Stony Brook University
Mathematics Department
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Seminar in Mathematics
Teaching/Learning
MAT 519
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Comments on answers to Questionnaire

The first three questions were about your personal information. There is nothing to comment.

Here are the answers to the questions 4-6. The numbers in parenthesis are the number of works with the corresponding answer.

*4. Which topics of the secondary school mathematics curriculum do you **like** to teach?*

Your answers:
Large topics: algebra (20, including 1 for pre-algebra, 1 for algebra1, 6 for algebra 2, and 2 for integrated algebra), geometry (14), trigonometry (8), precalculus (4), calculus (2), statistics (2), probability.
Algebraic topics: percentages, linear equations, radicals, pre-algebra, factoring.
Geometric topics: transformational geometry (2), conics, geometric formulas, geometric proofs, circles, Pythagorean theorem.
Calculus topics: functions, derivatives, differentiation, graphing.

*5. Which topics of the secondary school mathematics curriculum do you **not like** to teach?*

Your answers:
Large topics: trigonometry (7), statistics (6), probability (5), algebra (3 including 1 for integrated algebra), geometry (2), calculus, discrete mathematics.
Algebraic topics: factoring, logs, binomial expansion, algebraic fractions.
Geometric topics: proofs (2), geometric proofs, special right triangles, 7th grade geometry: surface area and volume, units of measurement, (geometric?) constructions (2),
Trigonometry topics: trigonometric identities.

*6. Which topics of the secondary school mathematics curriculum would you like to **be discussed** in this course?*

Your answers: *Large topics:* trigonometry (10), algebra (9), geometry (7), calculus (3), pre-calculus (2), statistics (2), probability, pre-algebra, AP calculus.
Algebraic topics: fractions.
Geometric topics: geometry proofs, logic, proofs.
Calculus topics: precalculus - conic sections, quadratics (is it Calculus?), functions.
Applications: real life applications of math, algebra/geometry applications.

Comments about the answers to questions 4 - 6.
Lots of topics are a way too broad, like Algebra, Geometry, or Calculus. They do

not give me an idea what topics that would be useful and desirable in the class. For this, more detailed formulations of topics would be useful.

Looking more attentively at these answers, we see that the statistics of them give an idea about your attitudes towards parts of elementary mathematics. In average, you know algebra better than geometry or trigonometry.

There are courses in the MAT 511-516 family devoted to the broad topics. MAT 515 is specifically devoted to Elementary Geometry. Similarly the material covered in MAT 516 is very close to your high school needs in Probability and Statistics. Algebra and Calculus are less served, although many algebraic topics are considered in MAT 512 and topics from calculus, in MAT 513 and MAT 514.

Trigonometry does not seem to be covered in MAT 511-516 courses at all, and we will need to consider trigonometry in details.

I do appreciate more specific answers like graphing, factoring, surface area and volume, geometric formulas, conic sections. We already started graphing. I plan to discuss factoring soon. It is a sharply determined topic. I can add to it a good stuff (collection of useful theorems and tricks). Area and volume is a great topic. I plan to discuss it. It will open an access to geometric formulas.

Conics is another great topic. Please, tell me what you know about conics. We may have a good discussion on them.

I will be happy to discuss units of measurement, fractions, radicals, special right triangles, constructions, real life applications, but I would like to hear more specific requests and explanations what are problems do you face when teaching them.

Please, let me know what else would be useful to study. My email address is oleg.viro@gmail.com

7. Simplify the following expressions:

(a) $\dfrac{\frac{5}{2}-\frac{5}{3}}{\frac{1}{2}+\frac{1}{3}}$; *(b)* $\sqrt{x^2}$; *(c)* $\dfrac{1}{n}-\dfrac{1}{n+1}$.

Correct answers: (a) 1; *(b)* $|x|$; *(c)* $\dfrac{1}{n(n+1)}$.

Comments about answers to questions 7 (a), (b), (c).
Everybody gave the correct answers to (a) and (c).
Nobody gave the correct answer to (b).

Your answers to 7 (b):
19 students gave answer x,
4 students gave answer $\pm x$
and 1 student gave a sort of combined answer: $x \ (\pm x)$.

I do not know how to interpret the combined answer. You have to be more specific what it means. Sometimes x, and sometimes $\pm x$? Then under what conditions is it x and under what conditions $\pm x$?

What is wrong with the answer x? If $x = -1$, then it means $\sqrt{(-1)^2} = -1$. However $\sqrt{(-1)^2} = \sqrt{1}$. Is $\sqrt{1} = -1$? Not 1, but -1! I guess, you agree that it's not.

Here we come to the question what is $\sqrt{a}$. What is the definition of $\sqrt{a}$?

Here it is: for a non-negative real number a, $\sqrt{a}$ is a non-negative number b such that $b^2 = a$.

Notice that the requirement $b \geq 0$ prohibits $\sqrt{1}$ to be -1.

If you never saw this definition, are surprised or do not believe, then take a look at *Wikipedia*. We live in the Internet epoch, and should be accustomed with using the tools that it provides.

Wikipedia says:
"**Square root.** In mathematics, a **square root** ($\sqrt{}$) of a number x is a number r such that $r^2 = x$, or, in other words, a number r whose square (the result of multiplying the number by itself, or $r \times r$) is x.

Every non-negative real number x has a unique non-negative square root, called the **principal square root**, denoted by a radical sign as $\sqrt{x}$. For positive x, the principal square root can also be written in exponent notation, as $x^{1/2}$. For example, the principal square root of 9 is 3, denoted $\sqrt{9} = 3$, because $3^2 = 3 \times 3 = 9$ and 3 is non-negative. Although the principal square root of a positive number is only one of its two square roots, the designation "*the* square root" is often used to refer to the *principal* square root..."

The only defect that I found in this text is appearance of the symbol ($\sqrt{}$) after the first mentioning of the square root. I just removed it from the page of Wikipedia (as an editor).

The Wikipedia is not yet the most reliable source. Let us check one with a well-known reputation.

Encyclopaedia Britannica says the same:
"If a is a positive real number and n a positive integer, there exists a unique positive real number x such that $x^n = a$. This number – the (principal) nth root of a – is written $\sqrt[n]{a}$ or $a^{1/n}$..."

Why is the notation $\sqrt{a}$ used for the principal square root of a? Because the notation presumes a functional dependence, which presumes being univalued.

I do not think that after this your have any doubts in the formula $\sqrt{x^2} = |x|$. By the way, you can find this formula in the same article of Wikipedia. However, anyway, let me provide a proof. The proof should rely not only on the definition of the symbol $\sqrt{}$, but also on a definition of the absolute value. The absolute value is defined by the formula

$$|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x \leq 0. \end{cases}$$

Therefore $|x| \geq 0$ for any x. Obviously, $x^2 = |x|^2$. Recall that $\sqrt{a}$ is a non-negative real number b such that $b^2 = a$. Let $b = |x|$ and $x^2 = a$, then b is non-negative real number with $b^2 = |x|^2 = x^2 = a$, that is $|x| = b = \sqrt{a} = \sqrt{x^2}$. □

8. Write down a formula for the roots of the equation $x^2 + px + q = 0$, where p and q are given numbers.

Most of the students (20 out of 24) wrote the correct formula

$$x = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}.$$

No answer was given only by one student. Two students forgot minus in front of the first term, and one made other miscalculation. So, the formula is known.

9. What is the set of real numbers satisfying the inequality $\frac{1}{x+1} \leq 1$?

Correct solution. The inequality $\frac{1}{x+1} \leq 1$ is equivalent to $0 \leq 1 - \frac{1}{x+1}$. Further, $1 - \frac{1}{x+1} = \frac{x+1-1}{x+1} = \frac{x}{x+1}$. Hence the original inequality is equivalent to the inequality $\frac{x}{x+1} \geq 0$.

Now we will use the following **general theorem:**
Let A and B are real numbers. Then $\frac{A}{B} \geq 0$ if and only if either

$$\begin{cases} A \geq 0 \\ B > 0 \end{cases} \quad \textit{or} \quad \begin{cases} A \leq 0 \\ B < 0. \end{cases}$$

In our situation $A = x$, $B = x + 1$, and we come to

$$\begin{cases} x \geq 0 \\ x + 1 > 0 \end{cases} \quad \text{or} \quad \begin{cases} x \leq 0 \\ x + 1 < 0. \end{cases}$$

The first of these two systems is equivalent to a single inequality $x \geq 0$, the second one, to $x < -1$. Thus x satisfies the original inequality if and only if either $x \geq 0$ or $x < -1$. In other words, the set of solutions of our inequality is $(-\infty, -1) \cup [0, \infty)$. □

The correct answer for this problem was formulated only by one third of students (8 out of 24). Only two students wrote nothing about it.

Six students claimed that any real number except -1 satisfies the inequality. They confused the set of solutions of the inequality with the set of numbers for which the function on the left hand side is defined.

A counter-example to this answer (that is $x \neq -1$ such that $\frac{1}{x+1} > 1$) is easy to find. For example, if $x = -1/2$, then $\frac{1}{x+1} = \frac{1}{-1/2+1} = \frac{1}{1/2} = 2 > 1$. Students in this group did not attempt to present any arguments justifying this answer.

More elaborate wrong solution was given by a group of six students. They just passed from the original inequality $\frac{1}{x+1} \leq 1$ to the inequality $1 \leq x+1$ by multiplying it by $x+1$ and then transformed it to $x \geq 0$.

The mistake in this arguments is that multiplication of the both sides of an inequality by the same number does not give an equivalent inequality.

Indeed, if the number by which the inequality was multiplied is negative, then the sign of inequality reverses. For instance, $1 < 2$, implies $(-1) \times 1 > (-1) \times 2$ that is $-1 > -2$.

The correct rule can be formulated as follows: the set of solutions of an inequality $A \leq B$ is equal to the union of the set of solutions of the system of inequalities

$$\begin{cases} AC \leq BC \\ C \geq 0 \end{cases}$$

and the set of solutions of the system

$$\begin{cases} AC \geq BC \\ C \leq 0 \end{cases}$$

This rule is as obvious as the general theorem we formulated above. However, it is applied usually only in the special case, namely, when C is a constant.

For our inequality $A = \frac{1}{x+1}$, $B = 1$ and $C = x+1$. Therefore the systems looks as follows:

$$\begin{cases} \frac{x+1}{x+1} \leq x+1 \\ x+1 \geq 0 \end{cases} \qquad \begin{cases} \frac{x+1}{x+1} \geq x+1 \\ x+1 \leq 0 \end{cases}$$

We must take into account that the denominator $x+1$ cannot vanish. Then the systems above turn into

$$\begin{cases} 1 \leq x+1 \\ x+1 > 0 \end{cases} \qquad \begin{cases} 1 \geq x+1 \\ x+1 < 0 \end{cases}$$

Clearly, the first system is equivalent to $0 \leq x$, the second system is equivalent to $x < -1$. So, the set of solutions of the original inequality is $(-\infty, -1) \cup [0, \infty)$, the same as above.

This solution is neither shortest nor standard. It is presented just to show how to repair the wrong solution cited above. The correct solution presented above is shorter and more straightforward.

None of the students who found the correct answer presented any argument in favor of it. So, no one gave a correct solution.

I have to announce that
I will not accept an answer to a problem, no matter correct or wrong, unless the work leading to the answer is shown.

For a comparatively simple inequality that we considered above one can easily find the correct answer without writing a solution. In more complicated cases it is more difficult.

Many students made strange calculations. They checked if the inequality is fulfilled for a few special integer values of x and tried to guess the right answer on the basis of this. Apparently, some succeeded. **This is a completely wrong approach!**

It may work in a multiple choice test, when you need to sort out wrong answers. But in any other situation it may give a wrong result.
I will not credit a solution by guessing, unless such solution is requested explicitly.

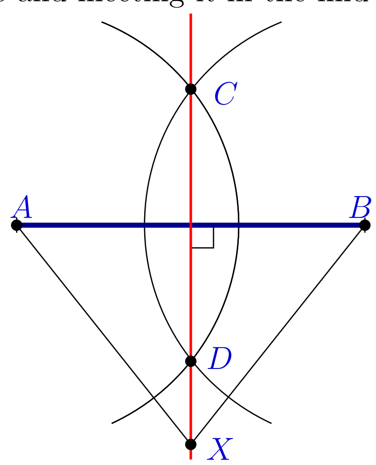
We should revisit the rules used for solving inequalities.

10. How to find the center of a circle passing through vertices of a given triangle?

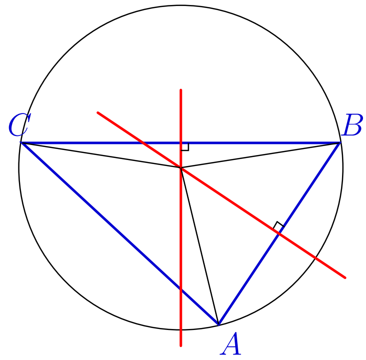
Fourteen works contain nothing concerning this problem, if not to count something like a picture with just a triangle, a circle that passes through the vertices of the triangle and a point which pretends to be the center of the circle.

Three suggested to find the center of gravity (that is the intersection point of medians). This is wrong. Two suggested to find the intersection point of bisectors, which is the center of inscribed circle, but for any non-regular triangle it is not the center of circumscribed one. Five works suggested to find the intersection point of bisector perpendiculars for the sides. This is the correct answer.

The set of points equidistant from two points is the line perpendicular to the segment connecting the two points and meeting it in the mid point.



The intersection point of two such bisector perpendiculars for two sides of the triangle is equidistant to all three vertices.



Question: what if we would choose other pair of sides and take intersection of their bisector perpendiculars?

11. For how many real values of k the graphs of functions $y = kx + 3$ and $y = \frac{1}{x}$ have exactly one common point? Sketch the graphs.

There are two approaches: geometric and algebraic. Let us start with the algebraic one. The intersection point of the graphs is the solution of the system

$$\begin{cases} y = kx + 3 \\ y = \dfrac{1}{x}. \end{cases}$$

The question is for how many values of k the number of solutions is one. Let us solve the system. First eliminate y:

$$kx + 3 = \frac{1}{x}.$$

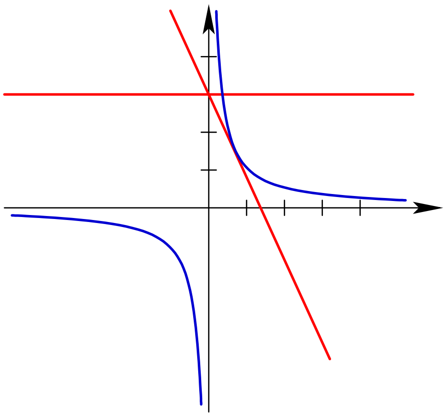
Obviously, $x = 0$ is not a solution of this equation. Multiply the equation by x. This gives $kx^2 + 3x - 1 = 0$. We need to find out for how many values of k this equation has exactly one root.

For a fixed k, what is the degree of this equation? If $k = 0$, then this is a linear equation, and it has a unique solution $x = 1/3$, and the original system, a unique solution $x = 1/3$, $y = 1/x = 3$.

If $k \neq 0$, then the equation $kx^2+3x-1 = 0$ is quadratic and it has a unique solution iff its discriminant vanishes. The discriminant is $3^2+4k = 9+4k$. It vanishes if and only if $k = -9/4$.

Thus we see that the answer is 2: for 2 values of k (namely, for $k = 0$ and $k = -9/4$) the system has exactly one solution.

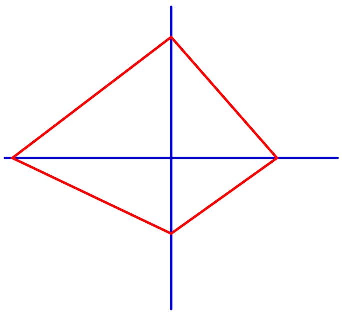
Geometric picture: the graph of $y = \frac{1}{x}$ is a hyperbola, the graph of $y = kx + 3$ is the line with slope k that passes through the point $(0, 3)$. If $k = 0$, then this curve is parallel to the x-axis and meet the hyperbola in a single point. If $k > 0$, then the line intersects each branch of the hyperbola $y = 1/x$ in a single point, so overall there are two common points. If $k < 0$, but close to the zero, then there are two intersection points with the branch of $y = 1/x$ in the positive quadrant. One is close to the intersection point of the hyperbola with the horizontal line passing through the point $(0, 3)$, another one is far near the x-axis. When the line rotates around $(0, 3)$, the two intersection points move towards each other, until meeting, when the line becomes tangent to the hyperbola. After moving through tangency, the line looses intersection with hyperbola.



12. Is it correct that if in a quadrilateral diagonals are perpendicular then the quadrilateral is a rhombus? Is the converse true?

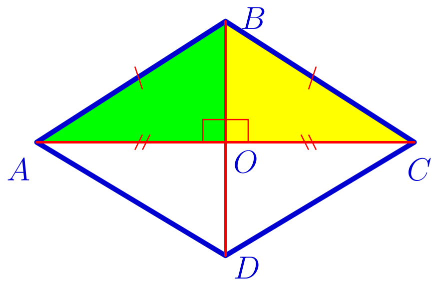
Correct answer. The answer to the first question is negative. There is a quadrilateral with perpendicular diagonals, which is not a rhombus. Most of quadrilaterals with perpendicular diagonals are not rhombuses.

In order to construct such a quadrilateral, just take two perpendicular lines and choose on them four points as shown on the picture below. The points should be chosen non-symmetric with respect to the intersection point of the lines. These four points are vertices of a required quadrilateral. It is not a rhombus, because a rhombus is symmetric about the intersection point of its diagonals.



The answer to the second question is positive. The diagonals of a rhombus are perpendicular. This is a standard theorem that can be found in any textbook in Elementary Geometry.

In order to prove it, let us recall that a rhombus is defined as a parallelogram in which all four sides are congruent to each other. Consider a rhombus $ABCD$. Let O be the intersection point of its diagonals AC and BD.



Then in the triangles AOB and BOC the sides AB and BC are congruent as sides of the rhombus, OB is their common side and sides AO and OC are congruent, because in any parallelogram diagonals bisect each other. Therefore by SSS test the triangles AOB and BOC are congruent and their corresponding angles AOB and BOC are congruent. On the other hand, these angles are adjacent and hence their sum is 180°. Therefore each of them is right. $\square$

Comments of students' works. The majority of answers were wrong (at least, half-wrong). Two thirds of all students (16 out of 24) replied that both statements are true. No prove supplied.

In 3 works the answers are correct, but the proofs show that the reasons behind the correct answers were wrong. For instance, how do you like this argument: "It is not correct that if the quadrilateral diagonals are perpendicular then the quadrilateral is a rhombus. It could be also a square."?

Well, a square is a rhombus! One of commonly used definitions of square is that this is a quadrilateral that is both rectangular and rhombus.

In one work both answers are wrong, no proof supplied.

Finally, in two works the answers are correct, and sketches of drawings demonstrate the correct understanding of the matter.

13. Formulate the Pythagorean theorem. Sketch a proof.

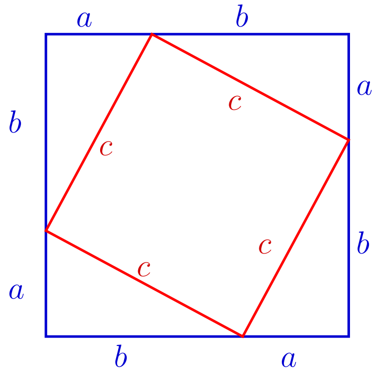
Correct solution.
The Pythagorean theorem states that in a right triangle the square of the length of the hypothenuse equals the sum of squares of the lengths of the other two sides.

The statement of the Pythagorean theorem can be written as a formula: if the length of the hypothenuse in a right triangle is c and the lengths of two other sides are a and b, then $c^2 = a^2 + b^2$.

This is one of the most important and useful results in Elementary Geometry. It has lots of proofs. 90 proofs can be found on
http://www.cut-the-knot.org/pythagoras/index.shtml

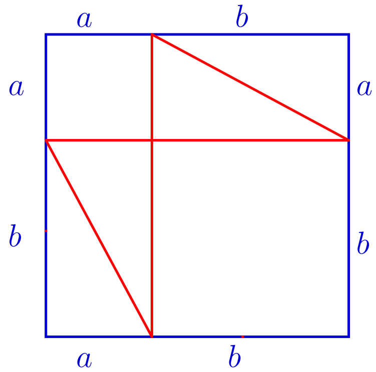
Comments on students' solutions.
In nine works there is a formulation together with a sketch of a proof. In all nine works the proof is the same. It is based on calculation of the area of a square with sides $a + b$ in two ways.

On one hand, the area is $(a+b)^2 = a^2 + b^2 + 2ab$. On the other hand, the square can be presented as the union of four copies of the right triangle with legs a and b positioned at the four corners of the square and the central square with side c:



Thus its area is $c^2 + 4\frac{1}{2}ab = c^2 + 2ab$. Compare these two calculations of the same area, we conclude that $c^2 = a^2 + b^2$. $\square$

This half-algebraic half-geometric proof can be made fully geometric by replacing the calculation $(a+b)^2 = a^2 + b^2 + 2ab$ with its pictorial version:



Five students gave a sketch (pictorial) of the statement, no proof sketched.

Three students sketched an Egyptian triangle (right triangle with sides 3, 4 and 5). It is of current political interest, but definitely not what was required. One student sketched an Egyptian triangle magnified with factor 2. One student just wrote $a^2 + b^2 = c^2$, nothing more. One student considered right triangle with legs equal 6, and got, as an answer (!) to the question $\sqrt{72}$.

None of these numerical illustrations is appropriate! You were asked about a statement and proof, examples were not solicited.