane projective curve of degree m is at most $(m-1)(m-2)/2+1$. On the other hand, for any natural number m he constructed a nonsingular real projective curve of degree m with $(m-1)(m-2)/2+1$ components, which shows that his estimate cannot be improved without introducing new ingredients.

It was D. Hilbert who made the first attempt to study systematically the topology of nonsingular real plane algebraic curves. The first difficult special problems he met were related to curves of degree 6.

Hilbert suggested that from a topological viewpoint the most interesting are the curves having the maximal number $(m-1)(m-2)/2+1$ of components among curves of a given degree m. Hilbert's guess was strongly confirmed by the whole subsequent development of the field. Now, following I. Petrovsky, these curves are called *M-curves*.

Hilbert succeeded in constructing M-curves of degree ≥ 6 with the mutual position of components different from the ones realized by Harnack. However, he realized only one new real scheme of degree 6. See Fig. 4, where the real schemes of Harnack's and Hilbert's curves of degree 6 are shown. Hilbert conjectured that these are the only real schemes realizable by M-curves of degree 6, and for a long time claimed that he had a (long) proof of this conjecture. Although false (it was disproved by D. A. Gudkov in 1969, who constructed a curve with the real scheme shown in Fig. 5), this conjecture captured the essence of what in the thirties and seventies became the core of the theory.

In fact, Hilbert invented a method which allows one to answer all questions on the topology of curves of degree 6. It involves a detailed analysis of singular curves which could be obtained from a given nonsingular one. The method required complicated fragments of singularity theory, which had not been developed at the time of Hilbert. It was only in the sixties that this project was completely realized. A complete table of real schemes of curves of degree 6 was obtained by Gudkov.

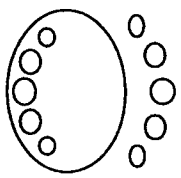

Figure 5. Real scheme of Gudkov's curve of degree 6.

We have mentioned that Hilbert's sixteenth problem is on this topic. Curious that he placed this sixteenth on his list! The number *sixteen* plays a very special role in the topology of real algebraic varieties. It is difficult to believe that Hilbert was aware of that. It became clear only in the beginning of the seventies (see Rokhlin's paper [15]).

In 1906 V. Ragsdale [14] made a remarkable attempt to analyze Harnack's and Hilbert's constructions in search of new restrictions on topology of curves. To a great extent, the success of her analysis was due to the right choice of parameters of a real scheme.

Ragsdale suggested considering separately the case of curves of even degree $m=2k$. Each connected component of the set of real points of a curve of even degree is an oval (i.e., divides $\mathbb{R}P^2$ into two parts). An oval of a curve is called *even* (resp. *odd*) if it lies inside of an even (resp. odd) number of other ovals of this curve. The number of even ovals of a curve is denoted by p, the number of odd ovals by n.

It was Ragsdale who suggested distinguishing even and odd ovals. Ragsdale provided good reasons why one should pay special attention to p and n. A curve of an even degree divides the plane $\mathbb{R}P^2$ into two pieces with a common boundary $\mathbb{R}A$ (these pieces are the subsets of $\mathbb{R}P^2$ where a polynomial defining the curve takes positive and negative values, respectively). One of these pieces is nonorientable; it is denoted by $\mathbb{R}P^2_-$. The other one is denoted by $\mathbb{R}P^2_+$. The numbers p and n are the fundamental topological characteristics of $\mathbb{R}P^2_+$ and $\mathbb{R}P^2_-$; namely, p is the number of connected components of $\mathbb{R}P^2_+$, and $n+1$ is the number of connected components of $\mathbb{R}P^2_-$ (exactly one component of $\mathbb{R}P^2_-$ is nonorientable, so n is the number of orientable components of $\mathbb{R}P^2_-$). Ragsdale singled out also the difference $p-n$, motivating this by the fact that it is the Euler characteristic of $\mathbb{R}P^2_+$. It is amazing that essentially these considerations were stated in a paper in 1906.

RAGSDALE OBSERVATION. *For any of Harnack's M-curves of even degree $m=2k$,*

$$p=\frac{3k(k-1)}{2}+1, \qquad n=\frac{(k-1)(k-2)}{2}.$$

For any of Hilbert's M-curves of even degree $m=2k$,

$$\frac{(k-1)(k-2)}{2}+1\leq p\leq \frac{3k(k-1)}{2}+1,$$

$$\frac{(k-1)(k-2)}{2}\leq n\leq \frac{3k(k-1)}{2}.$$

This motivated the following conjecture.

RAGSDALE CONJECTURE. *For any curve of even degree* $m = 2k$,

$$p \le \frac{3k(k-1)}{2} + 1, \qquad n \le \frac{3k(k-1)}{2}.$$

Writing cautiously, Ragsdale also formulated weaker conjectures. About 30 years later, I. G. Petrovsky [11,12] proved one of these weaker conjectures.

PETROVSKY INEQUALITIES. *For any curve of even degree* $m = 2k$,

$$p - n \le \frac{3k(k-1)}{2} + 1, \qquad n - p \le \frac{3k(k-1)}{2}.$$

It is clear from [11] and [12] that Petrovsky was not familiar with Ragsdale's paper. But his proof runs along the lines indicated by Ragsdale. He also reduced the problem to estimates of the Euler characteristic of the pencil curves, but he went further: he proved these estimates using the Euler–Jacobi formula.

Petrovsky also formulated conjectures about the upper bounds for p and n. His conjecture about n was more cautious (by 1) than Ragsdale's.

Both the Ragsdale Conjecture above and Petrovsky's [12] are wrong. However, they stood for a rather long time: the Ragsdale Conjecture for n was disproved by O. Viro [16] in 1979. Viro's disproof looked rather like an improvement of the conjecture, since in the counter-examples $n = 3k(k-1)/2 + 1$. In 1993 the Ragsdale–Petrovsky bounds were disproven by a considerable margin by I. Itenberg [8]: in Itenberg's counterexamples the difference between p (or n) and $3k(k-1)/2 + 1$ is a quadratic function of k (see below).

The numbers p and n introduced by Ragsdale occur in many of the prohibitions that were subsequently discovered. While giving full credit to Ragsdale for her insight, we must also say that if she had looked more carefully at the experimental data available to her, she should have been able to find some of these prohibitions. For example, it is not clear what stopped her from making the conjectures which were made by Gudkov [2] in the late 1960s. In particular, the experimental data could suggest the formulation of the Gudkov–Rokhlin congruence [15]: *for any M-curve of even degree* $m = 2k$

$$p - n \equiv k^2 \mod 8.$$

Maybe mathematicians trying to conjecture restrictions on some integer should keep this case in mind as evidence that restrictions can have not only the shape of an inequality but also a congruence. Proof of these Gudkov conjectures, initiated by Arnold [1] and completed by Rokhlin [15], Kharlamov [9], and Gudkov and Krakhnov [3], marked the beginning of the most recent stage in the topology of real algebraic curves.

Which of Ragsdale's questions are still open now? The inequalities

$$p \le \frac{3k(k-1)}{2} + 1, \qquad n \le \frac{3k(k-1)}{2} + 1$$

have been neither proved nor disproved for M-curves. Ragsdale gave interesting reformulations of the first of these inequalities for M-curves. Below we present her reformulations together with the corresponding reformulations of the inequality $n \le 3k(k-1)/2 + 1$.

RAGSDALE CONJECTURE ON *M*-CURVES. *For any M-curve of degree* $2k$

$$|p - n| \le k^2,$$

or, equivalently,

$$p \ge \frac{(k-1)(k-2)}{2} \quad \text{and} \quad n \ge \frac{(k-1)(k-2)}{2}.$$

Patchworking Harnack Curves

In each area of mathematics there are objects which appear much more frequently than others. Some of them (like Dynkin diagrams) appear in several domains quite distant from each other. In the topology of real plane algebraic curves, Harnack curves play this role. It was not an accident that they were constructed in the first paper devoted to this subject. Whenever one tries to construct an M-curve, the first success provides a Harnack curve. Patchwork construction is no exception to the rule.

In this section we describe, using the Patchwork Theorem, the construction of some Harnack curves of an even degree $m = 2k$.

In what follows, all the triangulations satisfy an additional assumption: they are *primitive*, which means that all triangles are of area $1/2$ (or, equivalently, that all integer points of the triangulated area are vertices of the triangulation). A polynominal defining a T-curve contains the maximal collection of nonzero monomials if and only if the triangulation used in the construction of the T-curve is primitive.

A primitive convex triangulation of T is said to be equipped with a *Harnack distribution of signs* if:

> vertex (i, j) has the sign "$-$" if i, j are both even, and has the sign "$+$" in the opposite case.

A vertex (i, j) of a triangulation of T is called *even* if i and j are both even, and *odd* otherwise.

PROPOSITION. *Patchworking applied to an arbitrary primitive convex triangulation of T with the Harnack distri-*

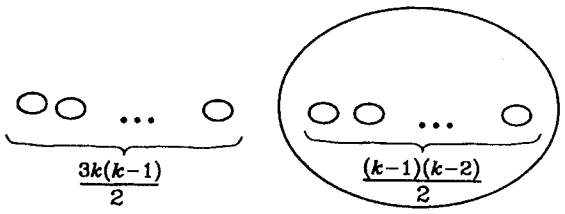

Figure 6. The real scheme of the simplest Harnack curve of degree $2k$.

bution of signs produces an M-curve with the real scheme shown in Fig. 6.

An example of the construction under consideration is shown in Fig. 7.

Proof of Proposition: First, note that the number of interior (i.e., lying in the interior of the triangle T) integer points is equal to $(m-1)(m-2)/2$, the number of even interior points is equal to $(k-1)(k-2)/2$, and the number of odd interior points is equal to $3k(k-1)/2$.

Take an arbitrary even interior vertex of a triangulation of the triangle T. This vertex has the sign "$-$". All adjacent vertices (i.e., the vertices connected with the vertex by edges of the triangulation) are odd, and thus they all have the sign "$+$." This means that the star of an even interior vertex contains an oval of the curve L. The number of such ovals is equal to $(k-1)(k-2)/2$.

Take now an odd interior vertex of the triangulation. It has the sign "$+$." There are two vertices with "$-$" and one vertex with "$+$" among the images of the vertex under $s = s_x \circ s_y$ and s_x and s_y (recall that s_x and s_y are reflections with respect to the coordinate axes). Consider the image with the sign "$+$." It is easy to verify that all its adjacent vertices have the sign "$-$." Again this means that the star of this vertex contains an oval of the curve L. The number of such ovals is equal to $3k(k-1)/2$.

But

$$\frac{(k-1)(k-2)}{2} + \frac{3k(k-1)}{2} = \frac{(m-1)(m-2)}{2}$$

so the curve can only have one more oval. This oval exists because, for example, the curve L intersects the coordinate axes.

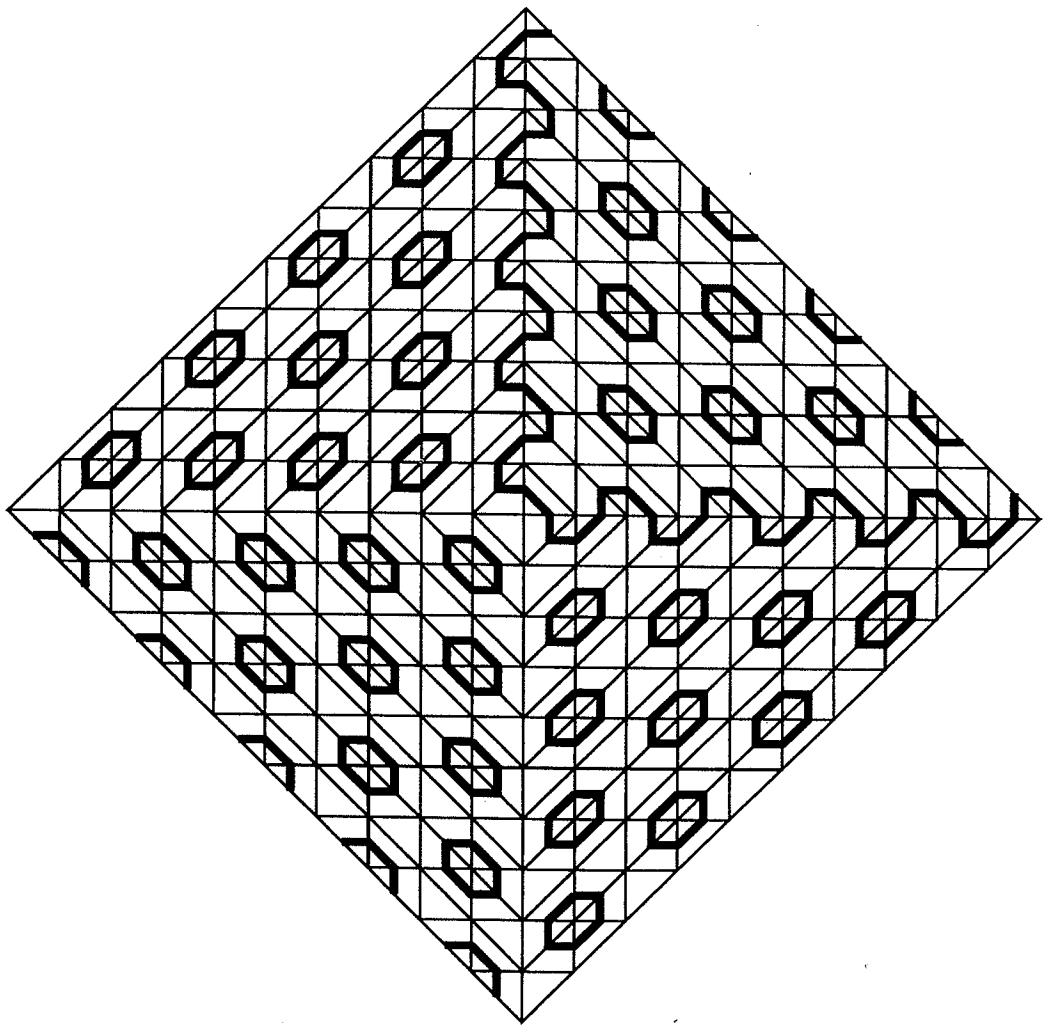

Figure 7. Patchwork of the simplest Harnack curve of degree 10.

Figure 8. Schemes of counterexamples to the Ragsdale Conjecture of degree 10.

To finish the proof, we need only note that the union of the segments

$$\{x - y = -m,\ -m \le x, y \le m\} \cup$$

$$\{x \le 0,\ y = 0,\ -m \le x,y \le m\} \cup$$

$$\{x = 0,\ y \le 0,\ -m \le x,y \le m\}$$

is not contractible in $\mathbb{T}$ and contains only minuses. This means that $3k(k-1)/2$ ovals corresponding to odd interior points and encircling pluses are situated outside of the nonempty oval. □

Counterexamples to the Ragsdale Conjecture

The following theorem gives counterexamples to the Ragsdale Conjecture (or the conjecture of Petrovsky) [8].

THEOREM: *For each integer* $k \ge 1$

(a) *there exists a nonsingular real algebraic plane projective curve of degree* $2k$ *with*

$$p = \frac{3k(k-1)}{2} + 1 + \left[\frac{(k-3)^2+4}{8}\right];$$

(b) *there exists a nonsingular real algebraic plane projective curve of degree* $2k$ *with*

$$n = \frac{3k(k-1)}{2} + \left[\frac{(k-3)^2+4}{8}\right].$$

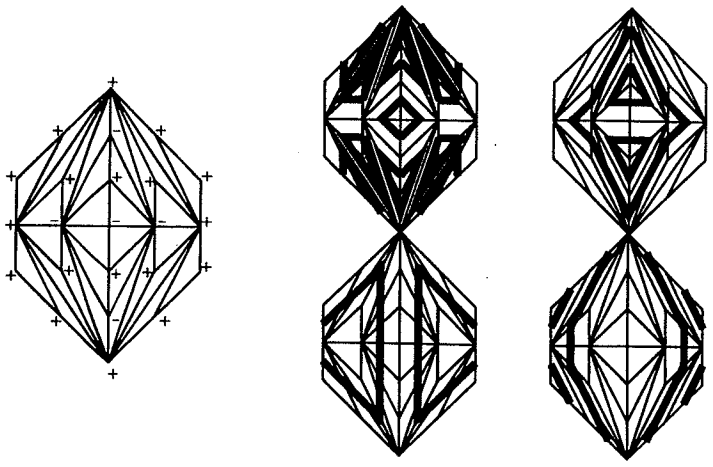

Figure 9. Hexagon S and the patchwork fragment produced by it.

Remarks: The term $[((k-3)^2+4)]/8$ is positive when $k \ge 5$ (i.e., we get counterexamples to the Ragsdale Conjecture in degree 10). On the other hand, it is known that there is no counterexample to the Ragsdale Conjecture among curves of lower degree. These counterexamples of degree 10 have the real schemes shown in Fig. 8. One of these counterexamples can be improved to obtain a curve of degree 10 with $n = 32$. The corresponding patchworks (for $p = 32$ and $n = 32$) are shown in Figs. 2 and 3.

Recently, B. Haas [4] improved the construction presented below and obtained T-curves of degree $2k$ with

$$p = \frac{3k(k-1)}{2} + 1 + \left[\frac{k^2 - 7k + 16}{6}\right].$$

Neither the counterexamples provided by the above theorem nor the curves constructed by Haas are M-curves. Moreover, as we mentioned above, it is not known if the conjecture of Petrovsky holds for M-curves. The only known counterexamples to the Ragsdale Conjecture among M-curves are the curves constructed by Viro [16] (see also [18]). It is curious that we did not succeed in presenting those M-counterexamples as T-curves.

Proof of Theorem: Let us show, first, how to construct a curve of degree $m = 2k$ with $p = 3k(k-1)/2 + 2$.

Suppose that the hexagon S shown in Fig. 9 is placed inside of the triangle T in such a way that the center of S has both coordinates odd. Any convex primitive triangulation of a convex part of a convex polygon is extendable to a convex primitive triangulation of the polygon. Inside of the hexagon S, let us take the convex primitive triangulation shown in Fig. 9 and extend it to T.

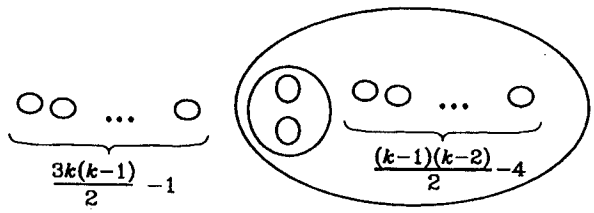

Figure 10. Real scheme of a curve of degree $2k$ with $p = 3k(k-1)/2 + 2$.

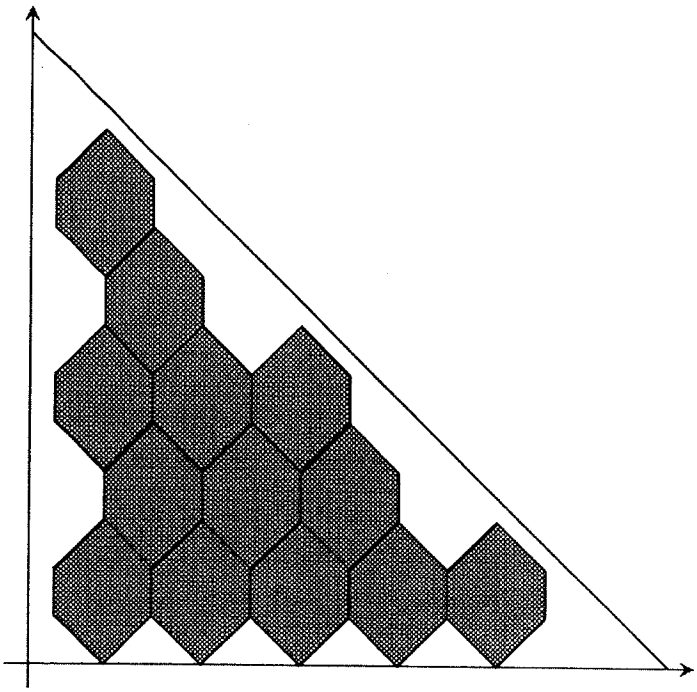

Figure 11. Partition of T for part (a).

To apply the Patchwork Theorem we need to choose signs at the vertices. Inside of S put signs according to Fig. 9; outside, use the Harnack rule of distribution of signs.

It is easy to calculate that the corresponding piecewise-linear curve $\mathfrak{L}$ has exactly one even oval more than the Harnack curve constructed above [i.e., now $p = 3k(k-1)/2 + 2$]. One can verify that the curve obtained has the real scheme shown in Fig. 10.

Consider the partition of the triangle T shown in Fig. 11. Let us take in each shaded hexagon the triangulation and the signs of the hexagon S. The triangulation of the union of the shaded hexagons can be extended to the primitive convex triangulation of T. Let us fix such an extension. Outside of the union of the shaded hexagons, choose the signs at the vertices of the triangulation using the Harnack rule.

Calculation shows that for the corresponding piecewise-linear curve $\mathfrak{L}$

$$p = \frac{3k(k-1)}{2} + 1 + a,$$

where a is the number of shaded hexagons, and

$$a = \left[\frac{(k-3)^2 + 4}{8}\right].$$

This curve has the real scheme shown in Fig. 12.

To prove part (b) of the theorem, let us take, again, the partition of the triangle T shown in Fig. 11 with the triangulation and the signs of each shaded hexagon coinciding with the triangulation and the signs of S. Fix, in addition, a triangulation of a neighborhood of the axis OY and the signs at the vertices of the triangulation as shown in Fig. 13 [the case $k \equiv 1 \pmod 4$]. The chosen triangulation of the union of the shaded hexagons and the neighborhood of the y-axis can be extended to a primitive convex triangulation of T. Outside of the union of the shaded hexagons and the neighborhood of the y-axis, let us again choose the signs at the vertices of the triangulation using the Harnack rule.

The corresponding piecewise-linear curve $\mathfrak{L}$ has the real scheme shown in Fig. 14. In this case $n = 3k(k-1)/2 + a$. □

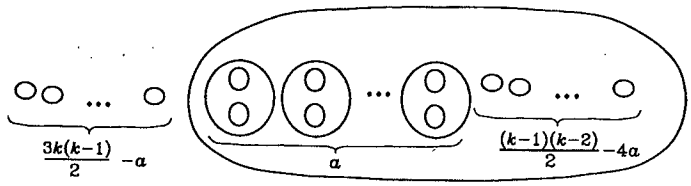

Figure 12. Real scheme of a curve of degree $2k$ with $p = 3k(k-1)/2 + 1 + a$.

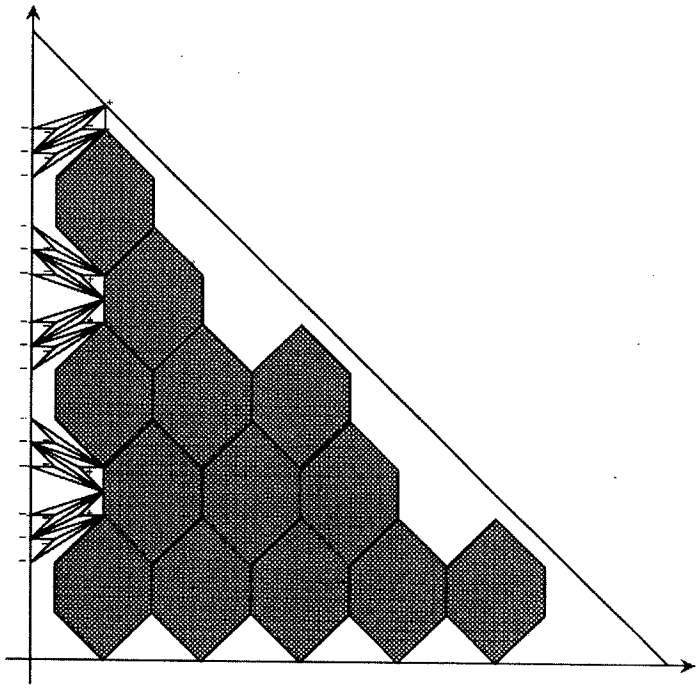

Figure 13. Partition of T for part (b).

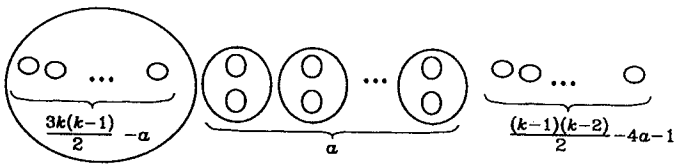

Figure 14. Real scheme of a curve of degree $2k$ with $n = 3k(k-1)/2 + a$.

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Institut de Recherche Mathématique de Rennes
Campus de Beaulieu
35042 Rennes Cedex
France

Box 480
Uppsala S-75106
Sweden

References

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Institut de Recherche Mathématique de Rennes
Campus de Beaulieu
35042 Rennes Cedex
France

Box 480
Uppsala S-75106
Sweden