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CURVES OF DEGREE 7, CURVES OF DEGREE 8, AND THE RAGSDALE CONJECTURE

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This article is devoted to the question of how the components of a nonsingular plane projective real algebraic curve of degree m are positioned with respect to one another. This question was included by D. Hilbert in his well-known sixteenth problem [1]. A complete answer was then known only for $m \leqslant 5$. In the late sixties D. A. Gudkov [2] resolved the case $m = 6$. The recent state of the subject was described in the surveys [3]–[5].

In this article we formulate a definitive answer for $m = 7$ and some new results on curves of higher degree. Among these results are the construction of M-curves refuting the well-known Ragsdale conjecture [6], the realization of 42 new isotopy types of M-curves of degree 8 (10 types were realized earlier), and a theorem on M-curves of degree 8 with three nests that excludes 36 isotopy types not previously excluded.

The new curves are constructed by a method which, as far as I know, has not been used before; it consists in perturbing curves with complicated singularities. The perturbation method (in a somewhat more general situation) is described in the last section. New exclusions are proved using recent results of V. A. Rohlin and of T. Fiedler on the complex orientations of real algebraic curves introduced by Rohlin [5].

2. *The method of describing a curve of isotopy type.* In this article, by a curve of degree m we mean a plane projective real algebraic curve of degree m. As is known, the components of a nonsingular curve of degree m are homeomorphic to a circle. If m is even, they are all positioned in $\mathbf{R}P^2$ two-sidedly; if m is odd, then there is exactly one one-sided component. Two-sided components are called ovals.

The isotopy type of a nonsingular curve of degree m is defined by the scheme of mutual placement of its ovals. For its description the literature contains two systems of notation, introduced by L. Brusotti [7] and by G. M. Polotovskii [8]. In this article we use a system of notation differing from Polotovskiĭ's only in being more compact.

The set consisting of one oval is encoded by the symbol $\langle 1\rangle$, the empty set by the symbol $\langle 0\rangle$. If the symbol $\langle A\rangle$ encodes some set of ovals, then the set obtained from it by adjoining one oval enclosing all the rest is encoded by the symbol $\langle 1\langle A\rangle\rangle$. A set of ovals presented as the union of two nonintersecting sets which are encoded by the symbols $\langle A\rangle$ and $\langle B\rangle$ and such that no oval of one set is enclosed by an oval of the other, is encoded by the symbol $\langle A \amalg B\rangle$. We shall use two abbreviations: first, if $\langle A\rangle$ is the code of a set of ovals, then a fragment of another code having the form $A \amalg \cdots \amalg A$, where A is repeated n times, is abbreviated by the notation $n \times A$; second, fragments of a code having the form $n \times 1$ are abbreviated by the notation n.

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3. *Curves of degree* 7.

THEOREM 1. *There exist nonsingular curves of degree* 7 *of the following isotopy types*:
(i) $\langle \alpha \amalg 1\langle\beta\rangle\rangle$ *with* $\alpha + \beta \leqslant 14$, $0 \leqslant \alpha \leqslant 13$, $1 \leqslant \beta \leqslant 13$;
(ii) $\langle\alpha\rangle$ *with* $0 \leqslant \alpha \leqslant 15$;
(iii) $\langle 1\langle 1\langle 1\rangle\rangle\rangle$.
Any nonsingular curve of degree 7 *belongs to one of these* 121 *types.*

Up to the time this work was being done it remained unknown whether there exist curves of the types $\langle 1\langle 14\rangle\rangle$, $\langle 10 \amalg 1\langle 4\rangle\rangle$ and $\langle \alpha \amalg 1\langle\beta\rangle\rangle$ with $13 \leqslant \alpha + \beta \leqslant 14$, $3 \leqslant \alpha$, $6 \leqslant \beta$.

The unrealizability of the type $\langle 1\langle 14\rangle\rangle$ is established as follows. Using Rohlin's theorem on complex orientations [5] it can be proved that if there were a curve of this type, it would intersect some curve of degree 2 in no less than 16 points, which would contradict Bézout's theorem.

The isotopy types $\langle \alpha \amalg 1\langle\beta\rangle\rangle$ with $6 \leqslant \alpha + \beta \leqslant 14$ are realized as follows. First, one constructs 4 curves of degree 7 having two singular points, at each of which three nonsingular branches are tangent (J_{10} singularities in V. I. Arnol'd's notation [10]), and then one perturbs these branches by the scheme described below in §7.

A curve of type $\langle 4 \amalg 1\langle 10\rangle\rangle$ not only was not, but, as noted by V. I. Zvonilov (personal communication) and by T. Fiedler [9], cannot be, constructed by previous methods, i.e. by a small perturbation of a curve decomposing into nonsingular curves transversal to each other.

4. *Counterexamples to Ragsdale's conjecture.* I recall that an oval is called even if the number of ovals enclosing it is even, and odd otherwise. The number of even ovals is denoted by p, the number of odd ovals by n.

Ragsdale's conjecture [6] says that for any nonsingular curve of even degree m,

$$(1) \qquad p \leqslant (3m^2 - 6m + 8)/8, \qquad n \leqslant (3m^2 - 6m)/8.$$

From well-known theorems it follows that the left-hand inequality holds for $m \leqslant 8$, and the right for $m \leqslant 6$.

THEOREM 2. *For any* $m \geqslant 8$ *which is a multiple of* 4, *there exists a nonsingular curve of degree* m *of isotopy type* $\langle (m^2 - 6m)/8 \amalg 1\langle (3m^2 - 6m + 8)/8\rangle\rangle$.

Thus the right-hand inequality (1) is violated for $m \geqslant 8$, $m \equiv 0 \pmod 4$.

5. *A weakening of Ragsdale's conjecture.* The question whether the inequalities

$$(2) \qquad p \leqslant (3m^2 - 6m + 8)/8, \qquad n \leqslant (3m^2 - 6m + 8)/8$$

are valid remains open. It admits a wider formulation: Whether it is true that if A is the set of fixed points of an antiholomorphic involution of a nonsingular simply connected compact complex surface $\mathbf{C}A$, then

$$(3) \qquad \dim H_1(A; \mathbf{Z}_2) \leqslant h^{1,1}(\mathbf{C}A).$$

If $\mathbf{C}A \longrightarrow \mathbf{C}P^2$ is a branched double covering with ramification over the complexification of a nonsingular curve of degree m, then inequality (2) for this curve is equivalent to inequality (3) for the two involutions $\mathbf{C}A \longrightarrow \mathbf{C}A$ covering the complex conjugation $\mathbf{C}P^2 \longrightarrow \mathbf{C}P^2$.

6. *M-curves of degree* 8. A special role in the topology of real algebraic curves is played by M-curves, i.e. curves of prescribed degree m having the greatest possible number of

components (equal to $(m^2 - 3m + 4)/2$). On the one hand, the most interesting restrictions on the topology of curves relate to M-curves; on the other hand, the isotopy types of curves with fewer components are usually easily realized if the corresponding M-curves can be constructed.

The topological properties of M-curves of degree 8 that are found in the literature reduce to the following three: (i) by the definition of an M-curve, $p + n = 22$; (ii) by Gudkov's comparison, $p - n \equiv 0 \pmod 8$, and so $p \equiv n \equiv 3 \pmod 4$; (iii) in view of obvious consequences of Bézout's theorem, the isotopy type of an M-curve of degree 8 has the form $\langle \alpha \amalg 1\langle\beta\rangle\rangle$ or $\langle \alpha \amalg 1\langle\beta\rangle \amalg 1\langle\gamma\rangle\rangle$ or $\langle \alpha \amalg \langle\beta\rangle \amalg 1\langle\gamma\rangle \amalg 1\langle\delta\rangle\rangle$ or $\langle \alpha \amalg 1\langle \beta \amalg 1\langle\gamma\rangle\rangle\rangle$, where α, β, γ and δ are nonnegative integers.

The following formulation contains an enumeration of the realized isotopy types of M-curves of degree 8. For completeness it includes the type from Theorem 2 and the 10 types realized before this paper. The latter are provided with references to the articles where their realizations are described. The rest are provided with notation (according to [10]) for the types of singularities of the curves which, when slightly perturbed according to the scheme of §7, give rise to curves of this type.

THEOREM 3. *There exist M-curves of degree 8 of the 52 isotopy types introduced in Table 1.*

CONJECTURE. *If $\langle \alpha \amalg 1\langle \beta \amalg 1\langle\gamma\rangle\rangle\rangle$ is the isotopy type of an M-curve of degree 8 and $\gamma \neq 0$, then the numbers β and γ are odd.*

TABLE 1

$p = 19,\ n = 3$	
$\langle 18 \amalg 1\langle 3\rangle\rangle$	([11])
$\langle 17 \amalg 1\langle 1\rangle \amalg 1\langle 2\rangle\rangle$	([11])
$\langle 16 \amalg 1\langle 1\rangle \amalg 1\langle 1\rangle \amalg 1\langle 1\rangle\rangle$	([13])
$\langle 1 \amalg 1\langle 2 \amalg 1\langle 17\rangle\rangle\rangle$	([12])
$\langle 9 \amalg 1\langle 2 \amalg 1\langle 9\rangle\rangle\rangle$	$2X_{21}$
$\langle 11 \amalg 1\langle 2 \amalg 1\langle 7\rangle\rangle\rangle$	$N_{16} + J_{10}$
$\langle 17 \amalg 1\langle 2 \amalg 1\langle 1\rangle\rangle\rangle$	([12])
$\langle 8 \amalg 1\langle 1\rangle \amalg 1\langle 3\rangle \amalg 1\langle 7\rangle\rangle$	$2X_{21}$
$\langle 8 \amalg 1\langle 1\rangle \amalg 1\langle 5\rangle \amalg 1\langle 5\rangle\rangle$	$2X_{21}$
$\langle 8 \amalg 1\langle 3\rangle \amalg 1\langle 3\rangle \amalg 1\langle 5\rangle\rangle$	$2X_{21}$
$\langle 1 \amalg 1\langle 10 \amalg 1\langle 9\rangle\rangle\rangle$	$J_{10}, 2X_{21}$
$\langle 3 \amalg 1\langle 10 \amalg 1\langle 7\rangle\rangle\rangle$	N_{16}
$\langle 5 \amalg 1\langle 10 \amalg 1\langle 5\rangle\rangle\rangle$	$2X_{21}$
$\langle 7 \amalg 1\langle 10 \amalg 1\langle 3\rangle\rangle\rangle$	$N_{16} + J_{10}$
$\langle 9 \amalg 1\langle 10 \amalg 1\langle 1\rangle\rangle\rangle$	$2J_{10}, N_{16}, 2X_{21}$
$p = 7,\ n = 15$	
$\langle 6 \amalg 1\langle 15\rangle\rangle$	$N_{16} + J_{10}$
$\langle 5 \amalg 1\langle 1\rangle \amalg 1\langle 14\rangle\rangle$	$J_{10}, 2X_{21}$
$\langle 5 \amalg 1\langle 2\rangle \amalg 1\langle 13\rangle\rangle$	$J_{10}, 2X_{21}$
$\langle 5 \amalg 1\langle 3\rangle \amalg 1\langle 12\rangle\rangle$	$2X_{21}$
$\langle 5 \amalg 1\langle 4\rangle \amalg 1\langle 11\rangle\rangle$	$2X_{21}$
$\langle 5 \amalg 1\langle 5\rangle \amalg 1\langle 10\rangle\rangle$	$J_{10}, 2X_{21}$
$\langle 5 \amalg 1\langle 6\rangle \amalg 1\langle 9\rangle\rangle$	$2X_{21}$
$\langle 5 \amalg 1\langle 7\rangle \amalg 1\langle 8\rangle\rangle$	$2X_{21}$
$\langle 4 \amalg 1\langle 1\rangle \amalg 1\langle 5\rangle \amalg 1\langle 9\rangle\rangle$	$2X_{21}$
$\langle 4 \amalg 1\langle 3\rangle \amalg 1\langle 5\rangle \amalg 1\langle 7\rangle\rangle$	$2X_{21}$
$\langle 1 \amalg 1\langle 14 \amalg 1\langle 5\rangle\rangle\rangle$	([12])
$\langle 3 \amalg 1\langle 14 \amalg 1\langle 3\rangle\rangle\rangle$	N_{16}
$\langle 5 \amalg 1\langle 14 \amalg 1\langle 1\rangle\rangle\rangle$	([12])
$p = 15,\ n = 7$	
$\langle 14 \amalg 1\langle 7\rangle\rangle$	([14])
$\langle 13 \amalg 1\langle 1\rangle \amalg 1\langle 6\rangle\rangle$	$N_{16}, 2X_{21}$
$\langle 13 \amalg 1\langle 2\rangle \amalg 1\langle 5\rangle\rangle$	([14])
$\langle 13 \amalg 1\langle 3\rangle \amalg 1\langle 4\rangle\rangle$	$2X_{21}$
$\langle 12 \amalg 1\langle 1\rangle \amalg 1\langle 1\rangle \amalg 1\langle 5\rangle\rangle$	$N_{16}, 2X_{21}$
$\langle 12 \amalg 1\langle 1\rangle \amalg 1\langle 3\rangle \amalg 1\langle 3\rangle\rangle$	$2X_{21}$
$\langle 1 \amalg 1\langle 6 \amalg 1\langle 13\rangle\rangle\rangle$	$J_{10}, 2X_{21}$
$\langle 5 \amalg 1\langle 6 \amalg 1\langle 9\rangle\rangle\rangle$	$2X_{21}$
$\langle 7 \amalg 1\langle 6 \amalg 1\langle 7\rangle\rangle\rangle$	$J_{10}, 2X_{21}$
$\langle 9 \amalg 1\langle 6 \amalg 1\langle 5\rangle\rangle\rangle$	$2X_{21}$
$\langle 11 \amalg 1\langle 6 \amalg 1\langle 3\rangle\rangle\rangle$	$J_{10} + N_{16}$
$\langle 13 \amalg 1\langle 6 \amalg 1\langle 1\rangle\rangle\rangle$	$J_{10}, 2X_{21}$
$p = 11,\ n = 11$	
$\langle 10 \amalg 1\langle 11\rangle\rangle$	$N_{16}, 2J_{10}, N_{16} + J_{10}$
$\langle 9 \amalg 1\langle 1\rangle \amalg 1\langle 10\rangle\rangle$	([15])
$\langle 9 \amalg 1\langle 2\rangle \amalg 1\langle 9\rangle\rangle$	$2X_{21}$
$\langle 9 \amalg 1\langle 3\rangle \amalg 1\langle 8\rangle\rangle$	$2X_{21}$
$\langle 9 \amalg 1\langle 4\rangle \amalg 1\langle 7\rangle\rangle$	$2X_{21}$
$\langle 9 \amalg 1\langle 5\rangle \amalg 1\langle 6\rangle\rangle$	$N_{16}, 2X_{21}$
$\langle 8 \amalg 1\langle 1\rangle \amalg 1\langle 1\rangle \amalg 1\langle 9\rangle\rangle$	$2X_{21}$
$p = 3,\ n = 19$	
$\langle 2 \amalg 1\langle 19\rangle\rangle$	$N_{16} + J_{10}$
$\langle 1 \amalg 1\langle 2\rangle \amalg 1\langle 17\rangle\rangle$	$2X_{21}$
$\langle 1 \amalg 1\langle 5\rangle \amalg 1\langle 14\rangle\rangle$	$2X_{21}$
$\langle 1 \amalg 1\langle 8\rangle \amalg 1\langle 11\rangle\rangle$	$2X_{21}$
$\langle 1 \amalg 1\langle 18 \amalg 1\langle 1\rangle\rangle\rangle$	$N_{16} + J_{10}$

Using complex orientations T. Fiedler (unpublished) has proved the nonexistence of M-curves of degree 8 of types $\langle 1\langle 1\rangle \amalg 1\langle\alpha\rangle \amalg 1\langle\beta\rangle\rangle$ with nonzero even α and β. The following theorem generalizes Fiedler's theorem. It was formulated as a conjecture by A. B. Korčagin in connection with the author's results contained in Theorem 3.

THEOREM 4. *If $\langle \alpha \amalg 1\langle\beta\rangle \amalg 1\langle\gamma\rangle \amalg 1\langle\delta\rangle\rangle$ is the isotopy type of an M-curve of degree 8 with β, γ and δ nonzero, then the numbers β, γ and δ are odd.*

The requirements (i), (ii), and (iii), and the restriction imposed by Theorem 4, are satisfied by 104 isotopy types, so that the question of the realizability of 52 types remains open. Of these the conjecture stated above excludes 25.

7. *The method of perturbing singularities.* Recall that by the *Newton polyhedron* $\Delta(p)$ of a polynomial $p(x_1, \dots, x_n) = \Sigma_{k=(k_1,\dots,k_n)} p_k x_1^{k_1} \cdots x_n^{k_n}$ we mean the convex hull of the set $\{k \in \mathbf{R}^n \mid p_k \neq 0\}$. For $\Gamma \subset \mathbf{R}^n$ by p^Γ we denote the polynomial $\Sigma_{k \in \Gamma} p_k x_1^{k_1} \cdots x_n^{k_n}$. A polynomial p (over $\mathbf{C}$ or $\mathbf{R}$) is called nondegenerate if for any side Γ (including $\Delta(p)$) of the polyhedron $\Delta(p)$ the variety defined in $(\mathbf{C}\backslash 0)^n$ by the equation $p^\Gamma(x) = 0$ is nonsingular. For a polynomial p of degree m, by $[p]_m$ we shall denote the homogeneous polynomial by $x_0^m p(x_1/x_0, \dots, x_n/x_0)$.

Let u be a nondegenerate real polynomial of degree m in n variables. We shall assume that the unique face of the polyhedron $\Delta(u)$ that is directed to the origin of the coordinates is an $(n-1)$-dimensional simplex Ω with vertices on the coordinate axes, lying in the hyperplane $\{k \in \mathbf{R}^n \mid \omega_1 k_1 + \cdots + \omega_n k_n = 1\}$. This assumption means that the hypersurface $U \subset \mathbf{R}P^n$ defined by the equation $[u]_m(x) = 0$ has (according to Arnol'd's terminology [10]) a half-quasi-homogeneous singularity with indices $\omega_1, \dots, \omega_n$ at the point $e_0 = (1: 0: \dots: 0)$ and that the coordinate axes passing through e_0 do not lie in U.

Let v be a real polynomial in n variables with $\Delta(v) = \{k \in \mathbf{R}^n \mid \omega_1 k_1 + \cdots + \omega_n k_n \leq 1\}$. We shall assume that the hypersurface $V \subset \mathbf{R}^n$ defined by it is nonsingular and that $v^\Omega = u^\Omega$.

We put $v_t(x_0, \dots, x_n) = t[v]_m(x_0, t^{-\omega_1}x_1, \dots, t^{-\omega_n}x_n)$ and $a_t = [u]_m + v_t - [v^\Omega]_m$ and denote by V_t and A_t the hypersurfaces defined in $\mathbf{R}P^n$ by the equations $v_t(x) = 0$ and $a_t(x) = 0$.

The hypersurface V_t with $t \neq 0$ is obtained from V by means of the linear transformation $(x_1, \dots, x_n) \mapsto (t^{\omega_1}x_1, \dots, t^{\omega_n}x_n)$, following inclusion in $\mathbf{R}P^n$ and the adjunction of the hyperplane $x_0 = 0$, taken with multiplicity $m - \deg v$. The hypersurface A_t coincides with U for $t = 0$, and for sufficiently small $t > 0$ is described as follows.

THEOREM 5. *There exist $\epsilon > 0$ and a neighborhood N of the point e_0 such that for any $t \in (0, \epsilon]$ the set of singular points SA_t of the hypersurface A_t coincides with the set of singular points of the hypersurface U different from e_0, and the variety $A_t \backslash SA_t$ has a tubular neighborhood T_t in $\mathbf{R}P^n \backslash SA_t$ such that the variety $U \backslash (N \cup SA_t)$ lies in $T_t \backslash N$ and is the image of some section of the fibering $T_t \backslash N \longrightarrow A_t \backslash (N \cup SA_t)$ and such that the variety $V_t \cap N$ lies in $T_t \cap N$ and is the image of some section of the fibering $T_t \cap N \longrightarrow A_t \cap N$.*

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