Head to tail compositions

Oleg Viro

November 27, 2014

Plane Isometries

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Lemma. *A plane isometry is determined by its restriction to any three non-collinear points.* $\square$

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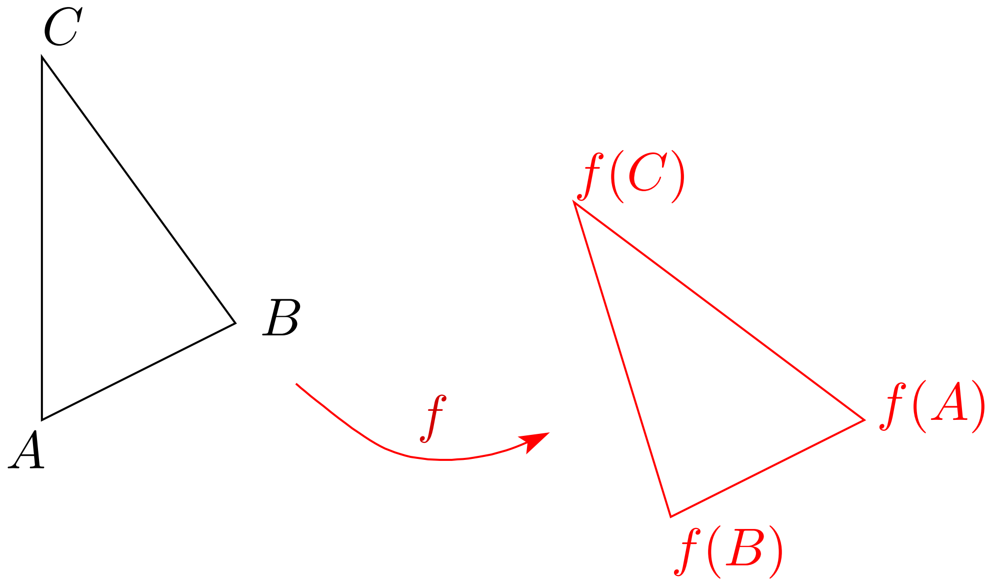
Proof of Theorem. Given an isometry:



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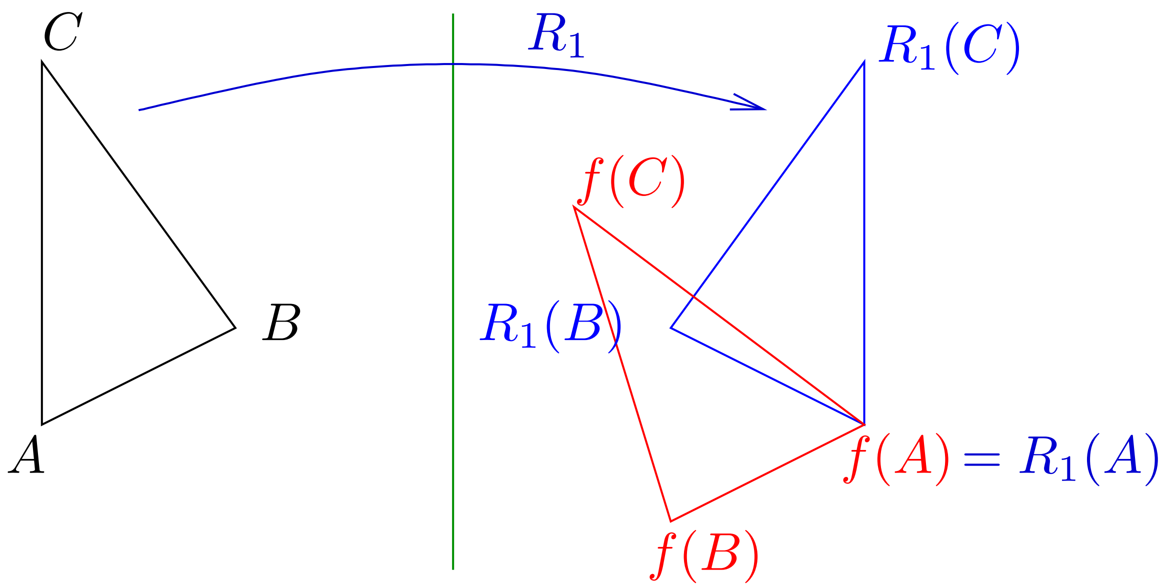

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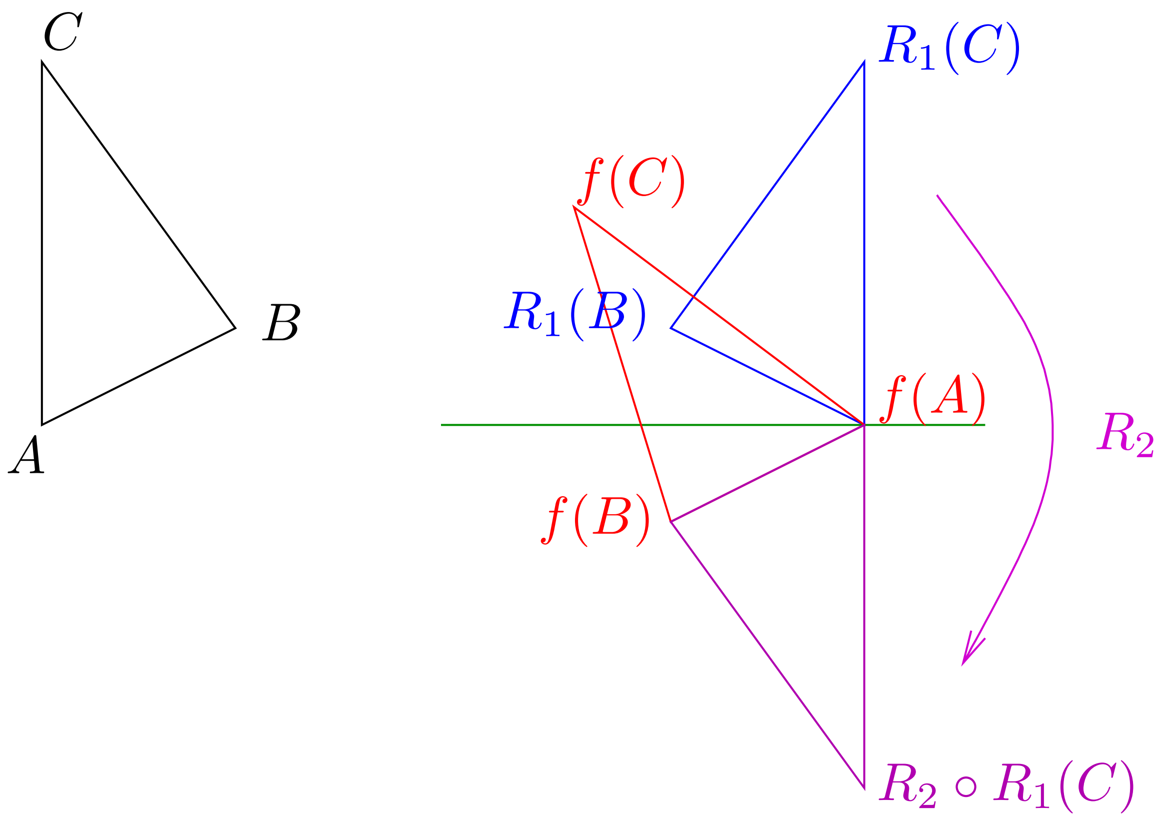

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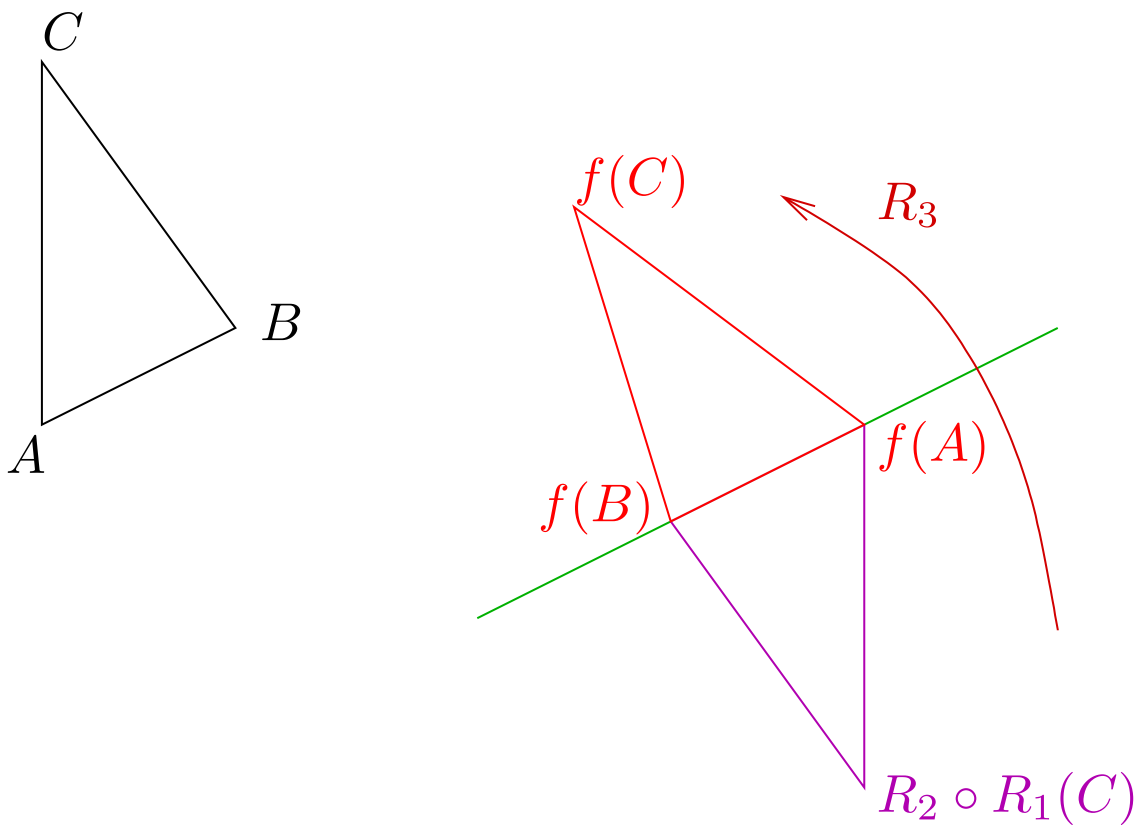

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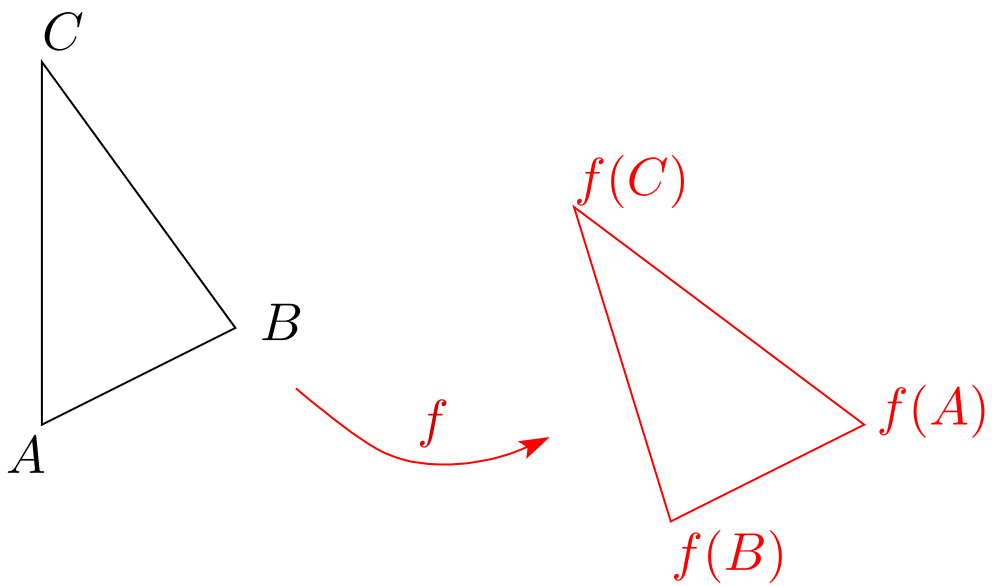
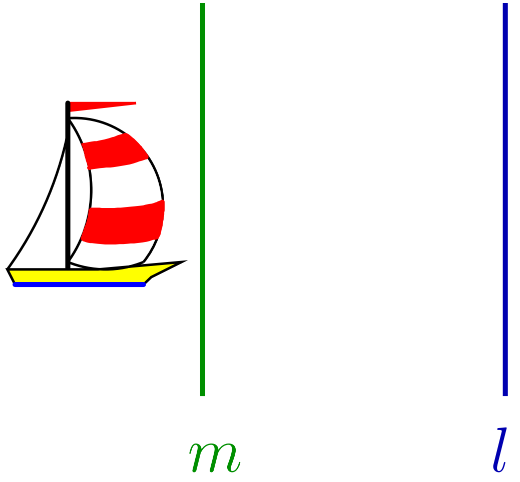


We are done. $\square$

Compositions of reflections in parallel lines



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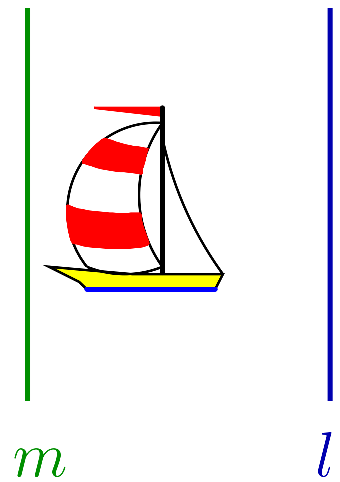

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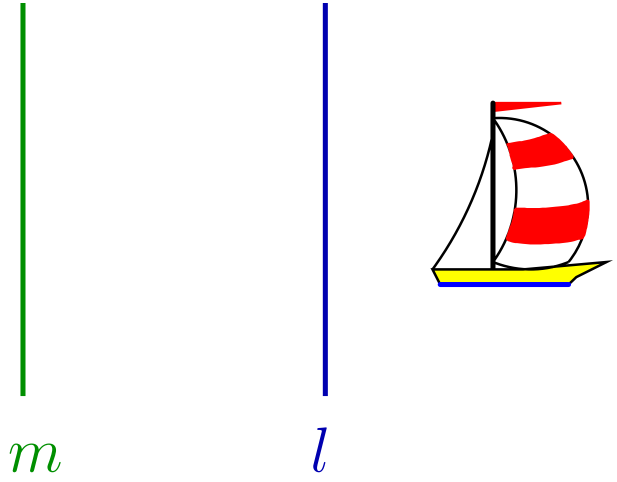

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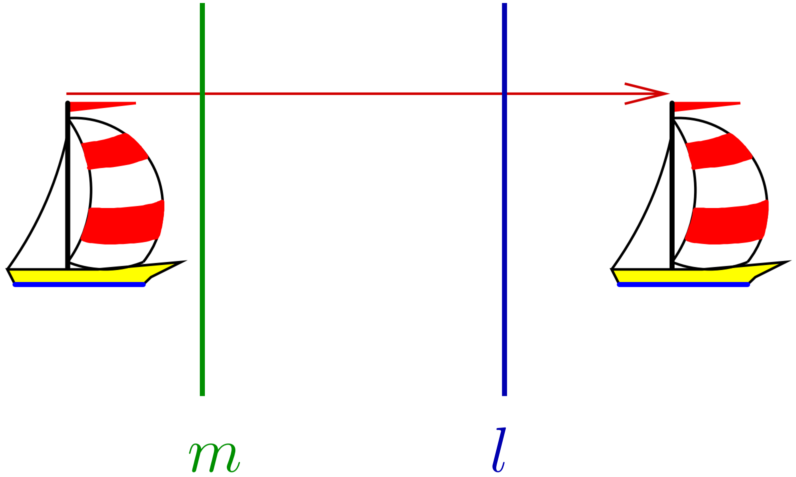

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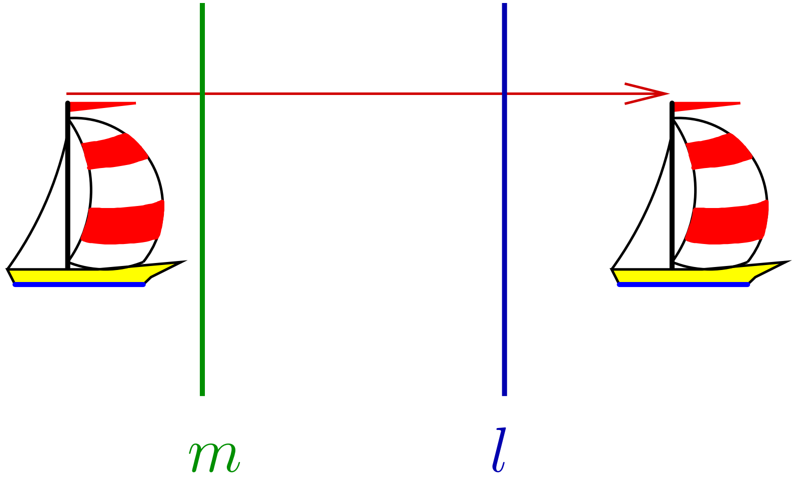

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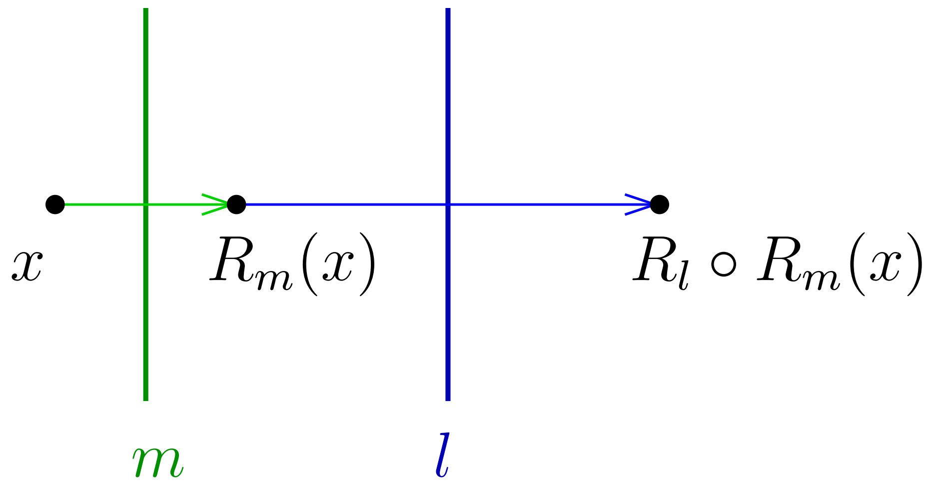

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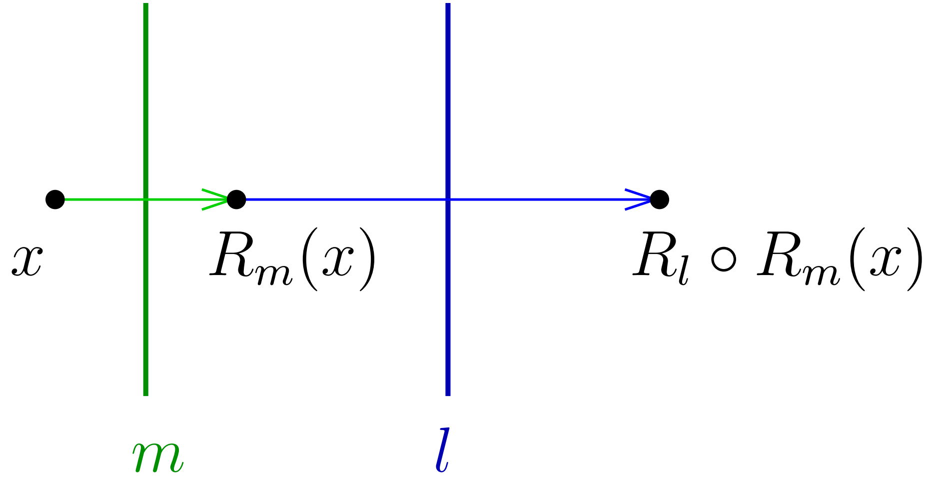

The decomposition is not unique:

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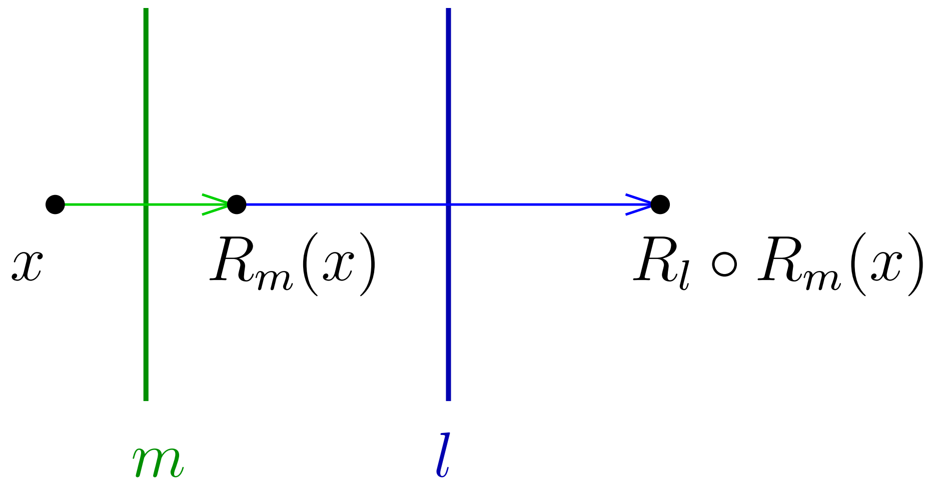

The decomposition is not unique:

$R_l \circ R_m = R_{l'} \circ R_{m'}$

iff l', m' can be obtained from l, m by a translation.

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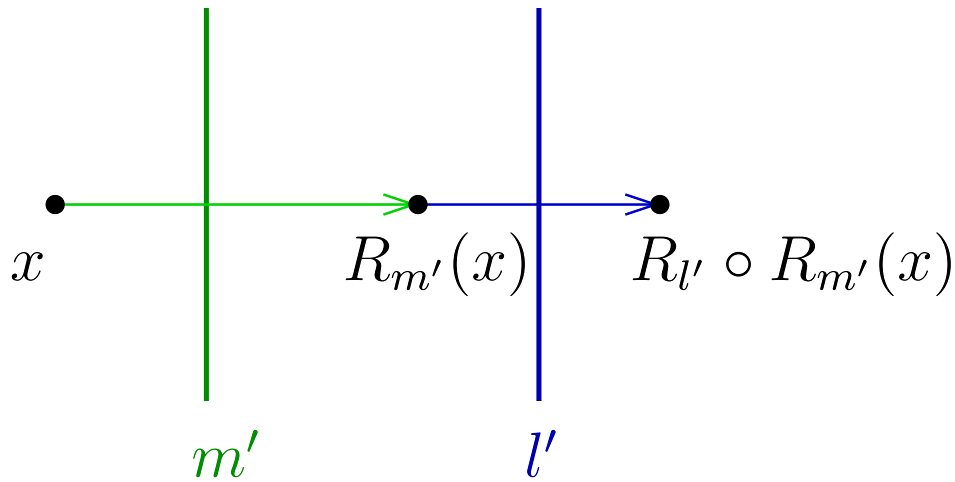

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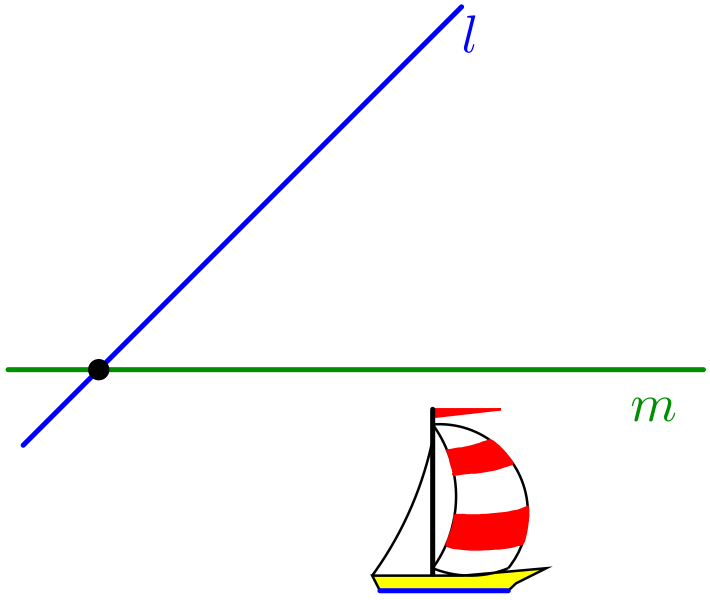

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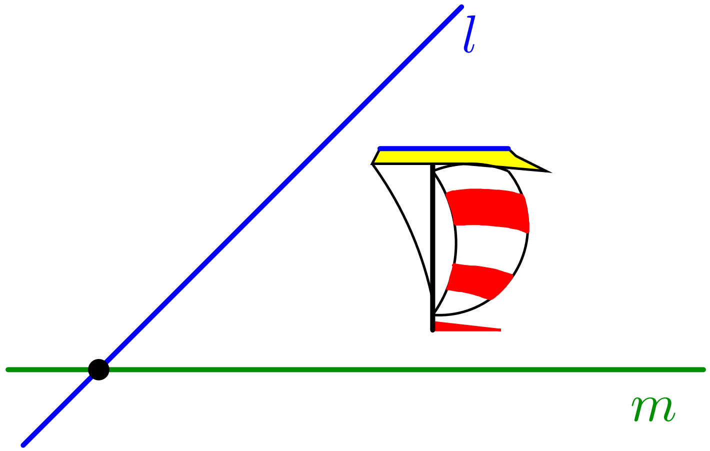

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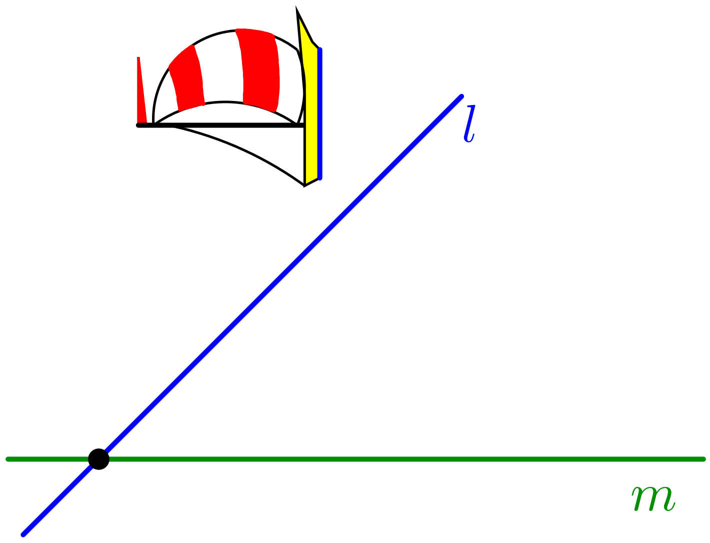

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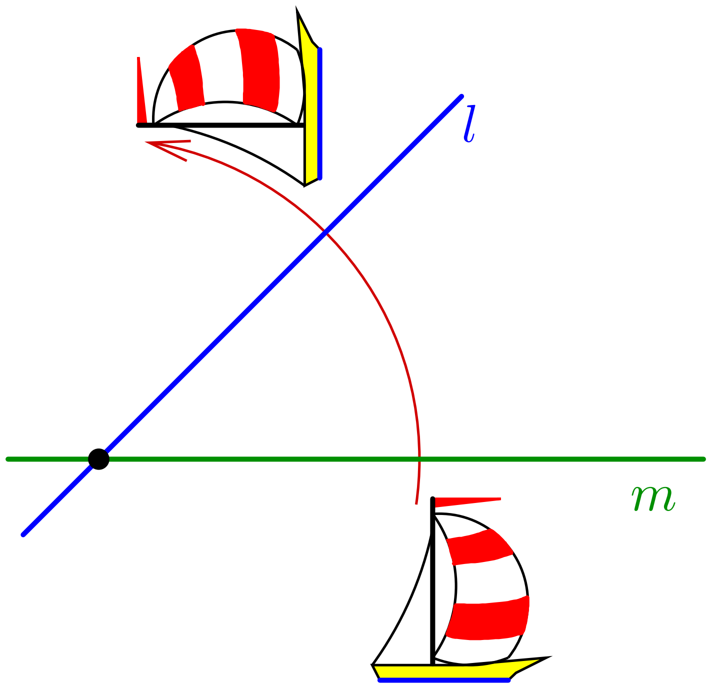

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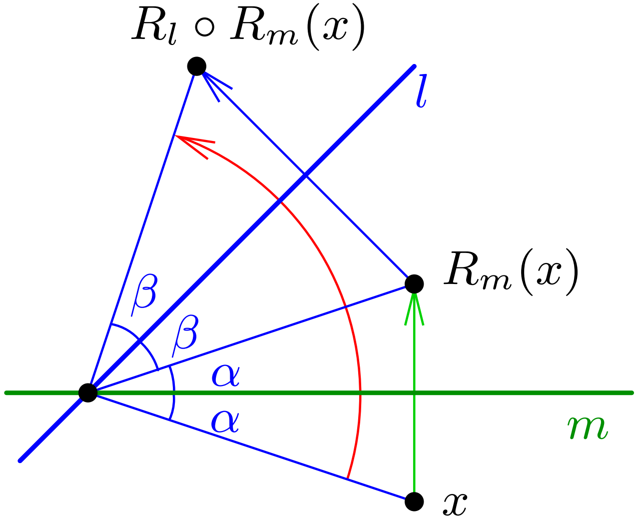

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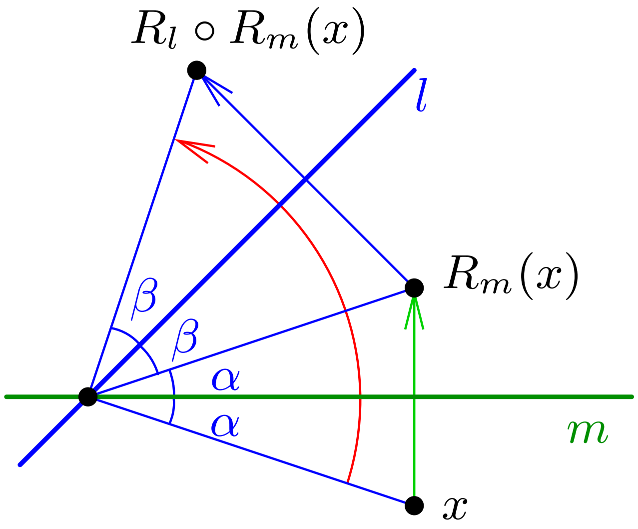

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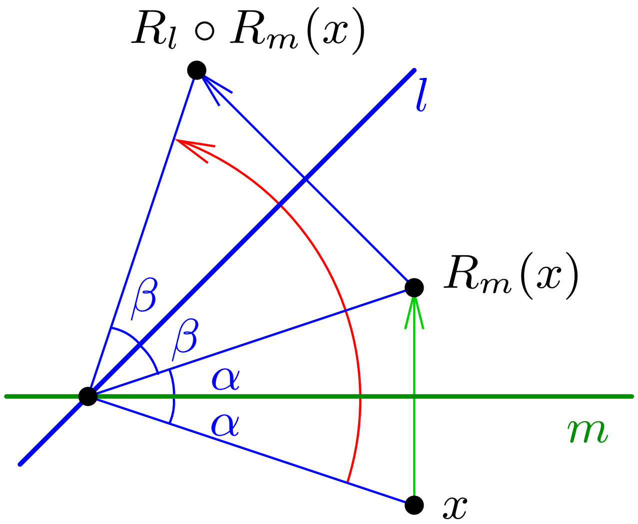

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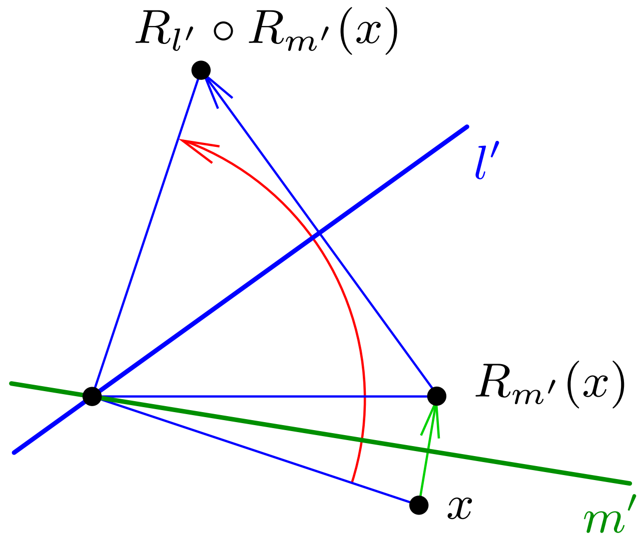

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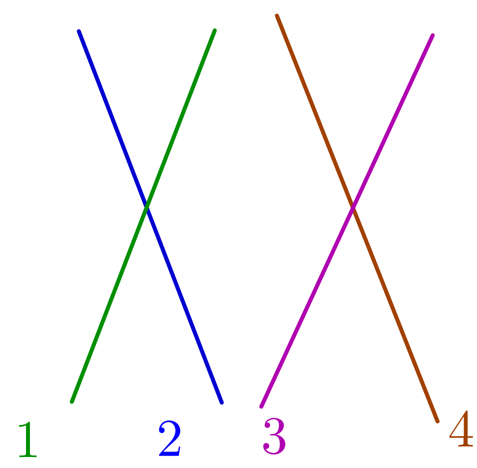

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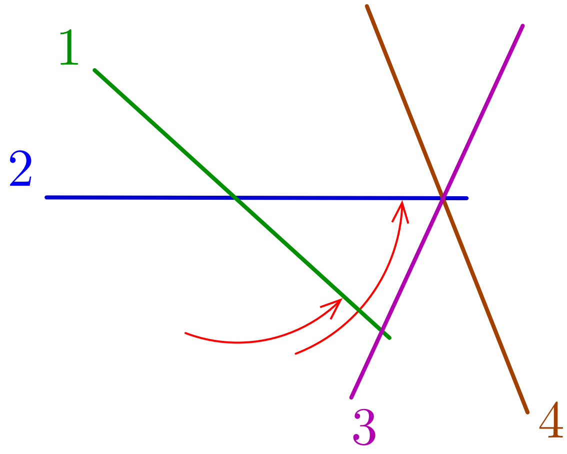

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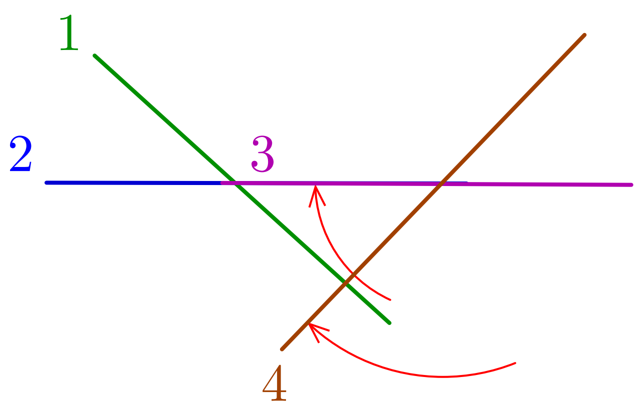

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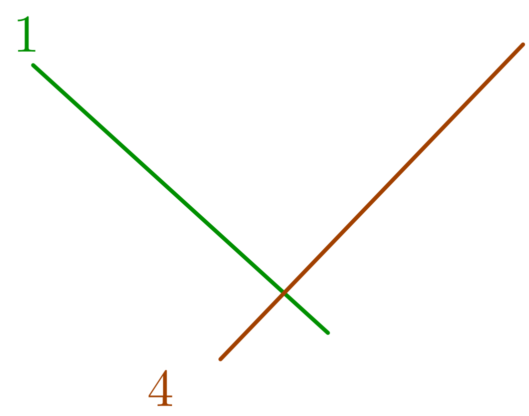

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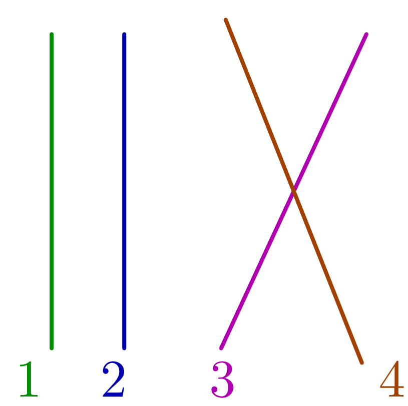

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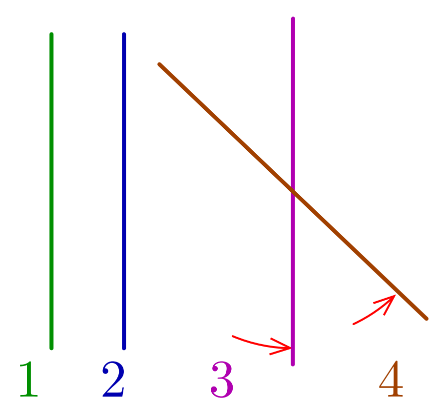

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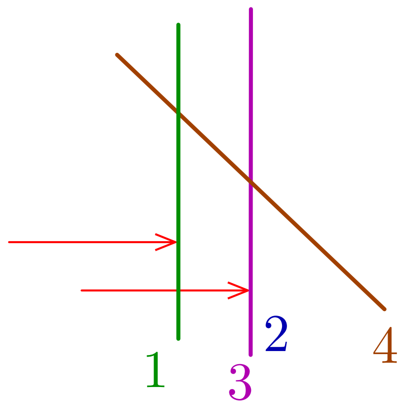

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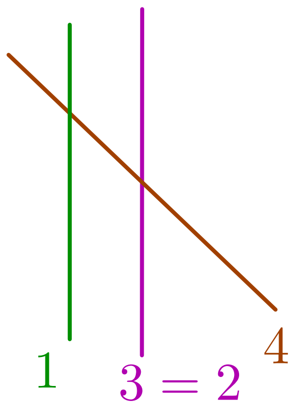

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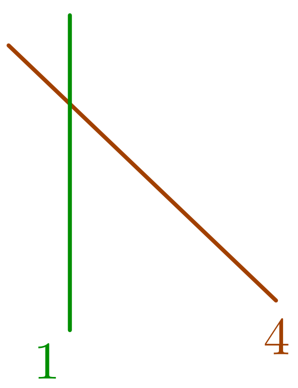

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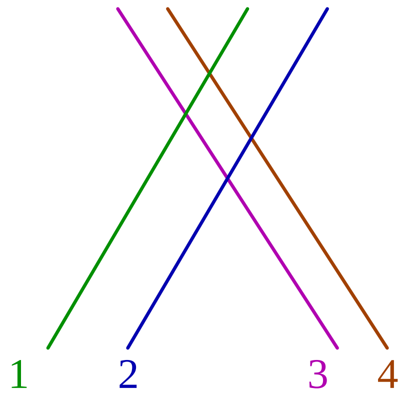

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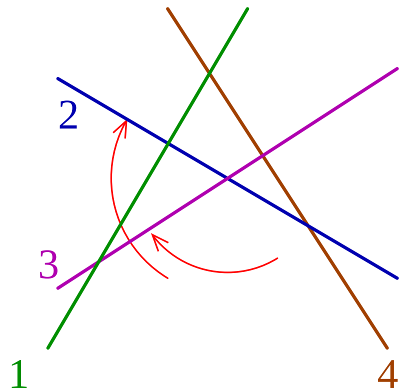

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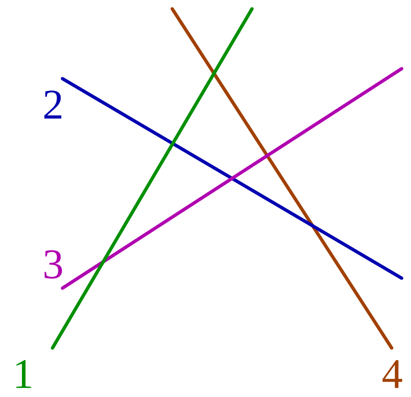

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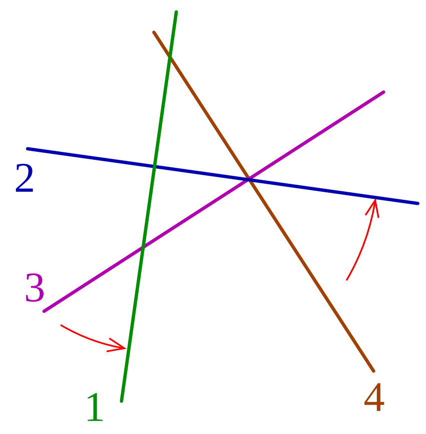

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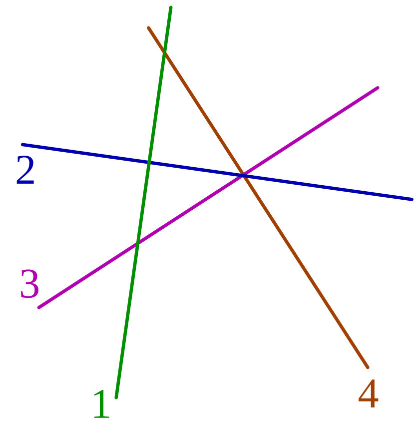

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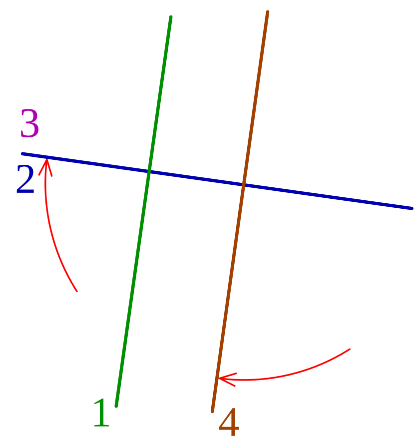

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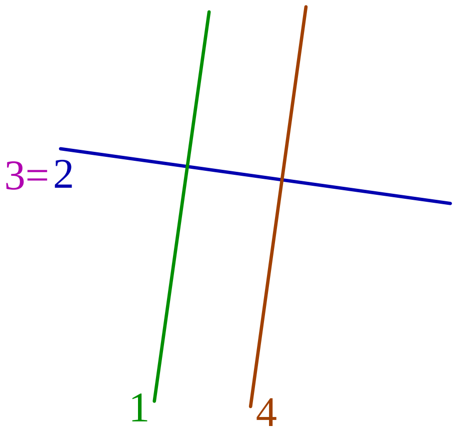

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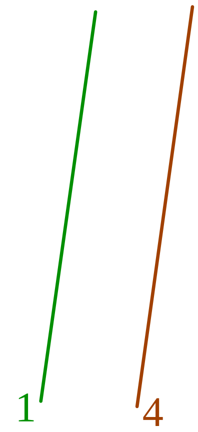

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Lemma. *A composition of any 4 reflections in lines can be transformed by these relations to a composition of 2 reflections in lines.* $\square$

Generalization of Lemma. In $\mathbb{R}^n$,
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Generalization of Theorem. Any relation among reflections in hyperplanes of $\mathbb{R}^n$ follow
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Key example: $\mathbb{R} \to \mathbb{R} : x \mapsto 2a - x$, the reflection of $\mathbb{R}$ in a point a.

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Correspondence Flipper $S \longleftrightarrow$ Flip in S is
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simple static geometric objects - flippers - and isometries.

Symmetry about a point

is a flip.

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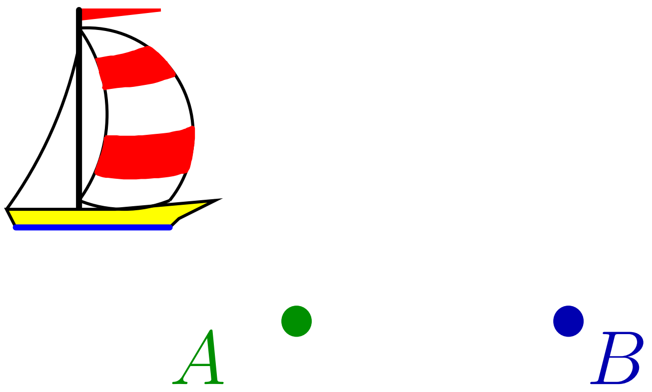
is a flip. Composition of flips in points



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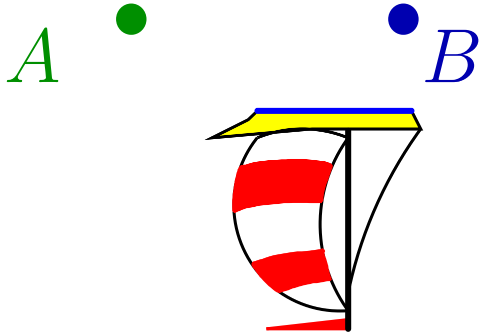

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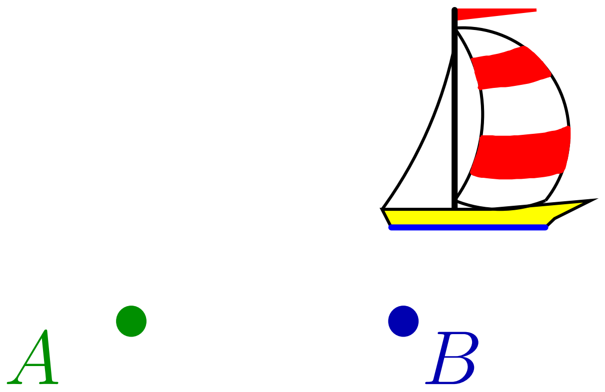

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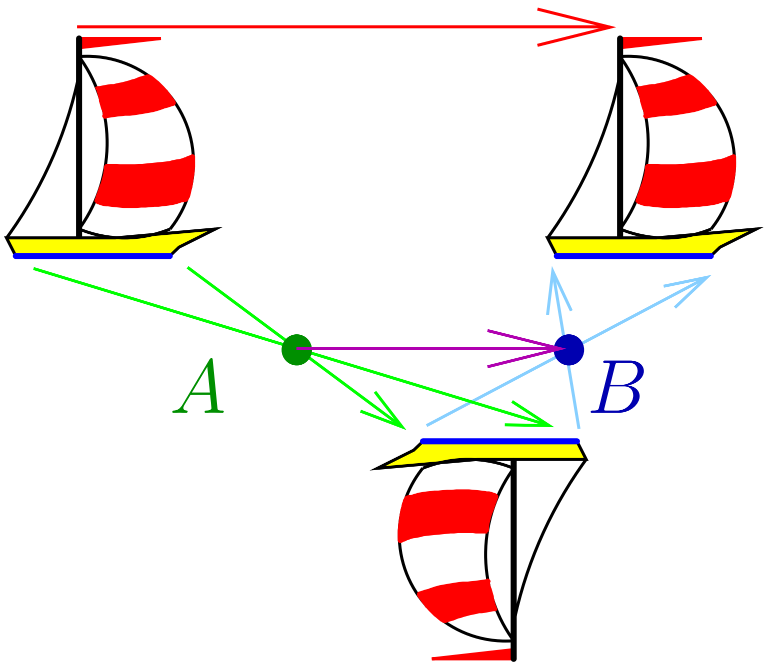

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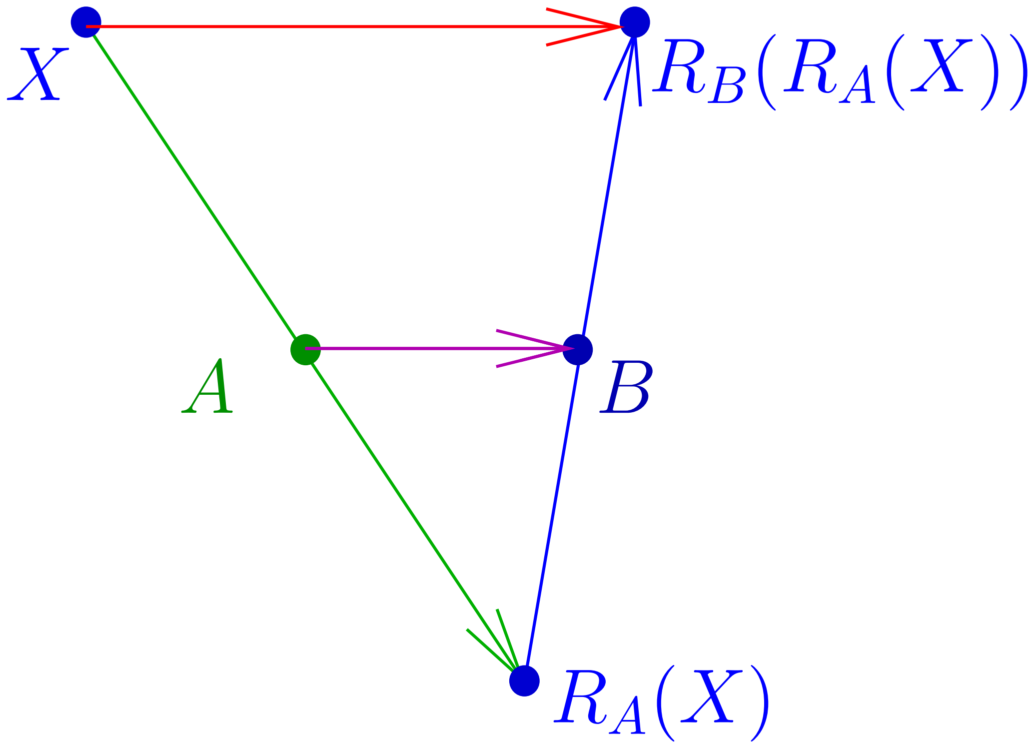

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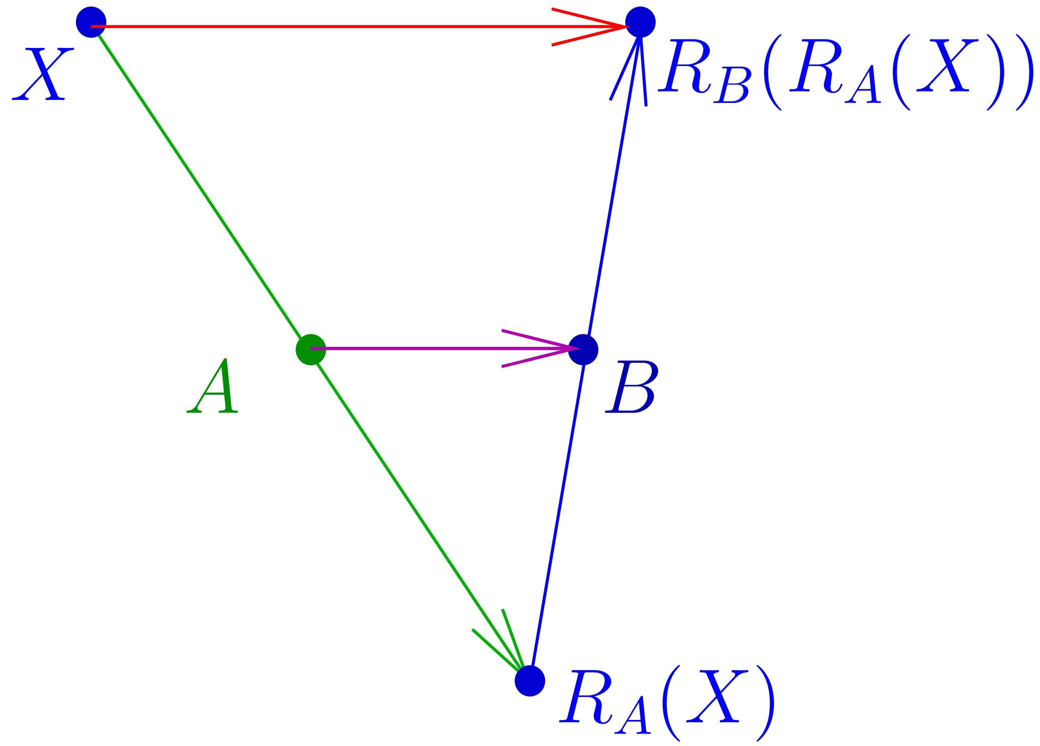

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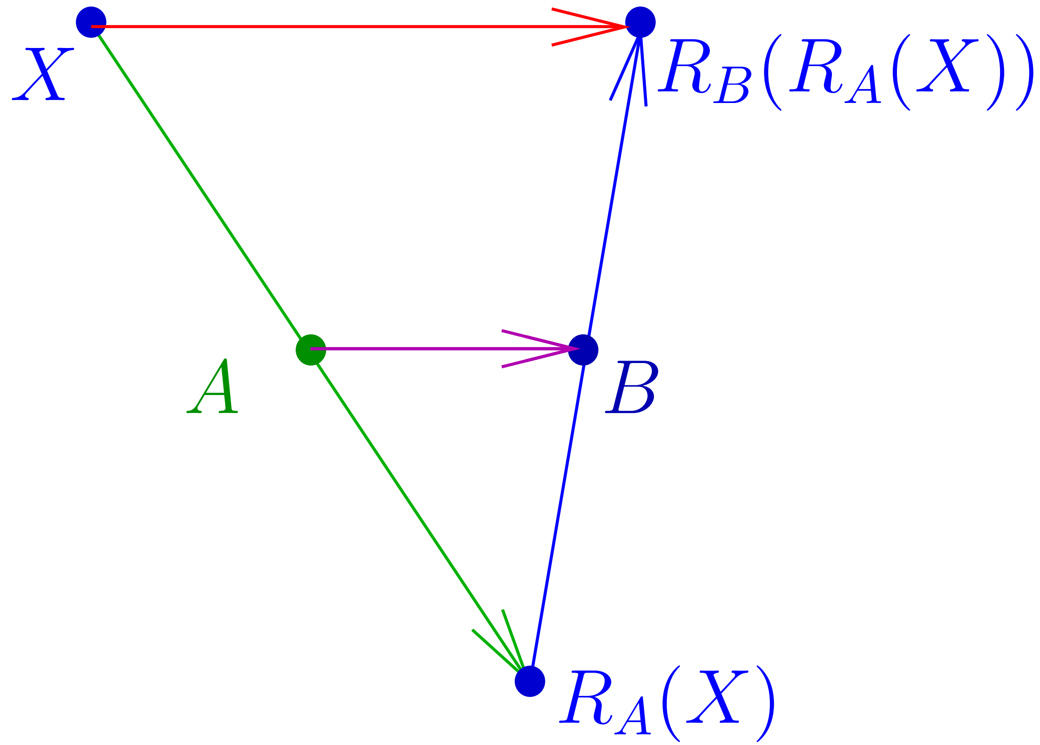

$\overrightarrow{AB} = \frac{1}{2}\overrightarrow{X\,R_B(R_A(X)}$

$\overrightarrow{AB}$ is half the arrow representing $R_B \circ R_A$.

Head to tail

Compare the head to tail addition $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

to $(R_C \circ R_B) \circ (R_B \circ R_A) = R_C \circ R_B^2 \circ R_A = R_C \circ R_A$.

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Corollary. *Any isometry of a hyperbolic space, sphere, projective space, etc. is a composition of two flips.*

A flip-flop decomposition.

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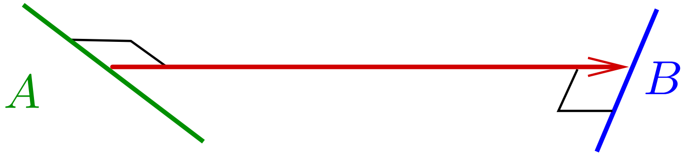

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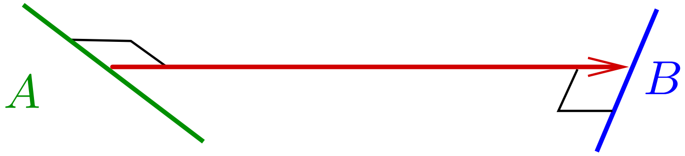

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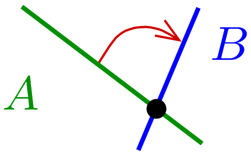

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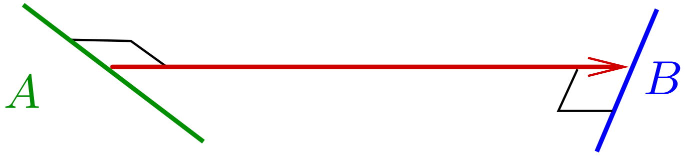

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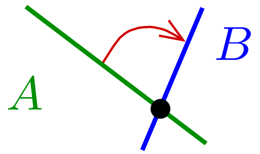

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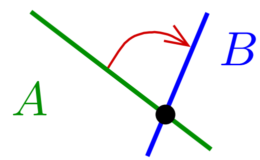

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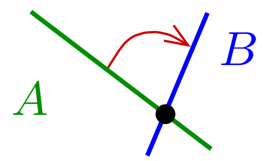

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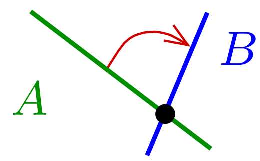

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Problem. Find an explicit description for the equivalence.

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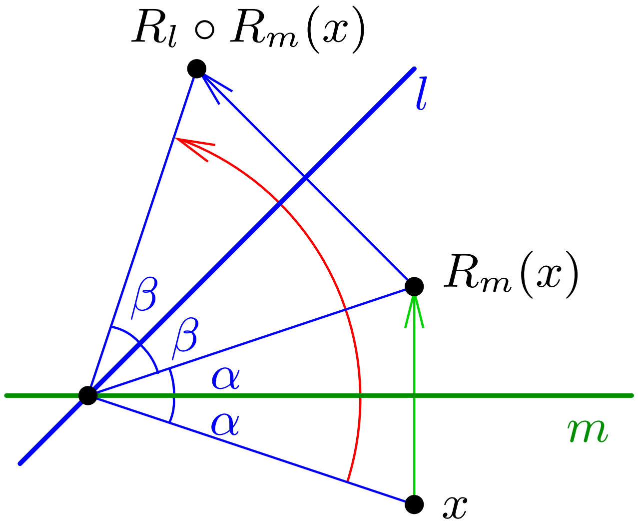

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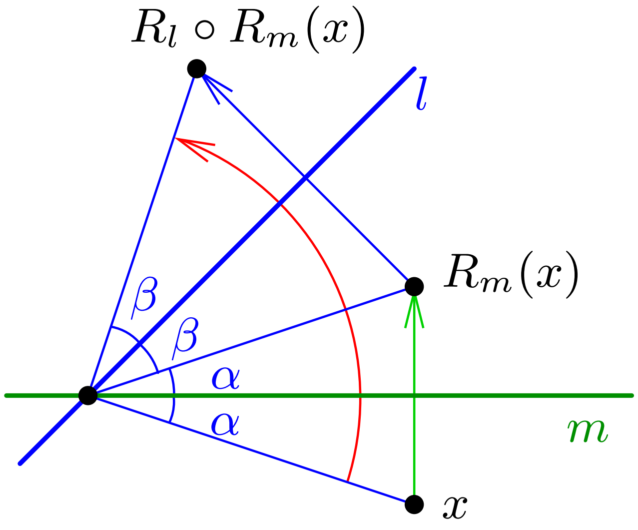

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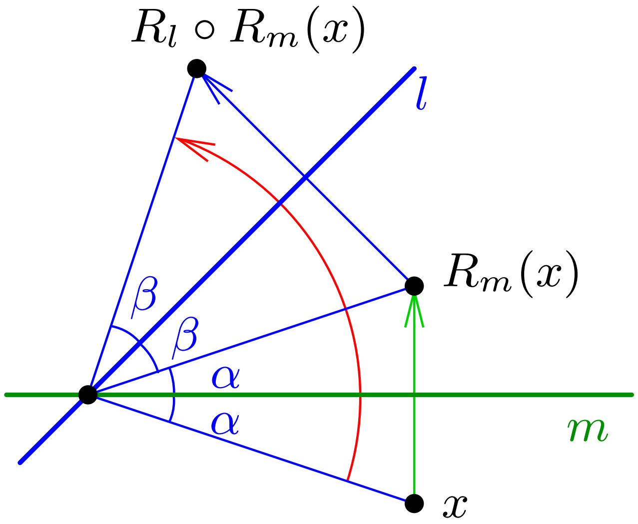

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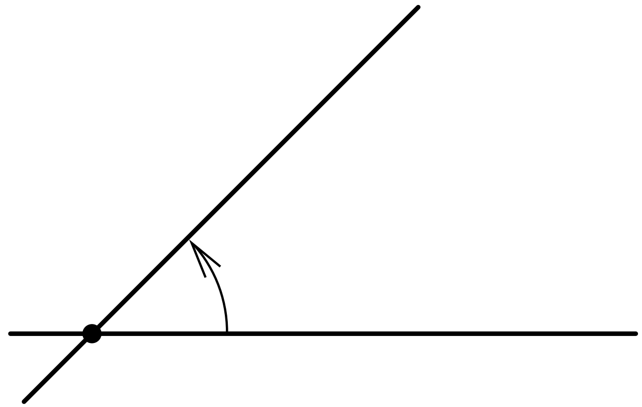

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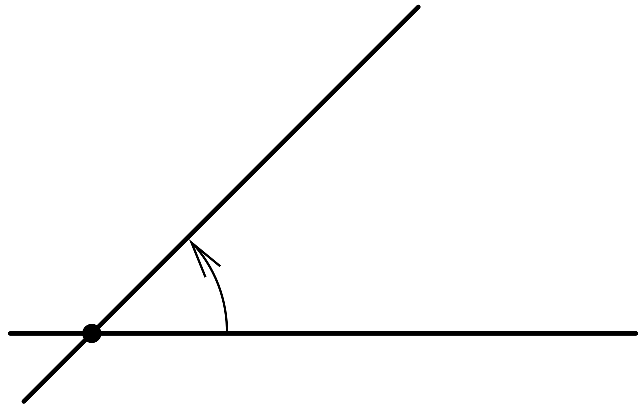

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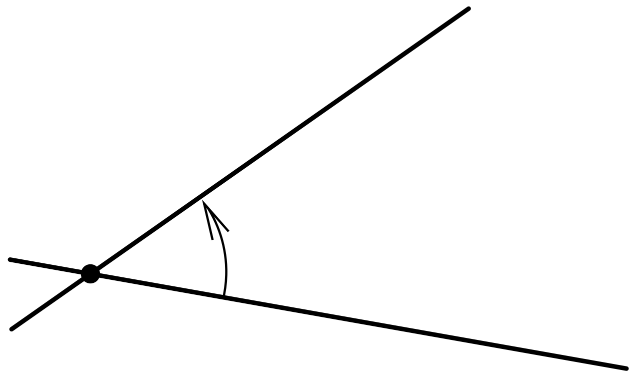

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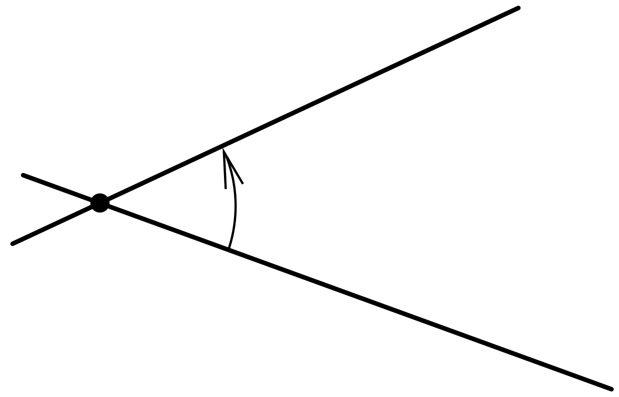

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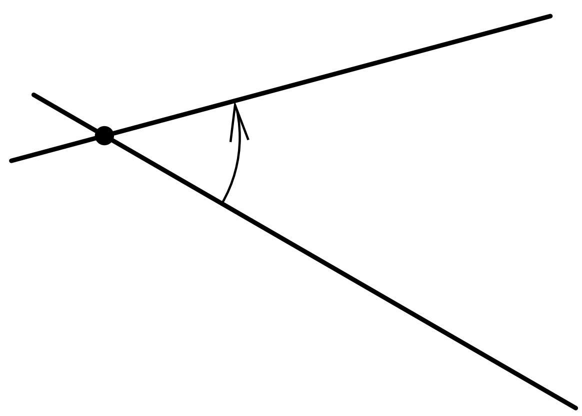

Head to tail for rotations

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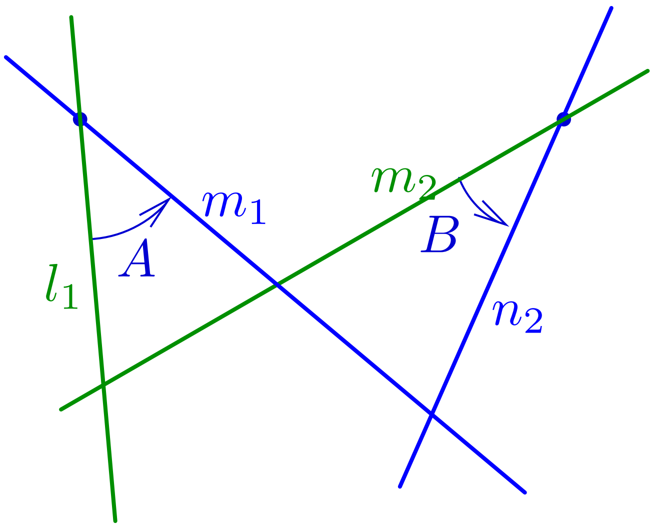

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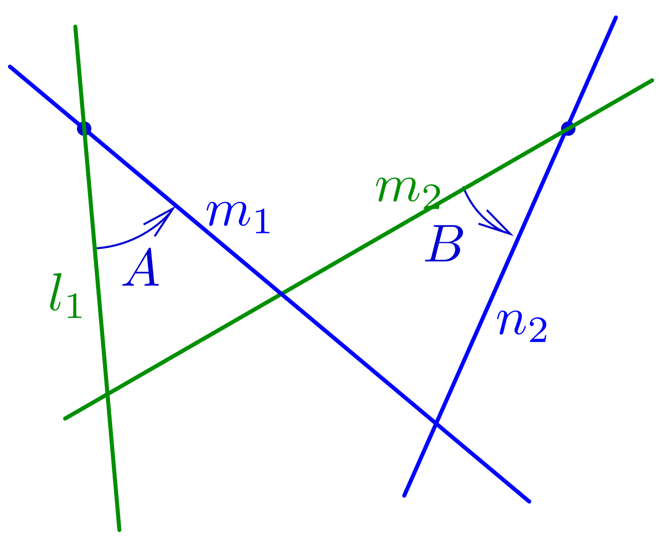

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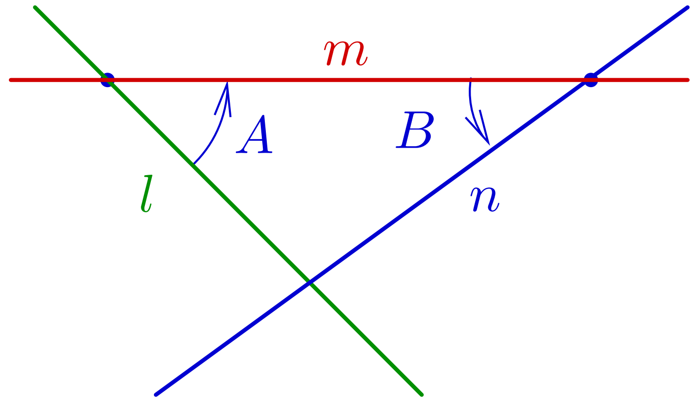

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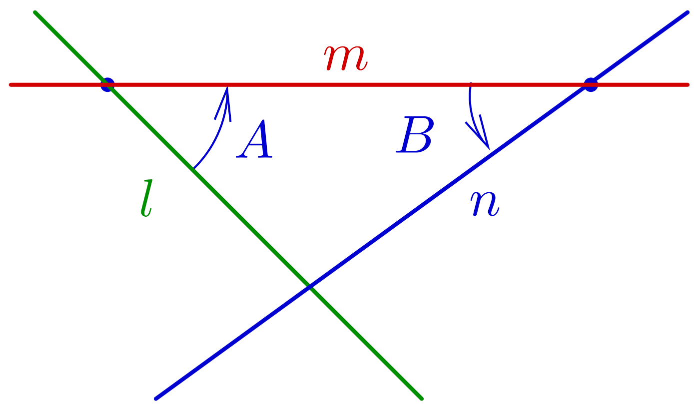

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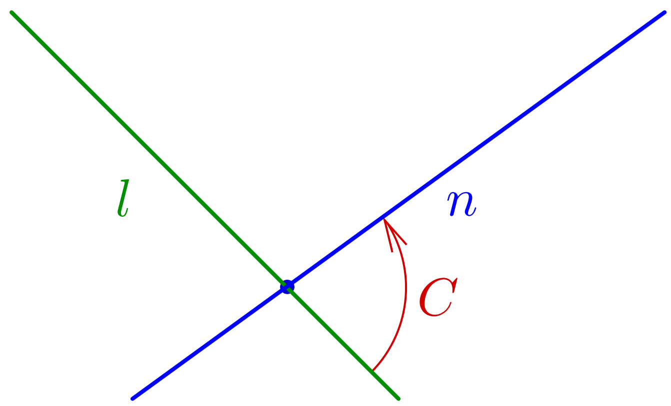
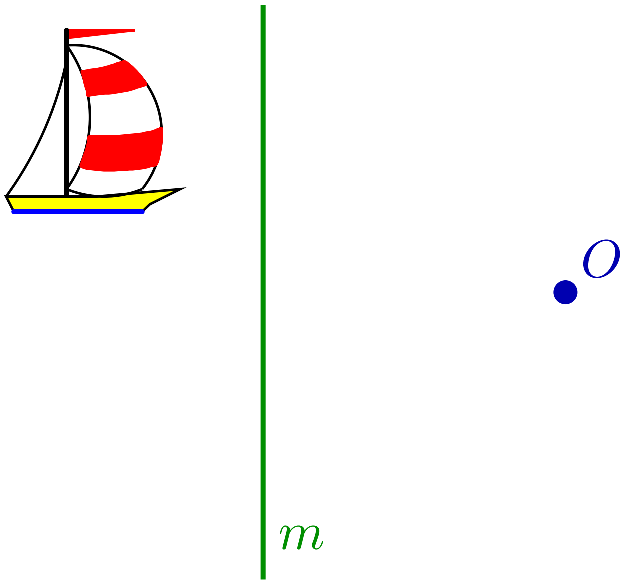


Composing reflections in line and point



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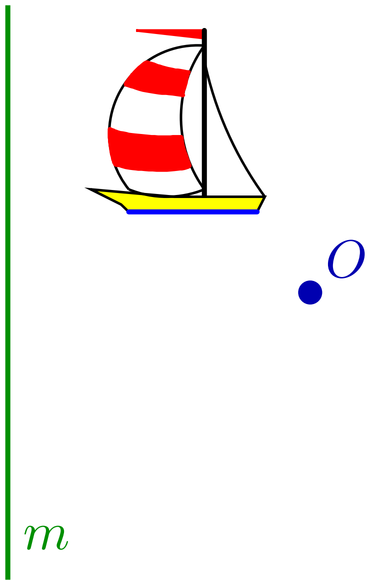

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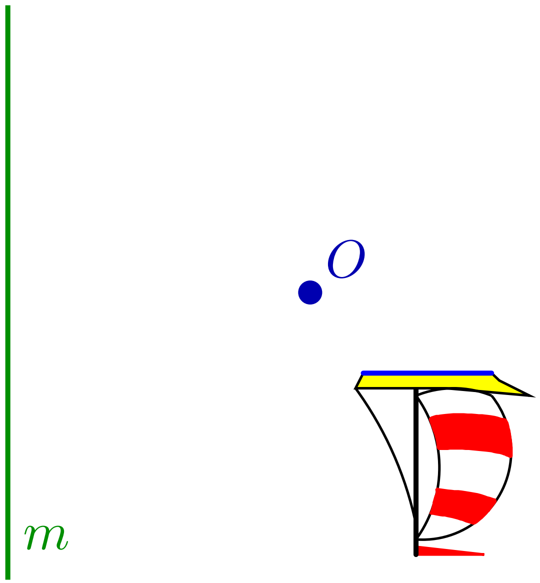

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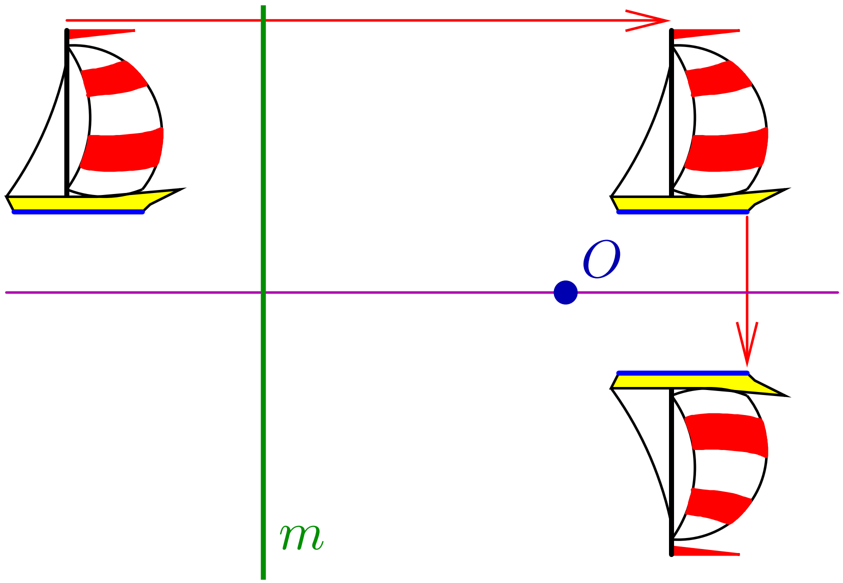

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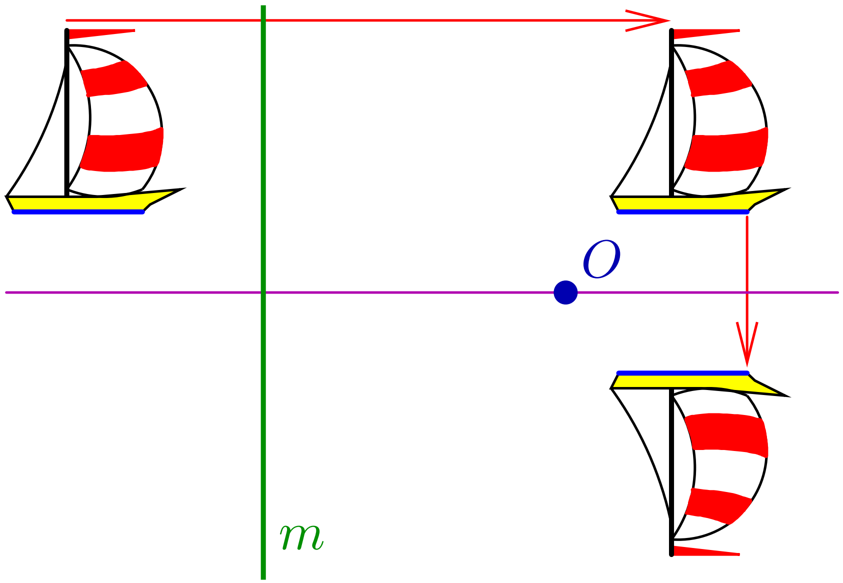

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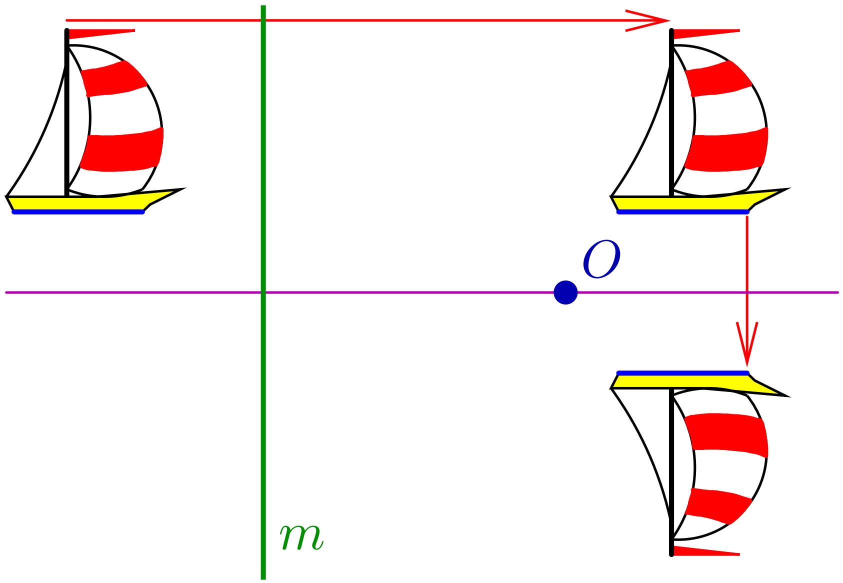

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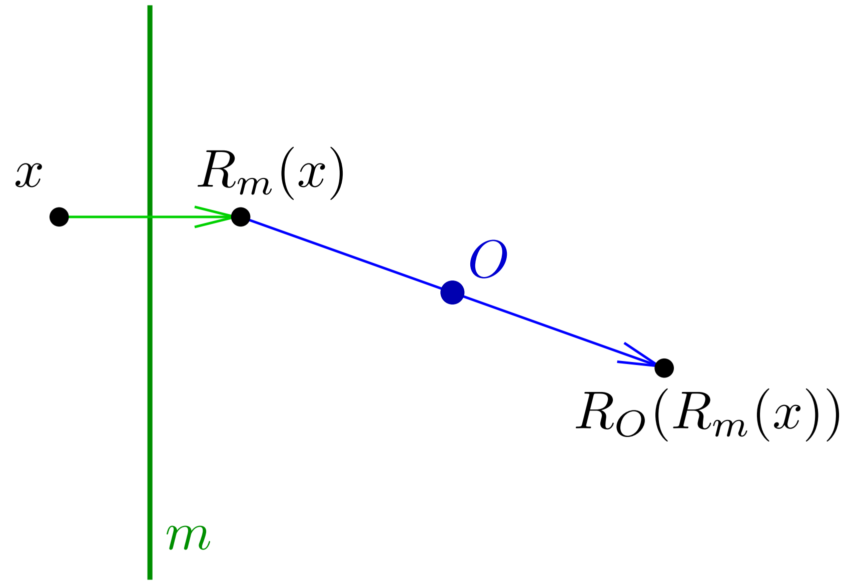

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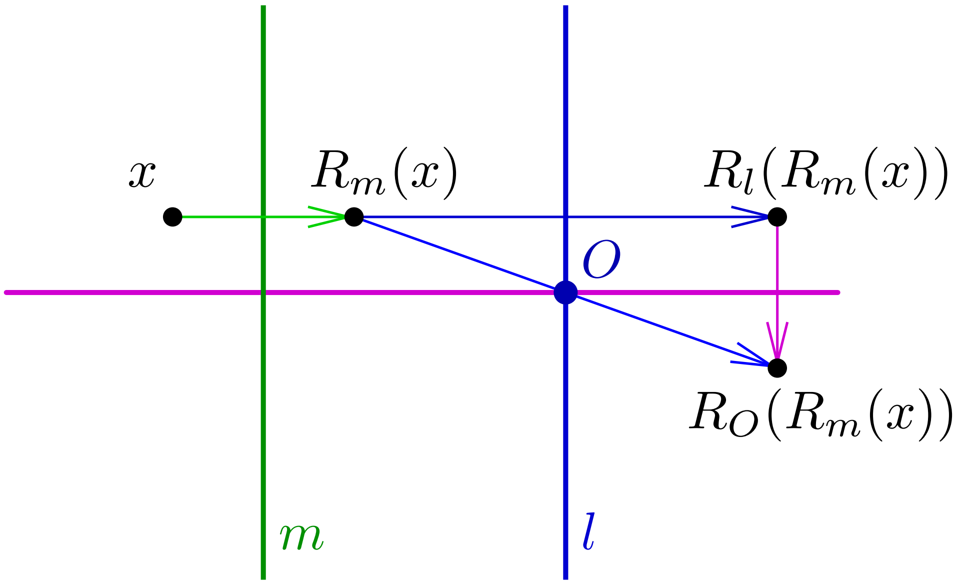

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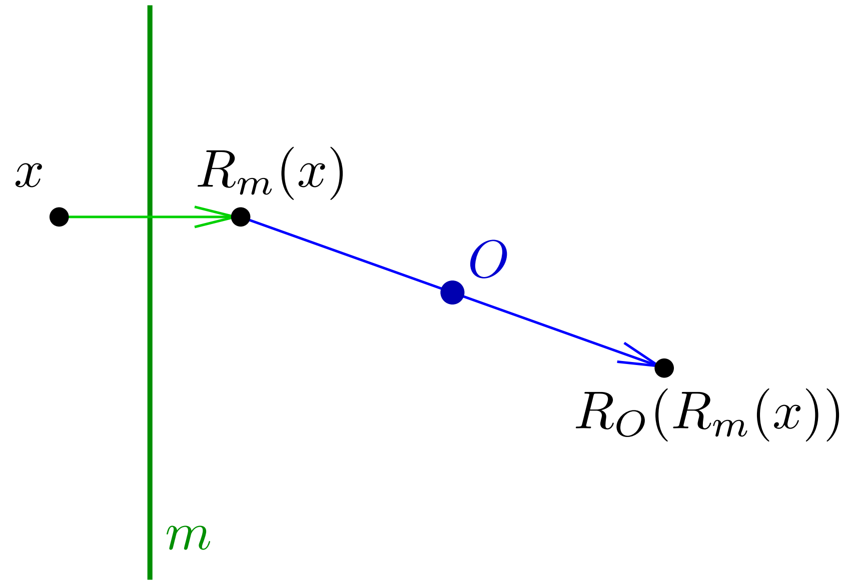

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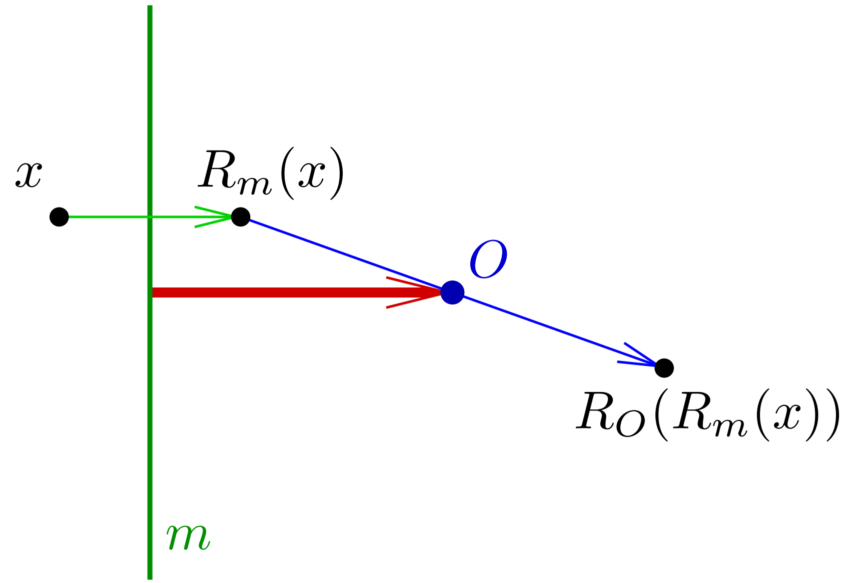

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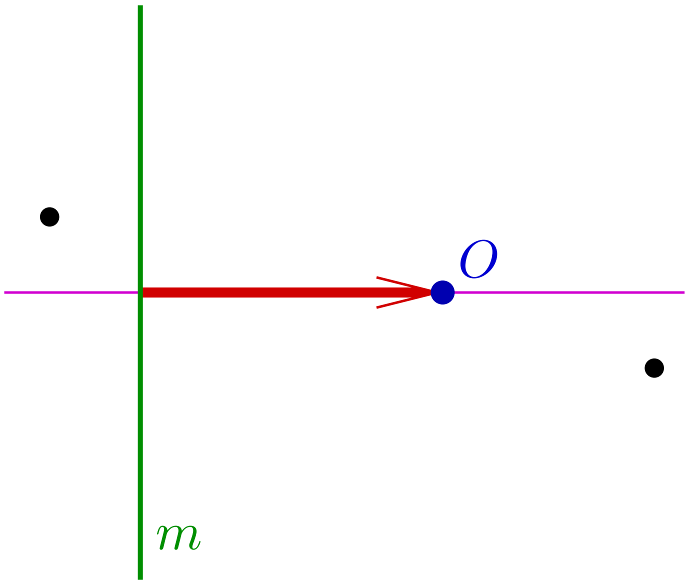

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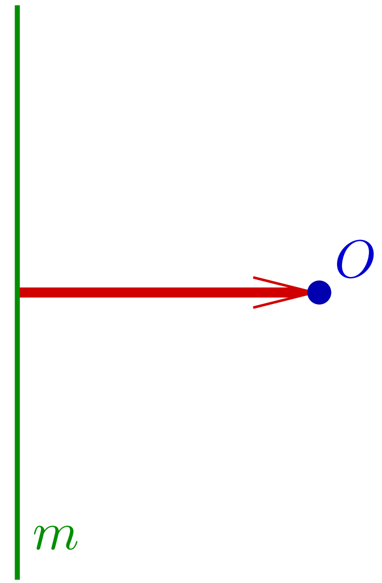

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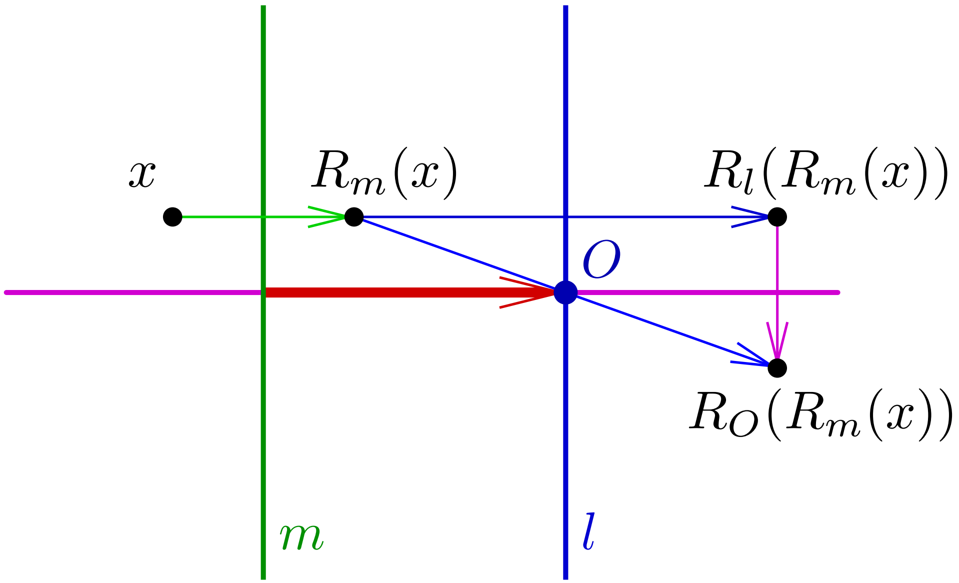

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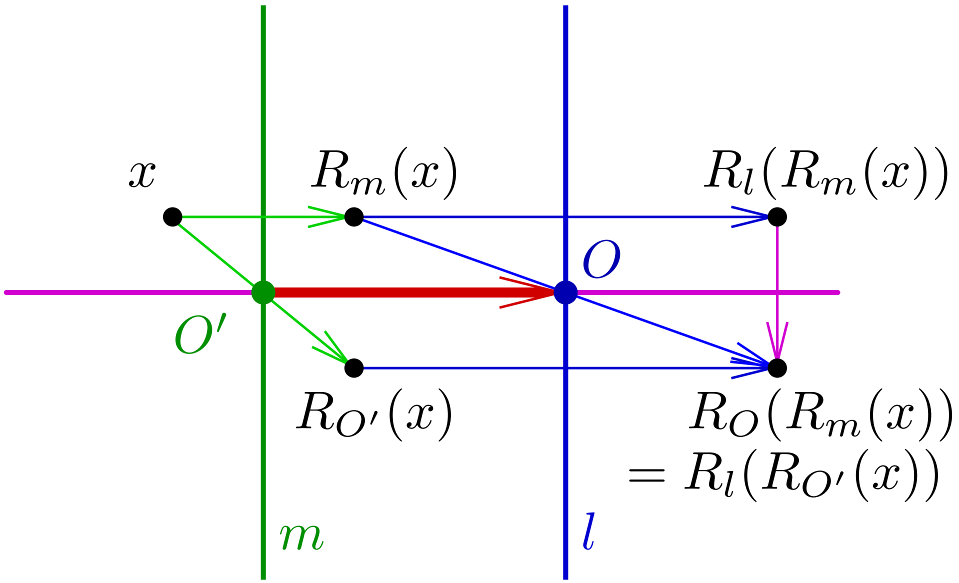

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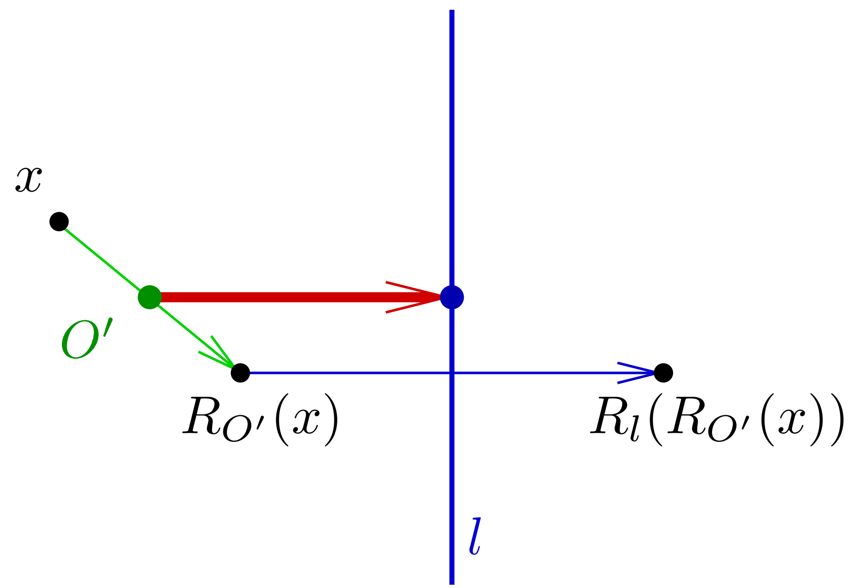

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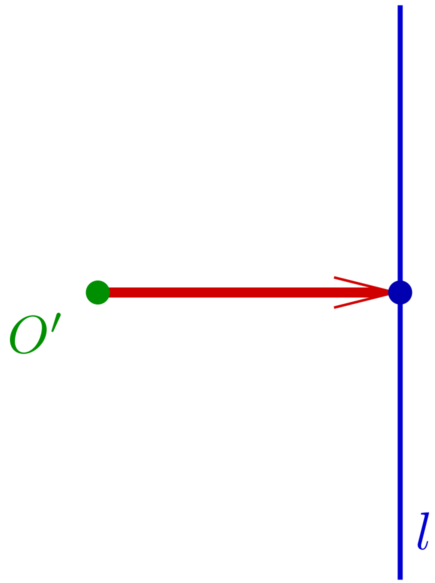

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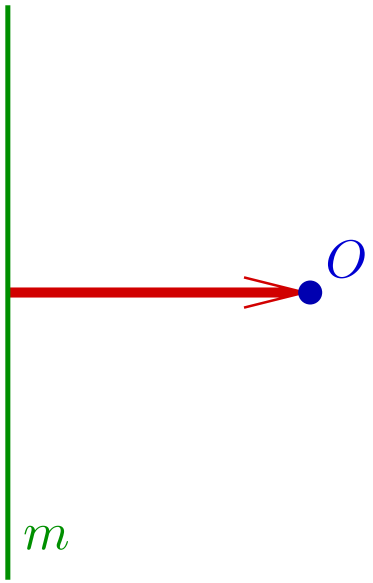

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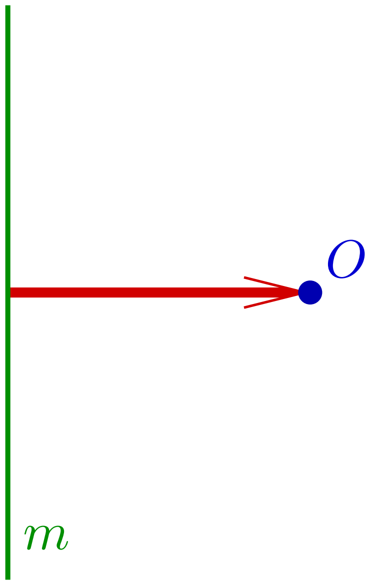

A biflipper for a glide reflection may glide along itself.

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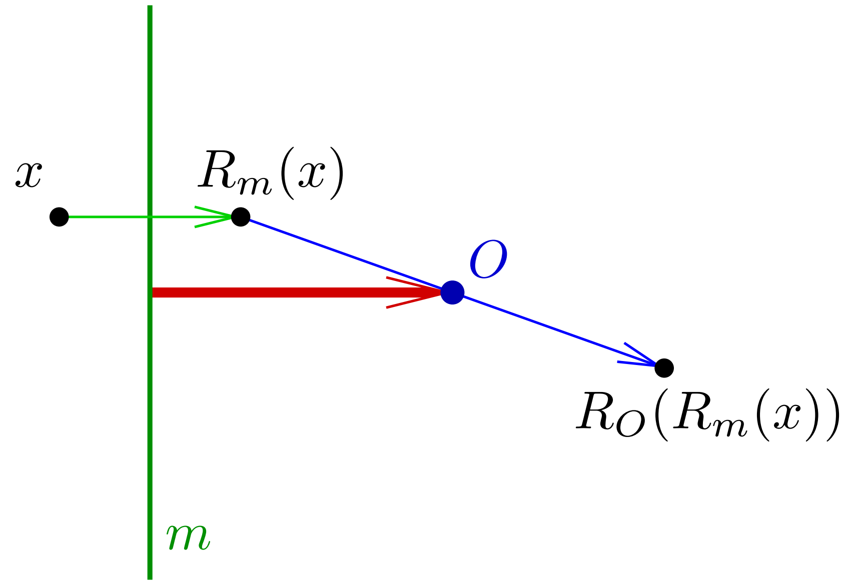

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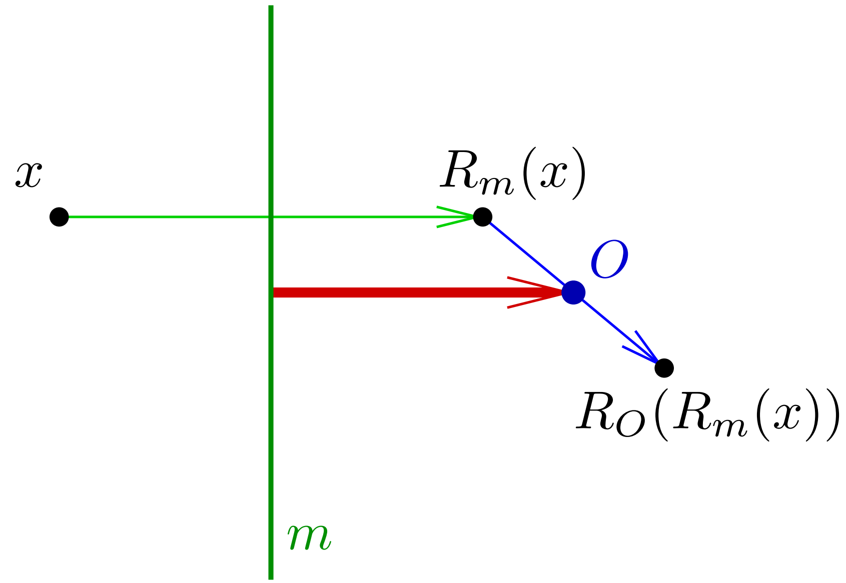
Biflippers for a glide reflection



A biflipper for a glide reflection may glide along itself.

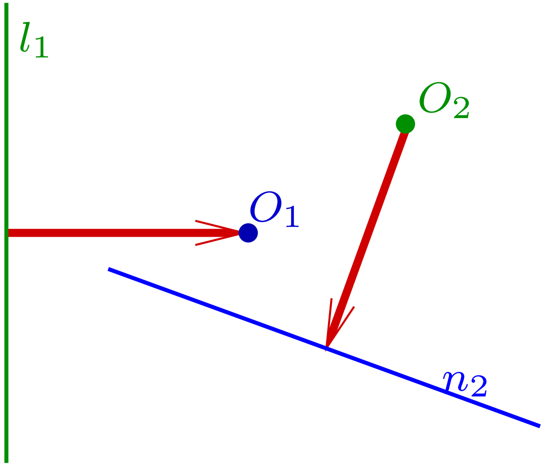
Head to tail for glide reflections

Head to tail for glide reflections

Given two glide reflections, present them by biflippers.

Head to tail for glide reflections

Given two glide reflections, present them by biflippers.
The head in the first biflipper and tail in the second one should **NOT** be lines.



Head to tail for glide reflections

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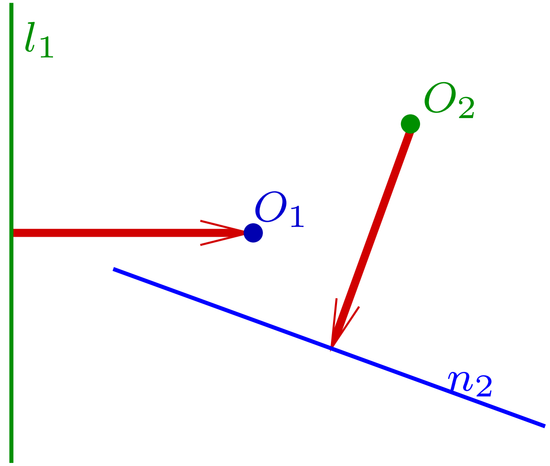
so that the biflippers are $\overrightarrow{lO}$ and $\overrightarrow{On}$.



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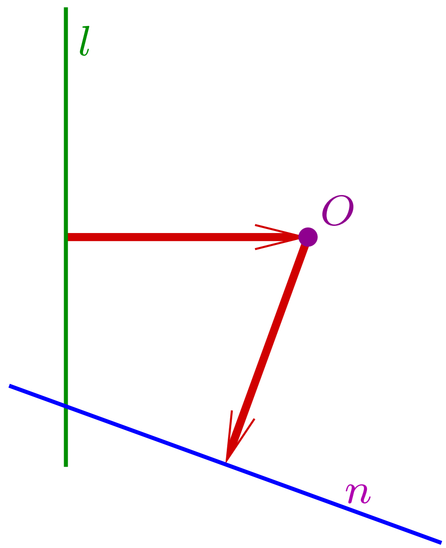

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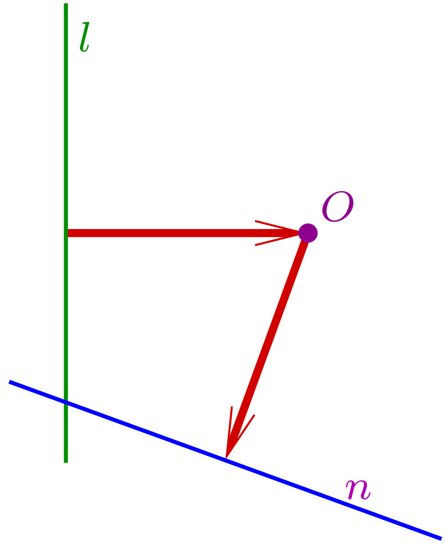
Draw an oriented arc from l to n



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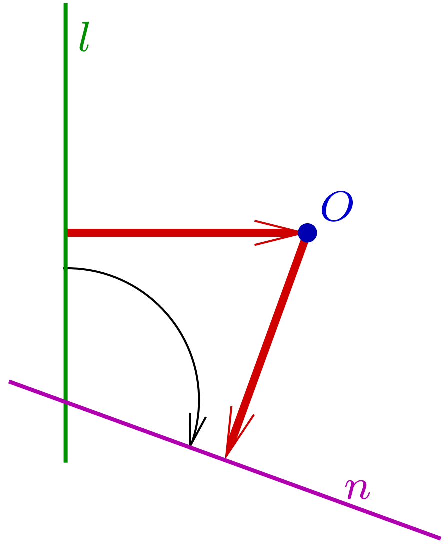

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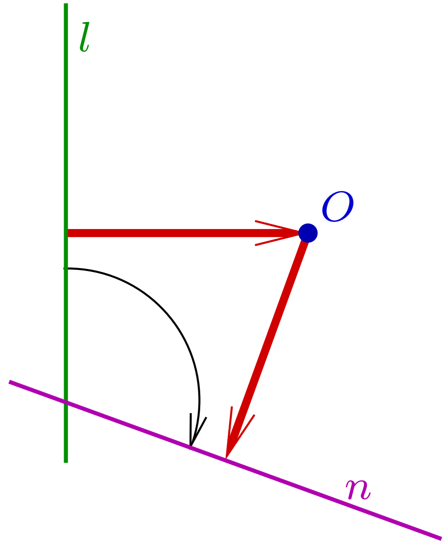
Draw an oriented arc from l to n and erase O.



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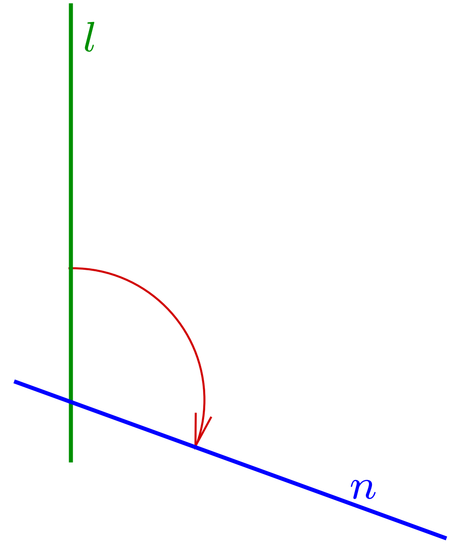

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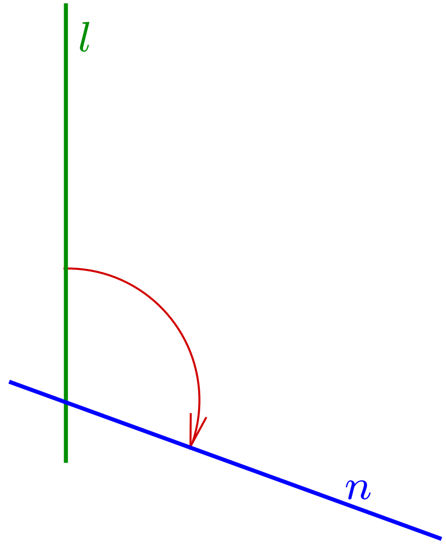

This is a rotation!

Head to tail for glide reflections

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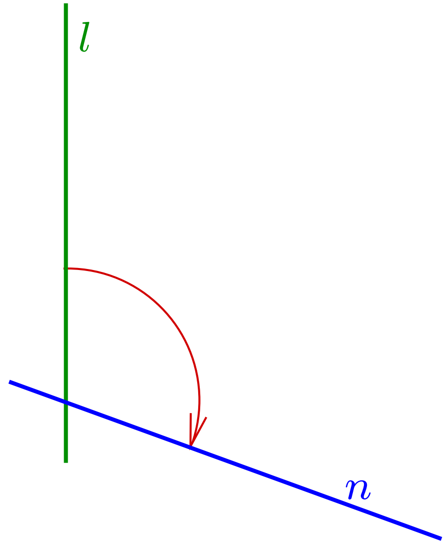
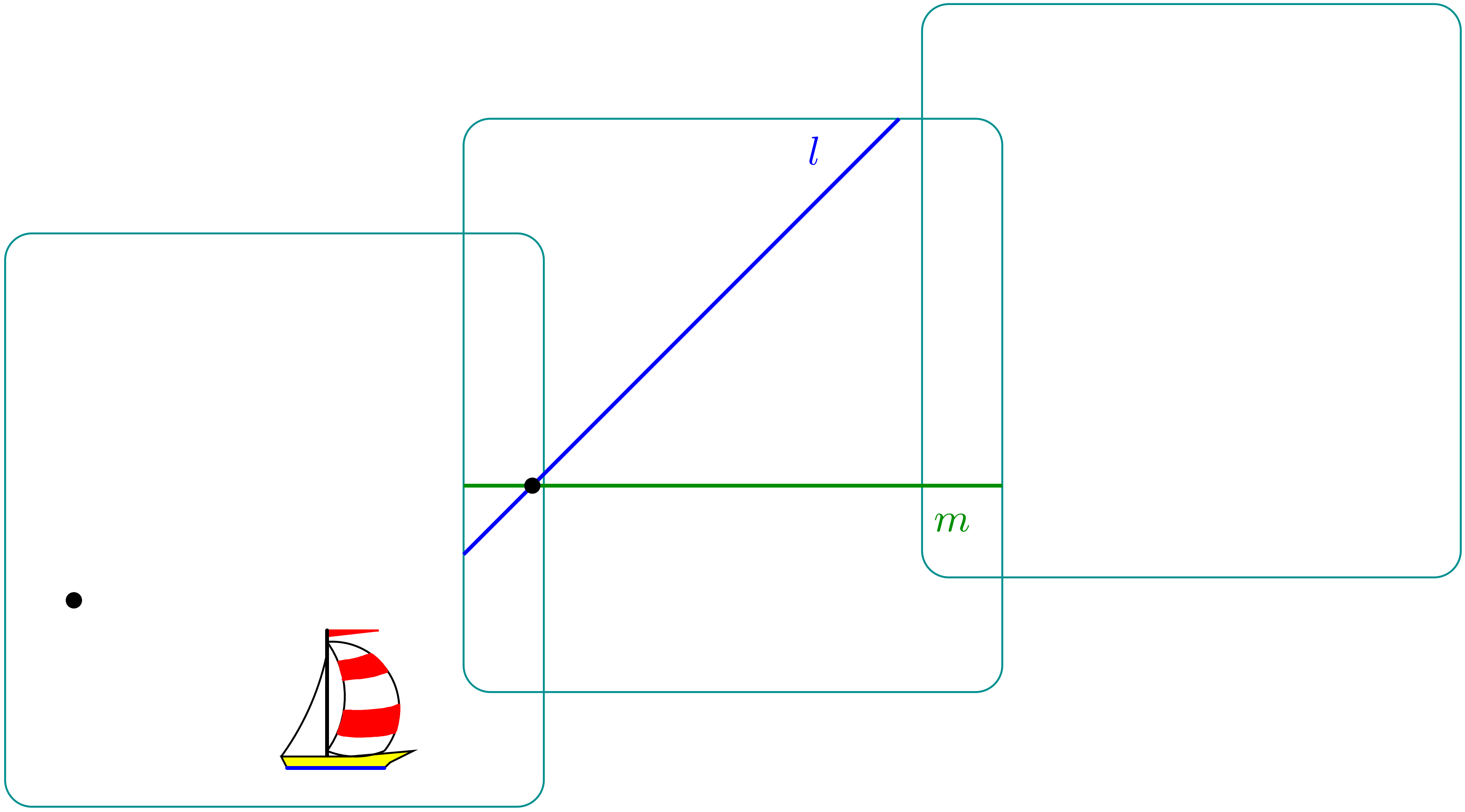


Exercise. *Find head to tail rules for rotation $\circ$ glide reflection.*

In the 3-space. Rotation



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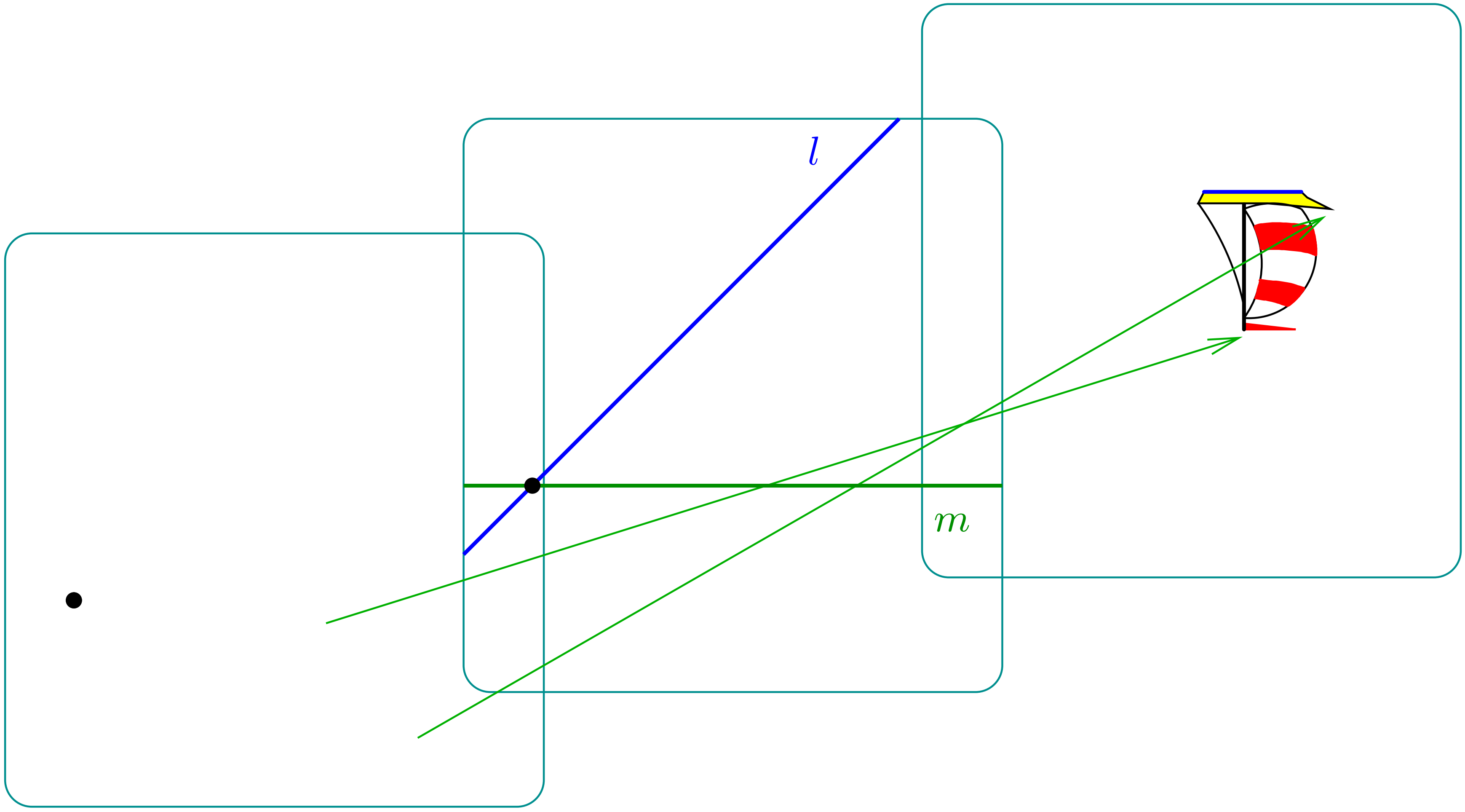
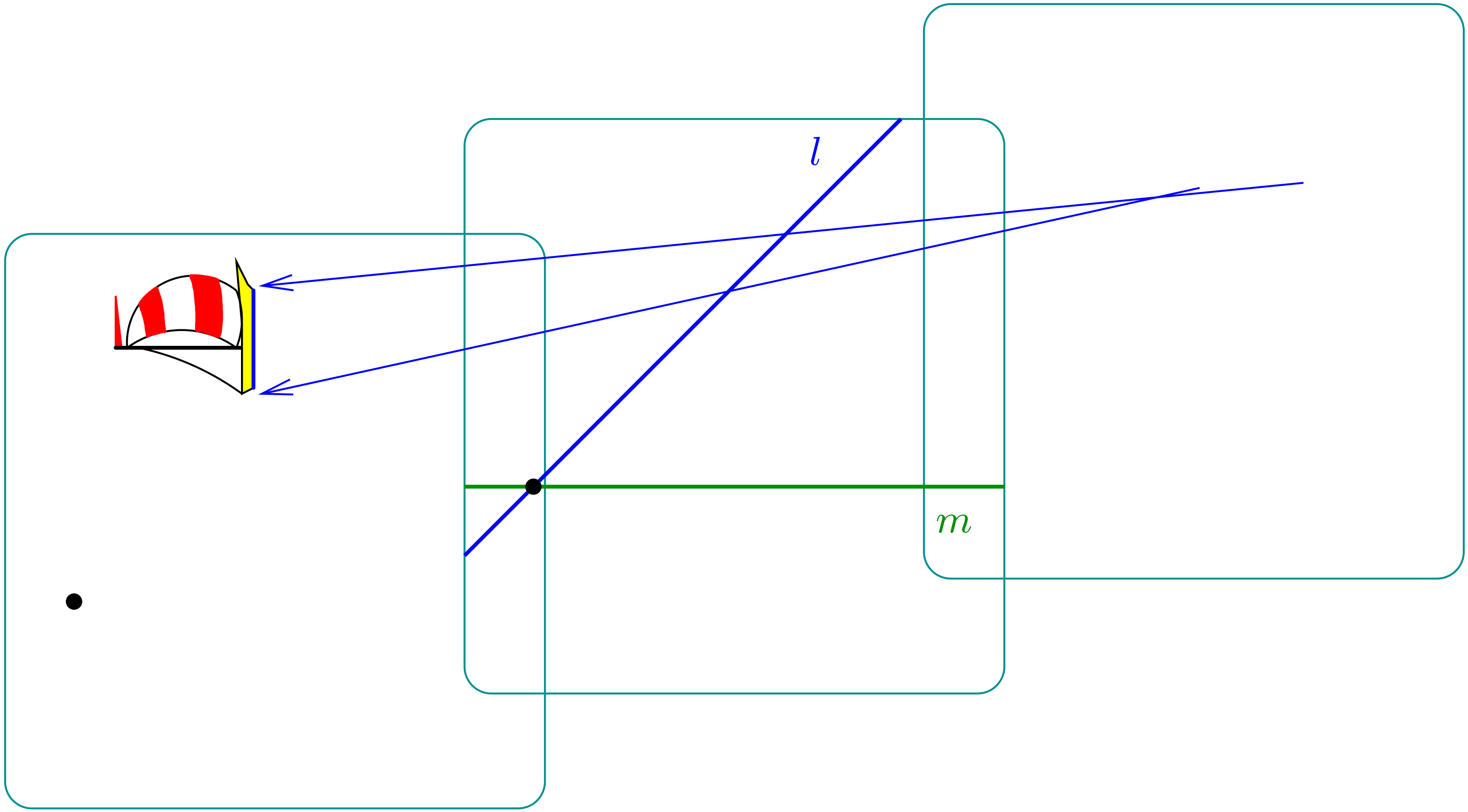
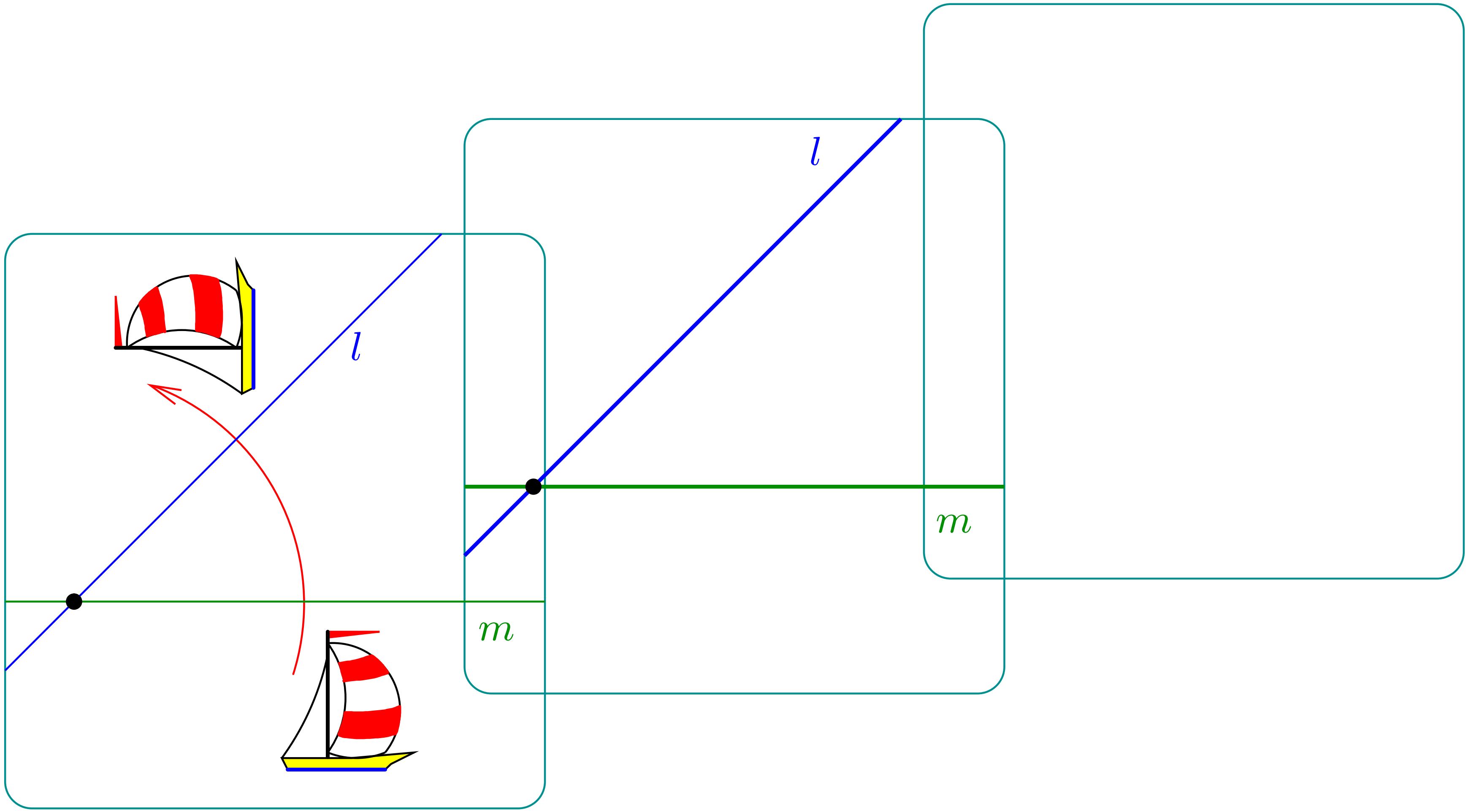


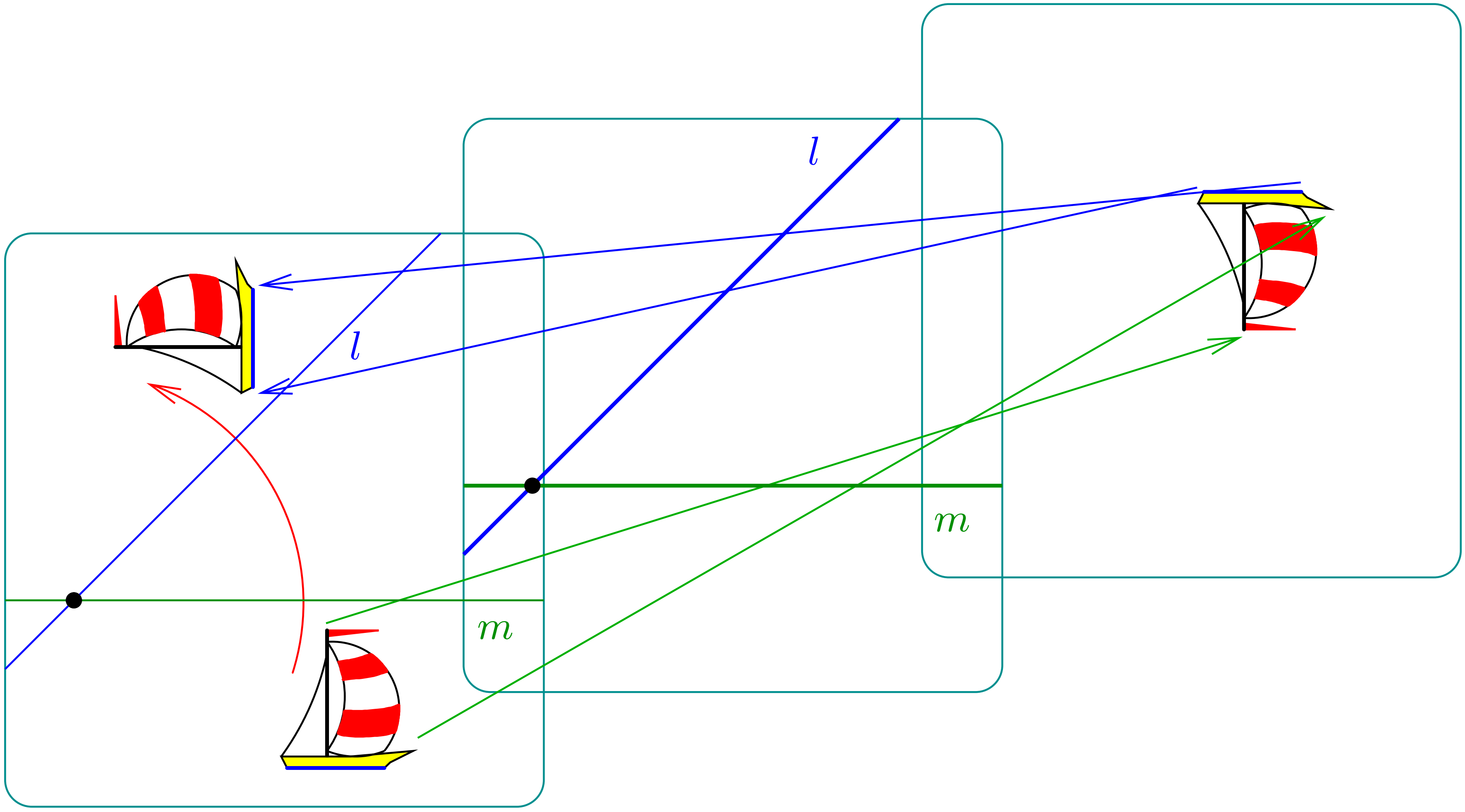
In the 3-space. Rotation



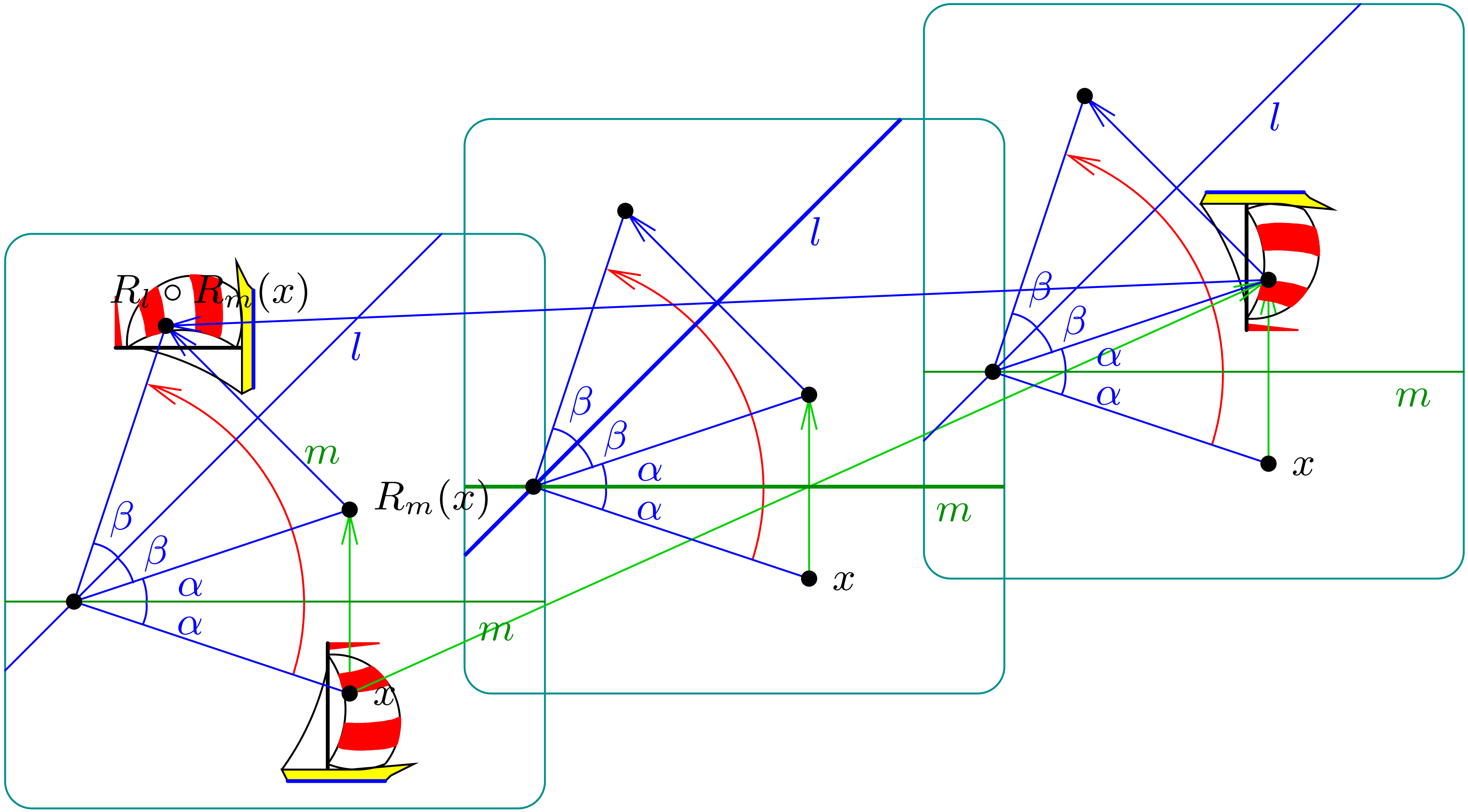
In the 3-space. Rotation



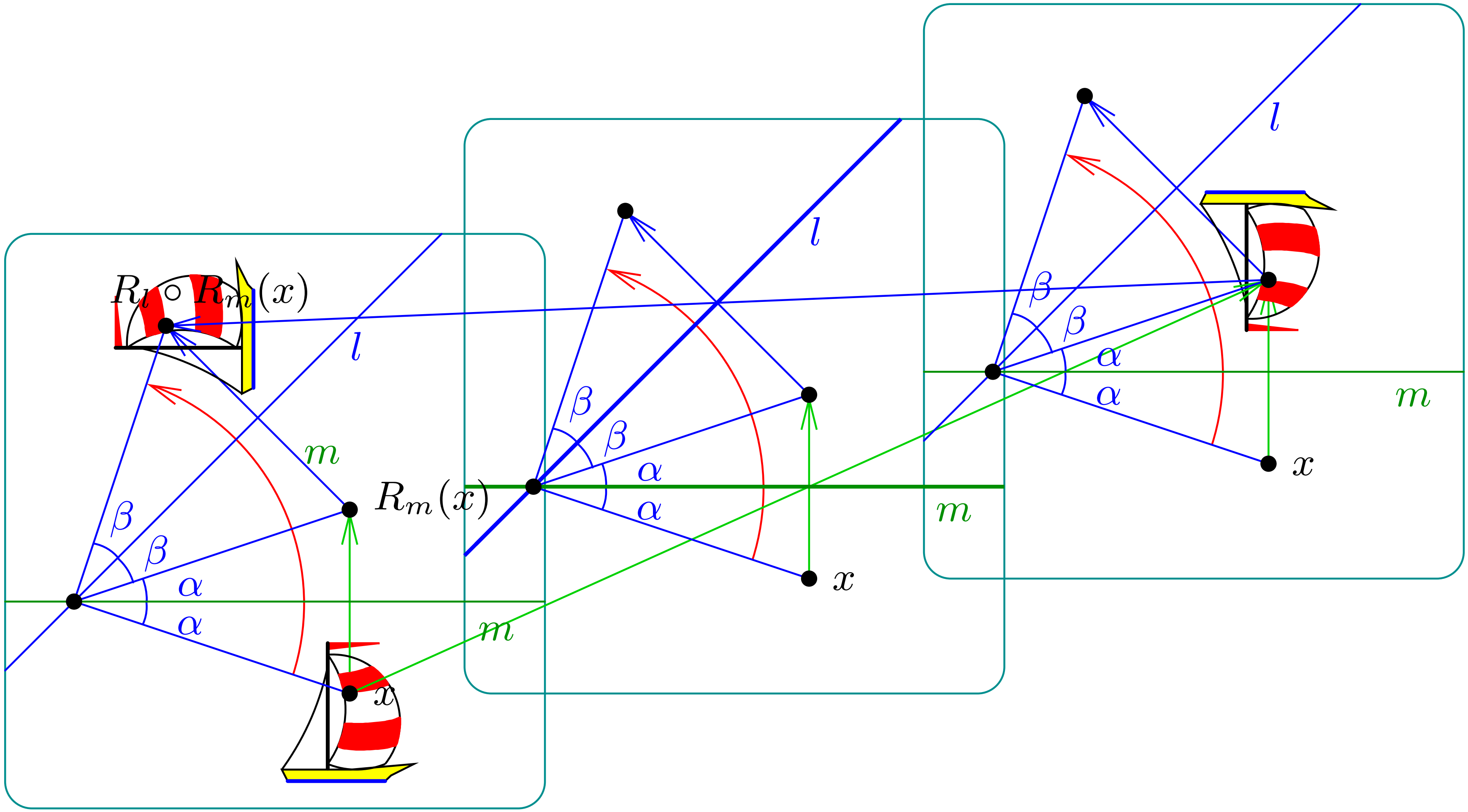
In the 3-space. Rotation



In the 3-space. Rotation

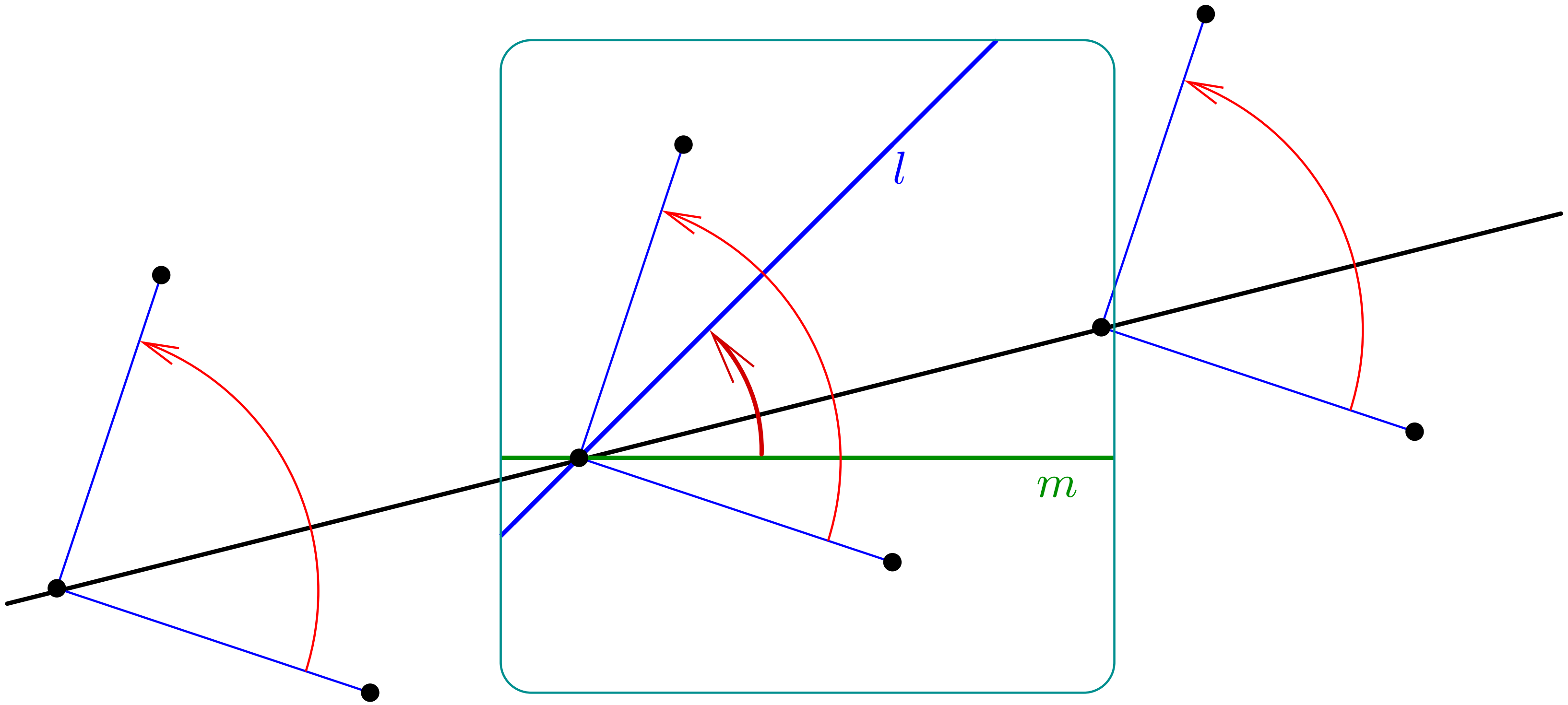


In the 3-space. Rotation



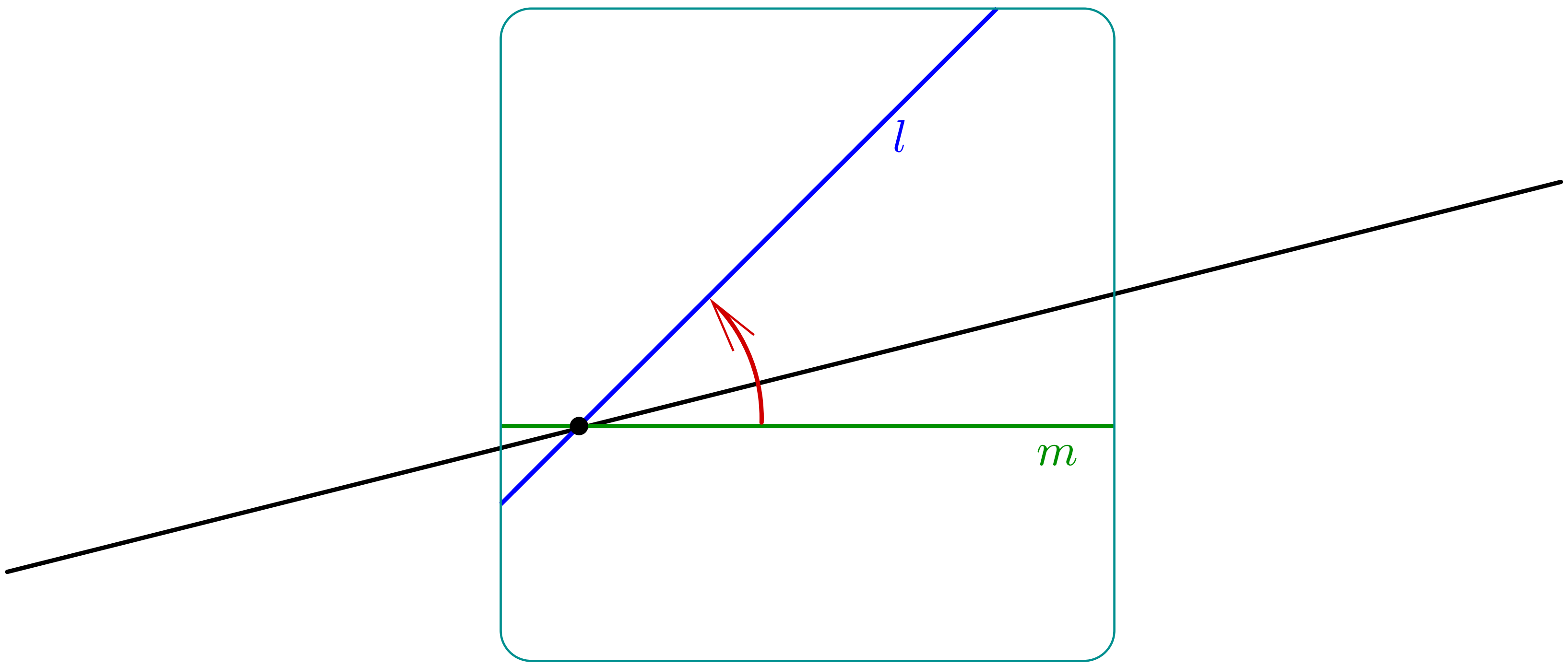
Everything like on the plane.

In the 3-space. Rotation



A biflipper formed by two intersecting lines defines a rotation of the 3-space about the axis $\perp$ to the plane of the lines.

In the 3-space. Rotation



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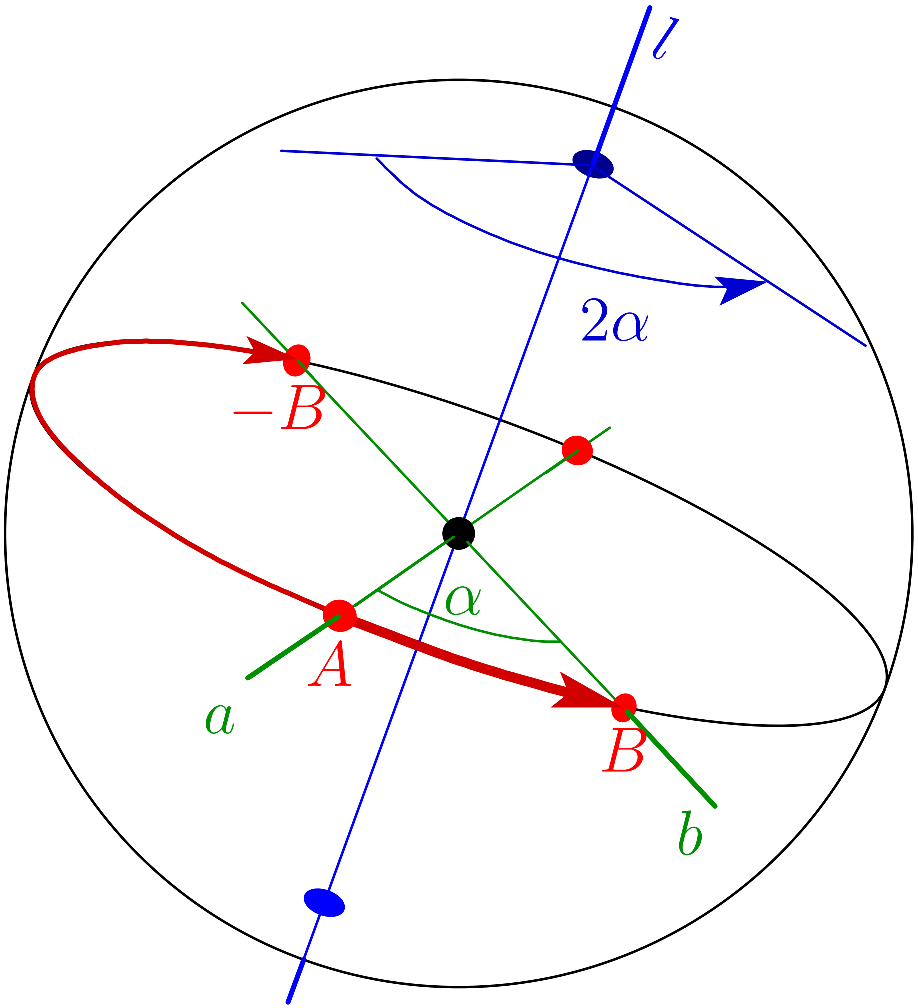
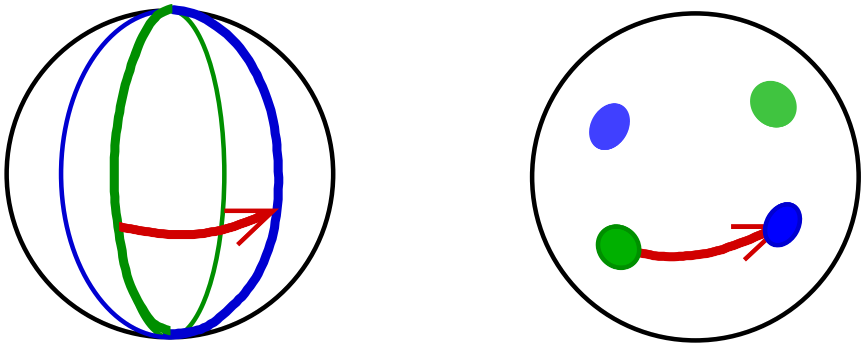
Rotations of 2-sphere



Rotations of 2-sphere

Biflippers:



Rotations of 2-sphere

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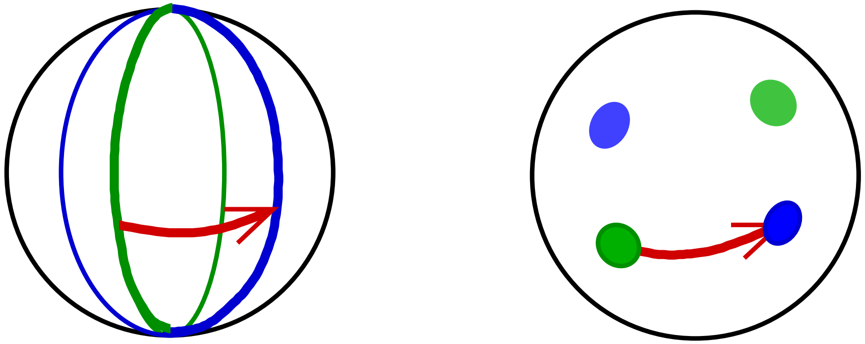
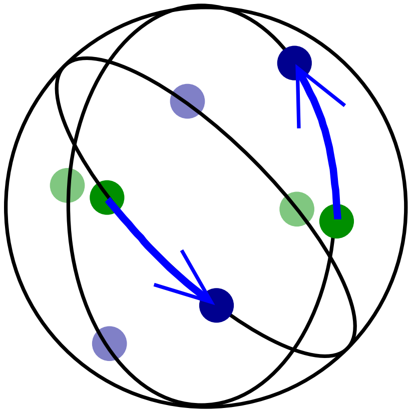


Head to tail for rotations:



Rotations of 2-sphere

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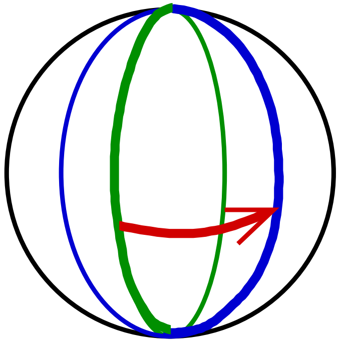

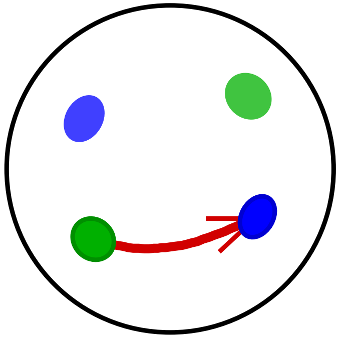

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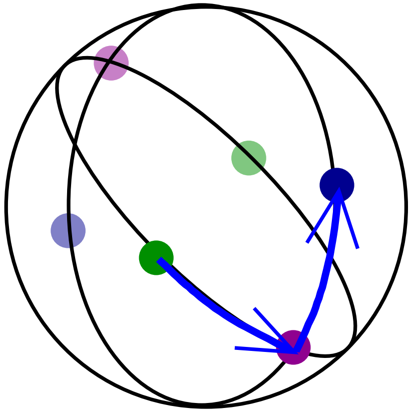

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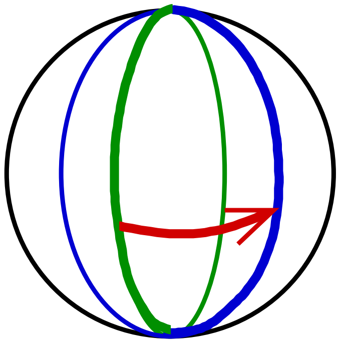

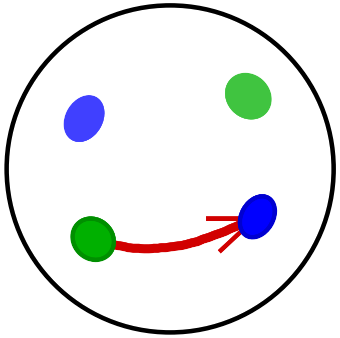

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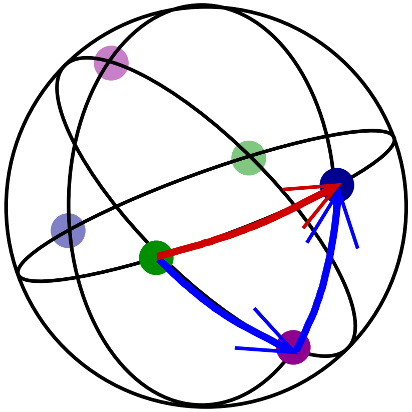

Rotations of 2-sphere

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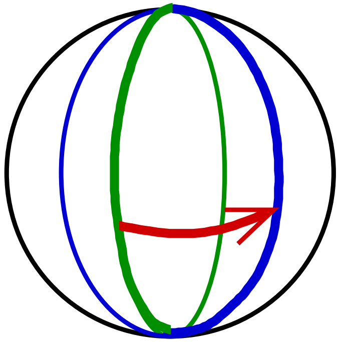

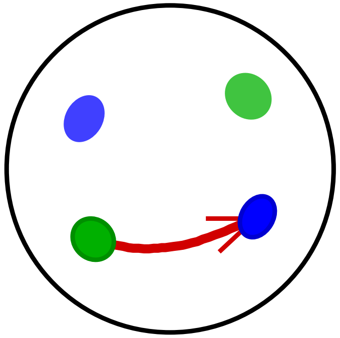

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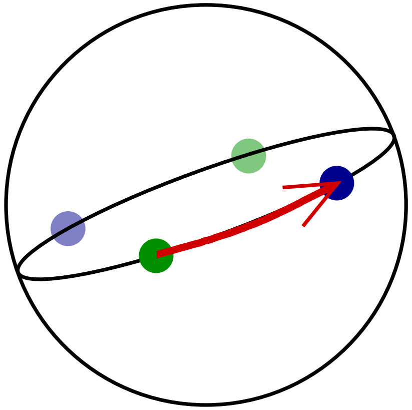

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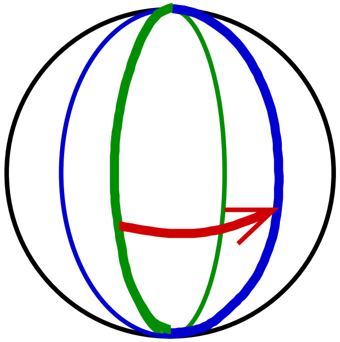

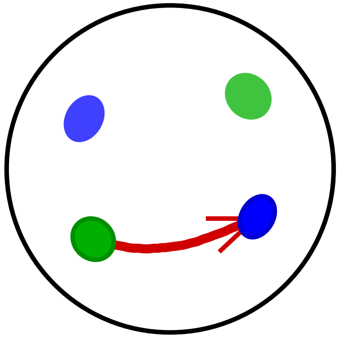

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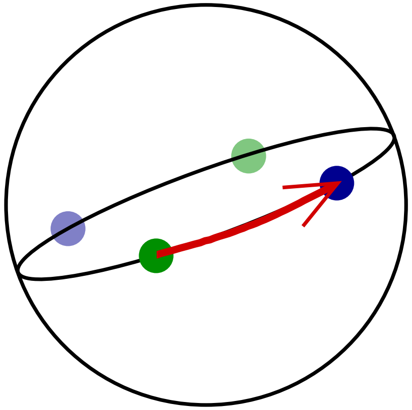

Biflipper vs. angular displacement vector vs. unit quaternion.

Rotations of 2-sphere

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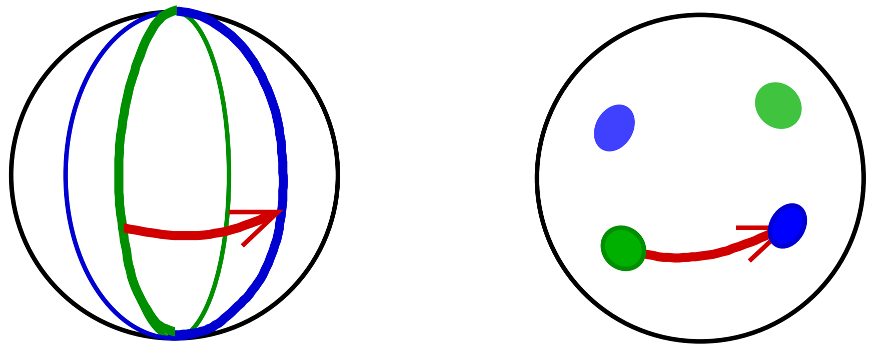

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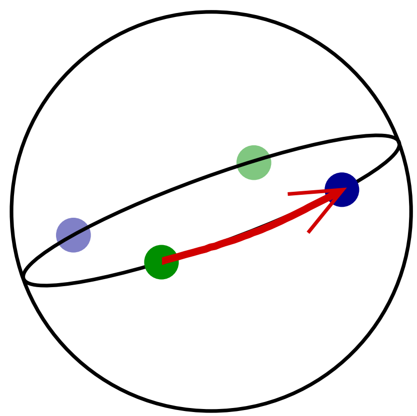
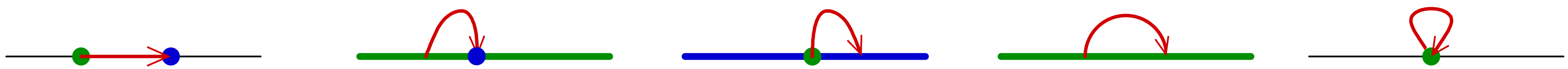


Biflipper vs. angular displacement vector vs. unit quaternion.

The rotation encoded by bilipper $\overrightarrow{wv}$ is defined by quaternion $vw = v \times w - v \cdot w$.

Parade of biflippers

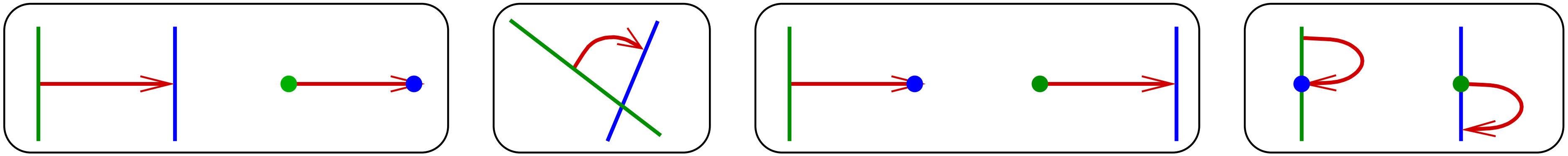
On line:



translation *reflections in points* *the identity*

Parade of biflippers

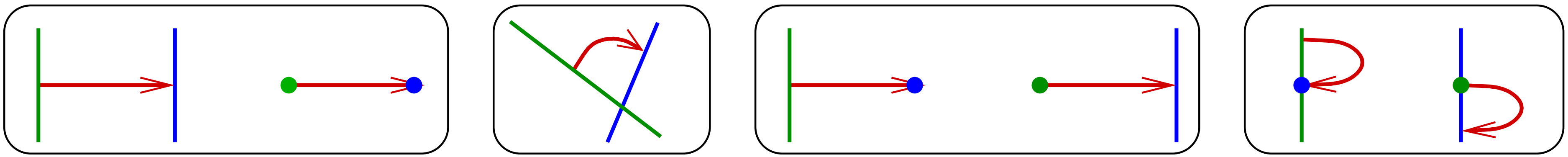
On plane:



translations *rotation* *glide reflections* *reflections*

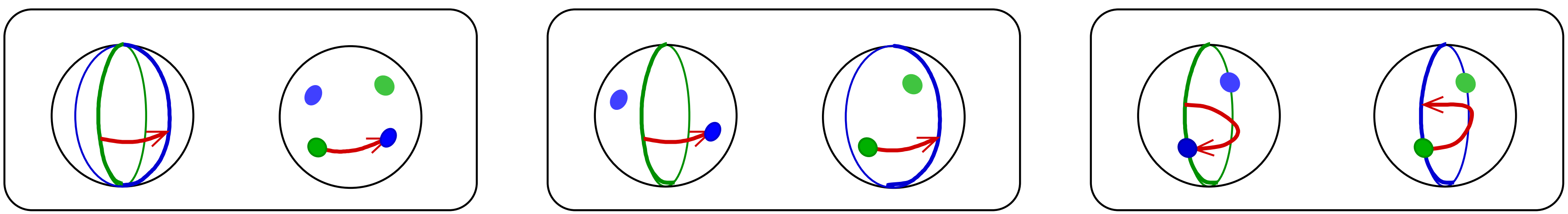
Parade of biflippers

On plane:



translations *rotation* *glide reflections* *reflections*

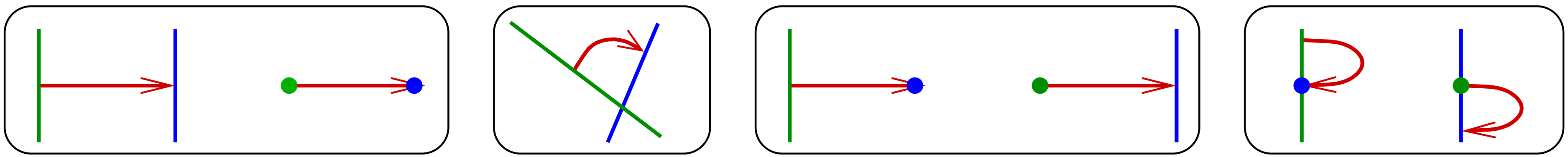
On sphere:



rotations *rotary reflections* *reflections*

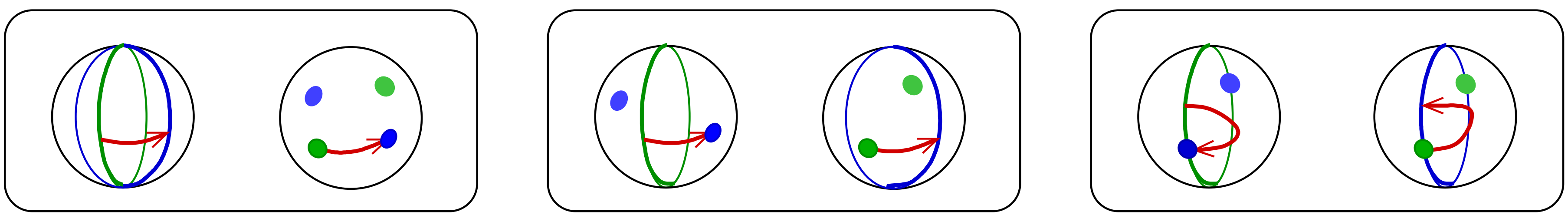
Parade of biflippers

On plane:



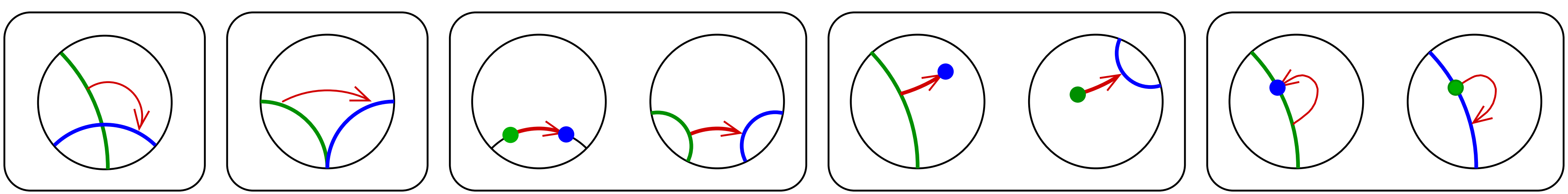
translations *rotation* *glide reflections* *reflections*

On sphere:



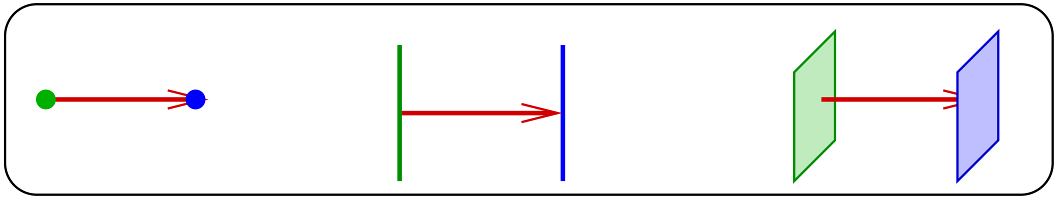
rotations *rotary reflections* *reflections*

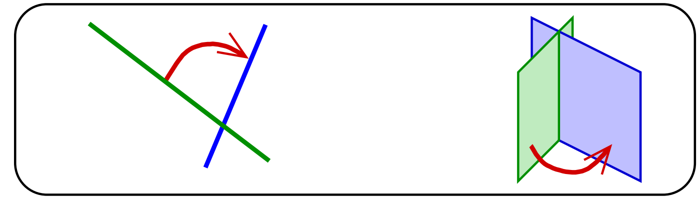
On the hyperbolic plane:



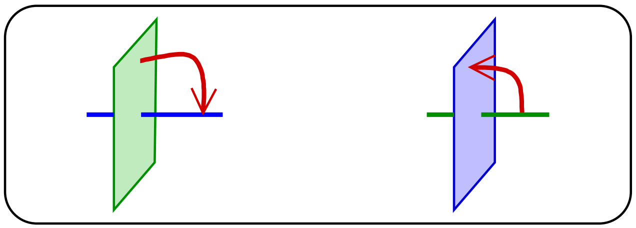
rotation *parallel motion* *translation* *glide reflections* *reflections*

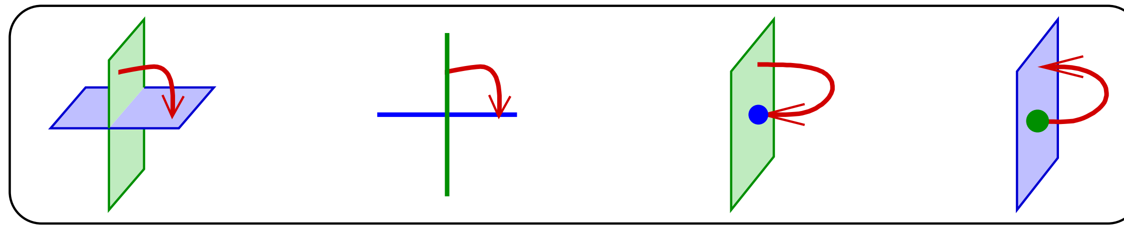
Biflippers in the 3-space



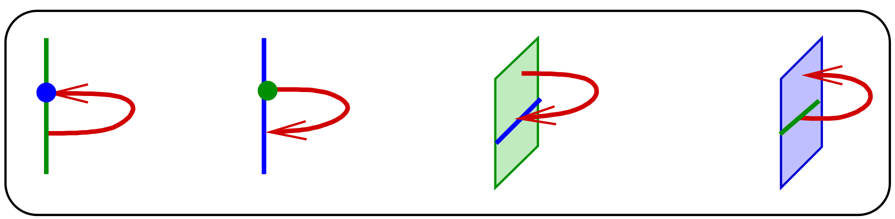


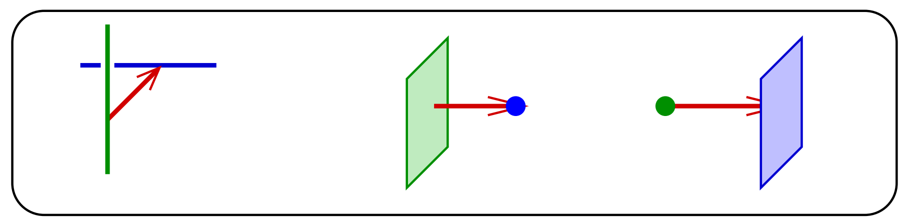
translations *rotations*



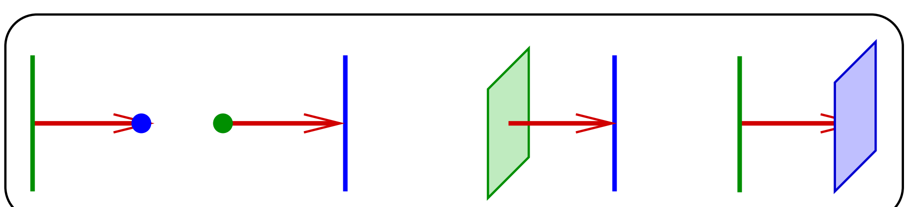


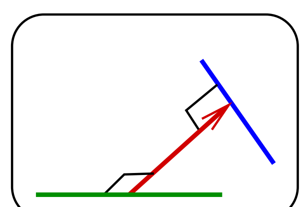
central symmetries *symmetries about a line (half−turns)*

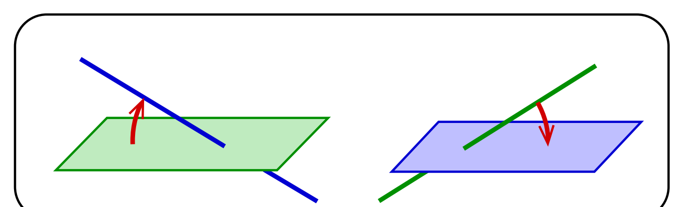




reflections *glide symmetries about a line*

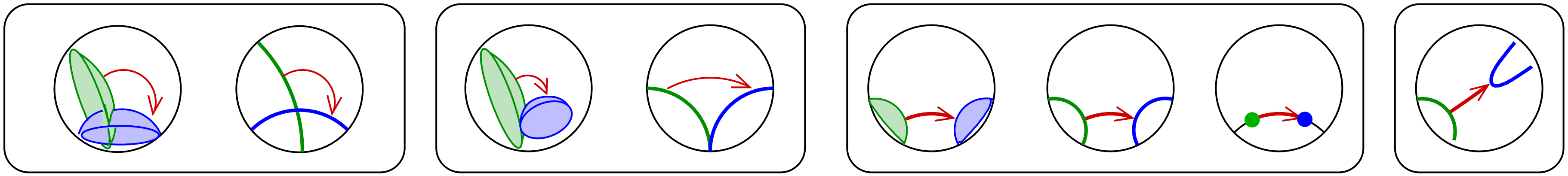




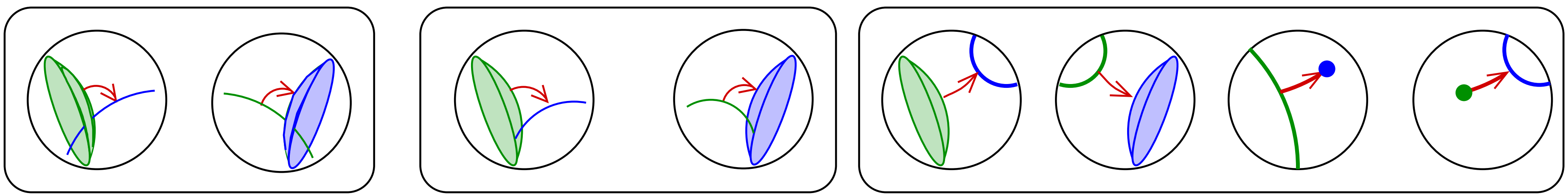


glide reflections *screw motion* *rotary reflections*

In hyperbolic 3-space

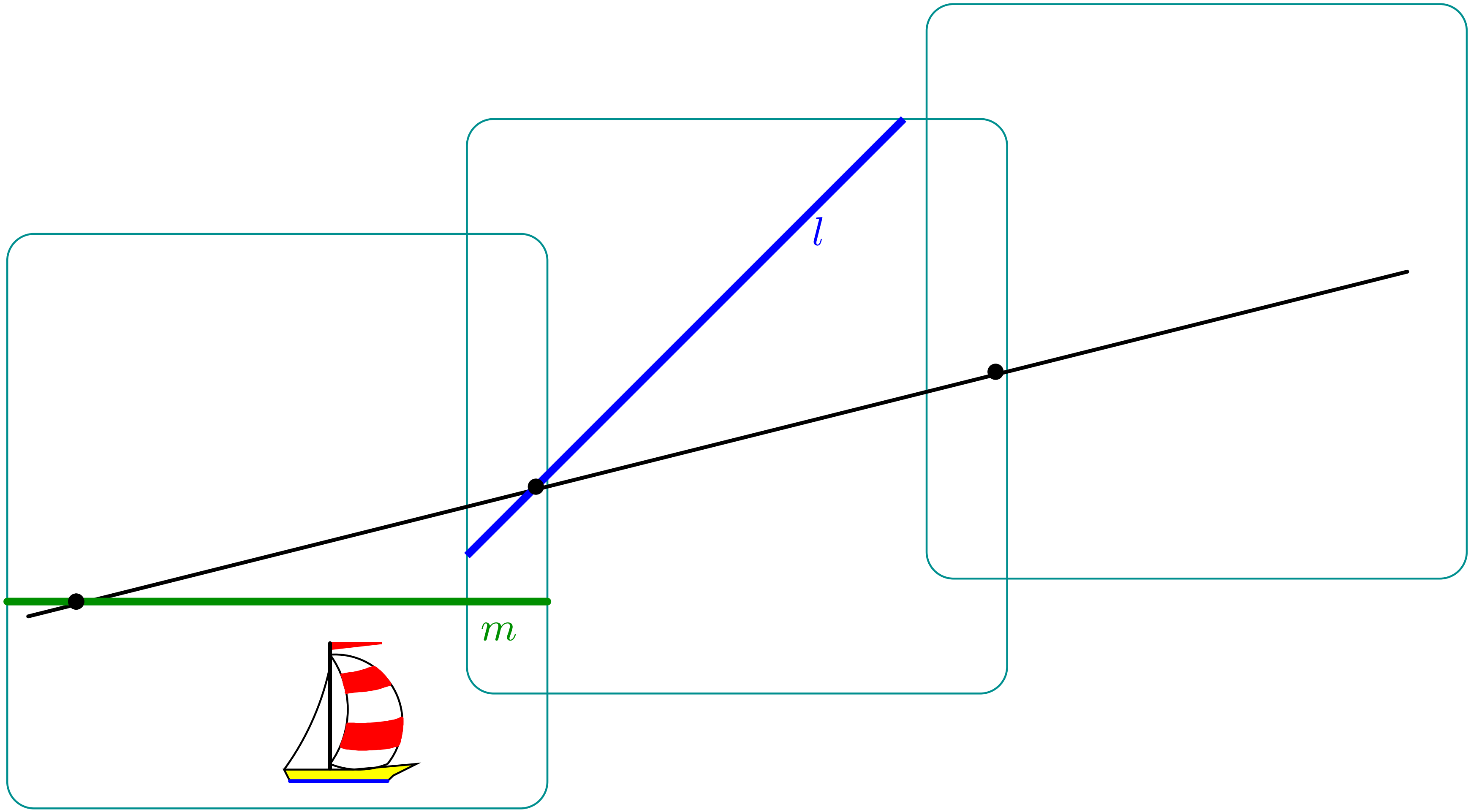


rotation *parallel motion* *translation* *screw motion*

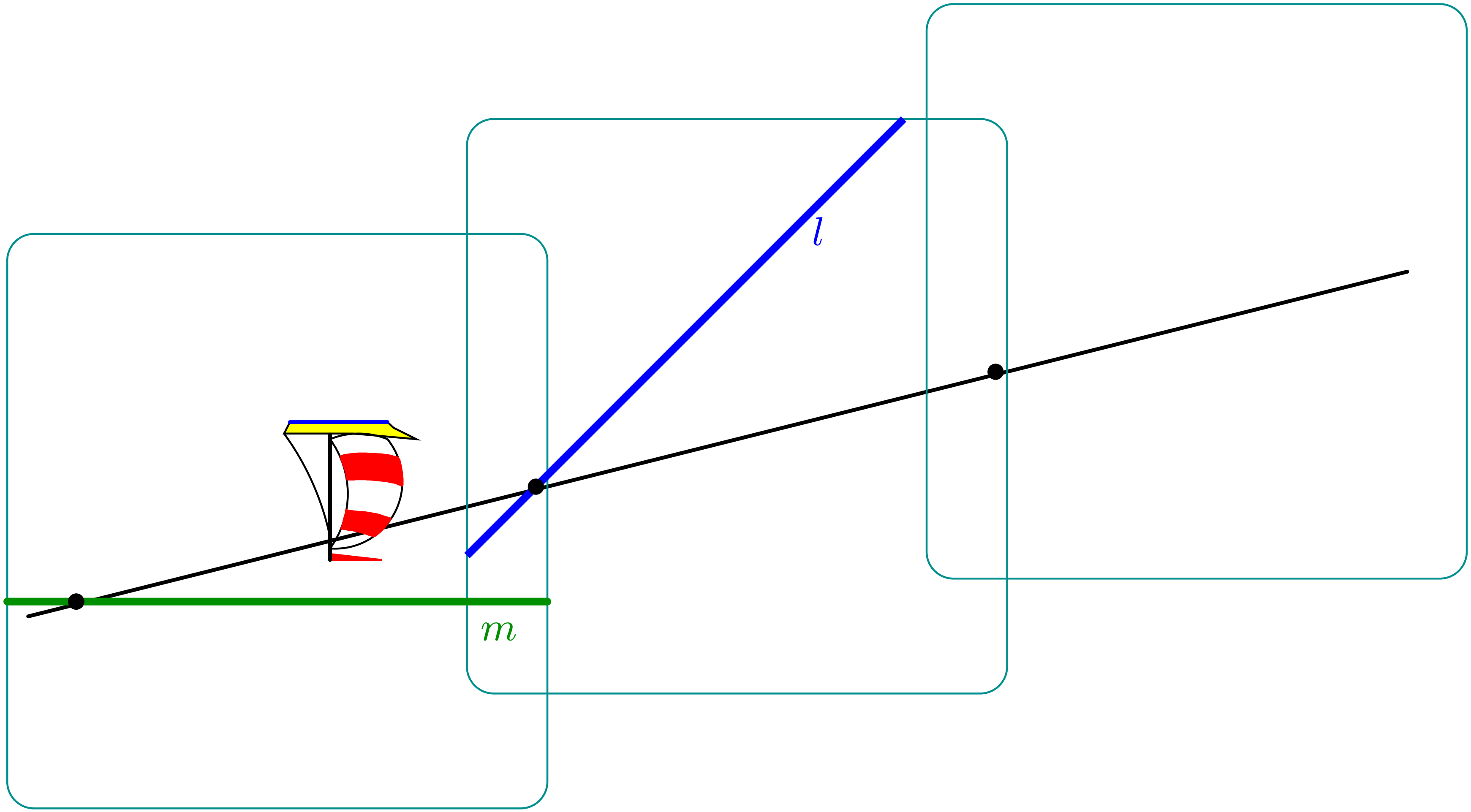


rotary reflections *parallel reflections* *glide reflections*

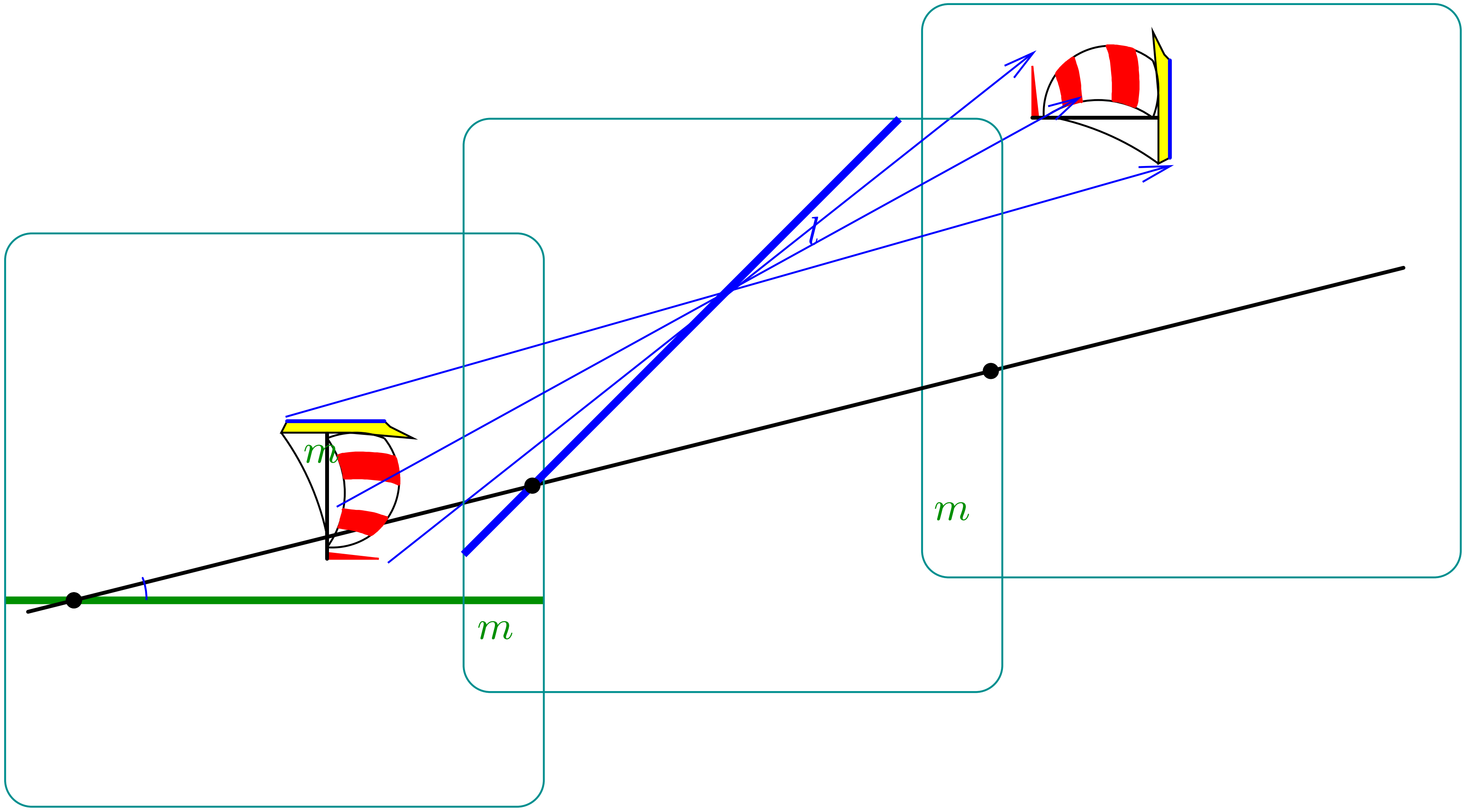
Screw displacement



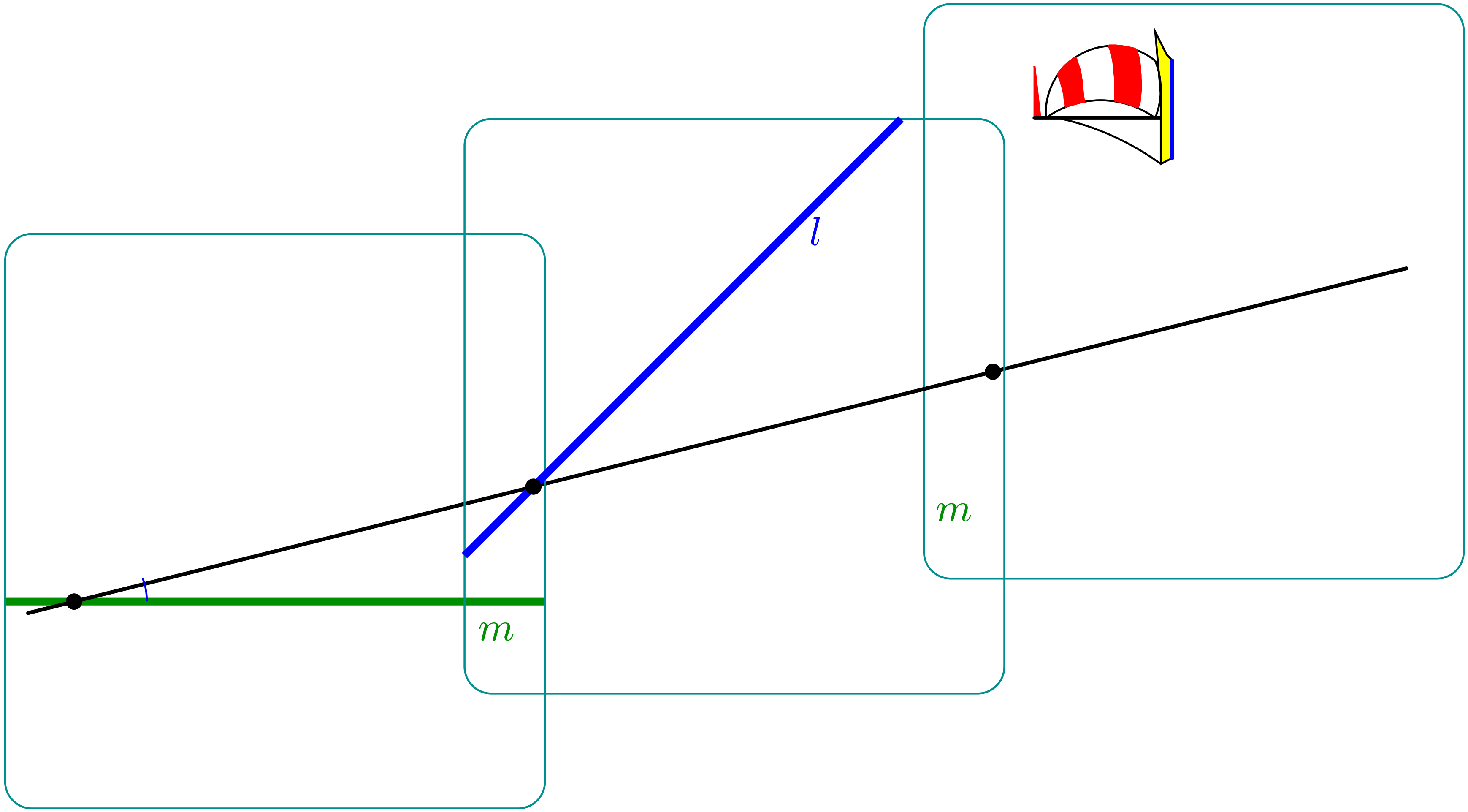
Screw displacement



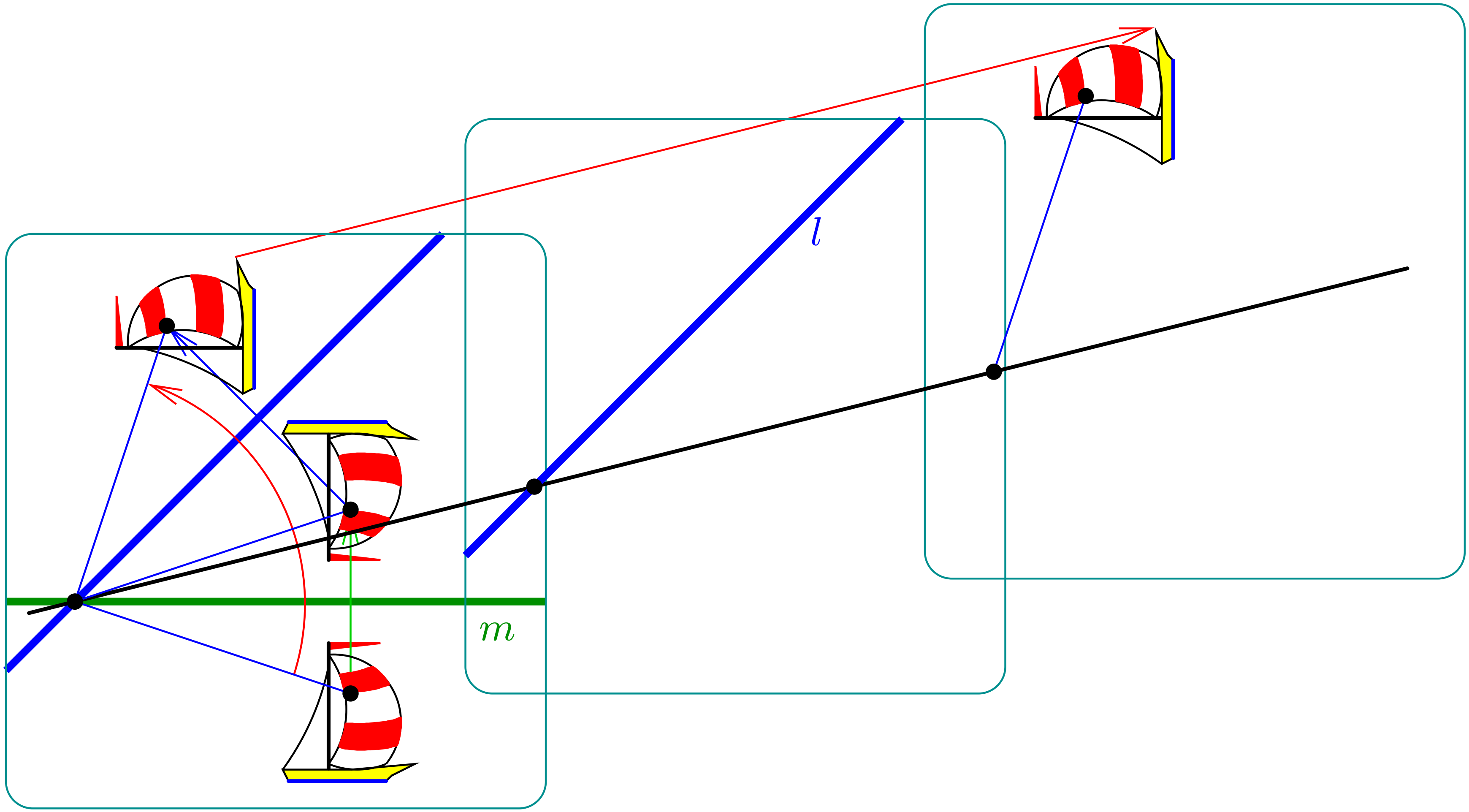
Screw displacement



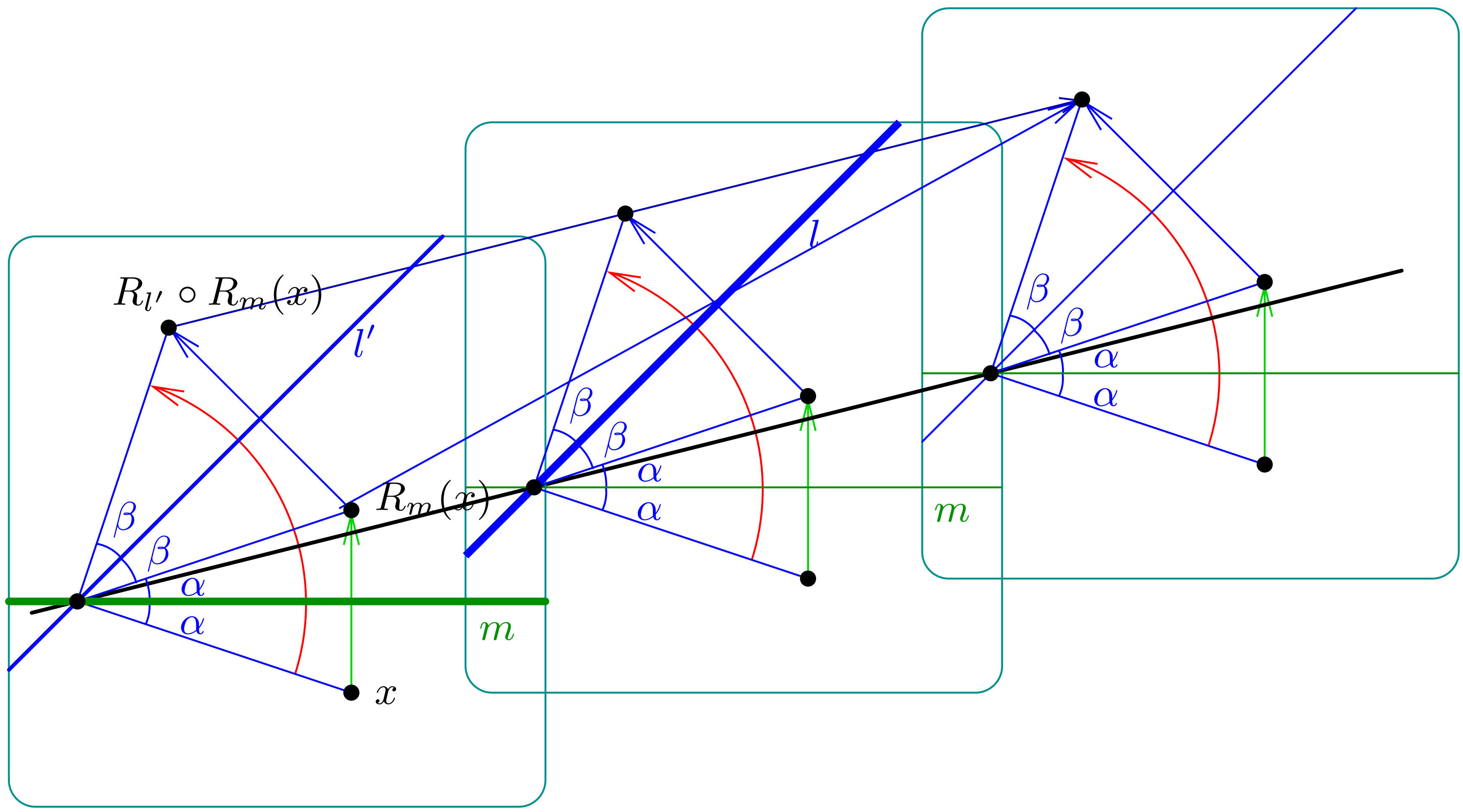
Screw displacement



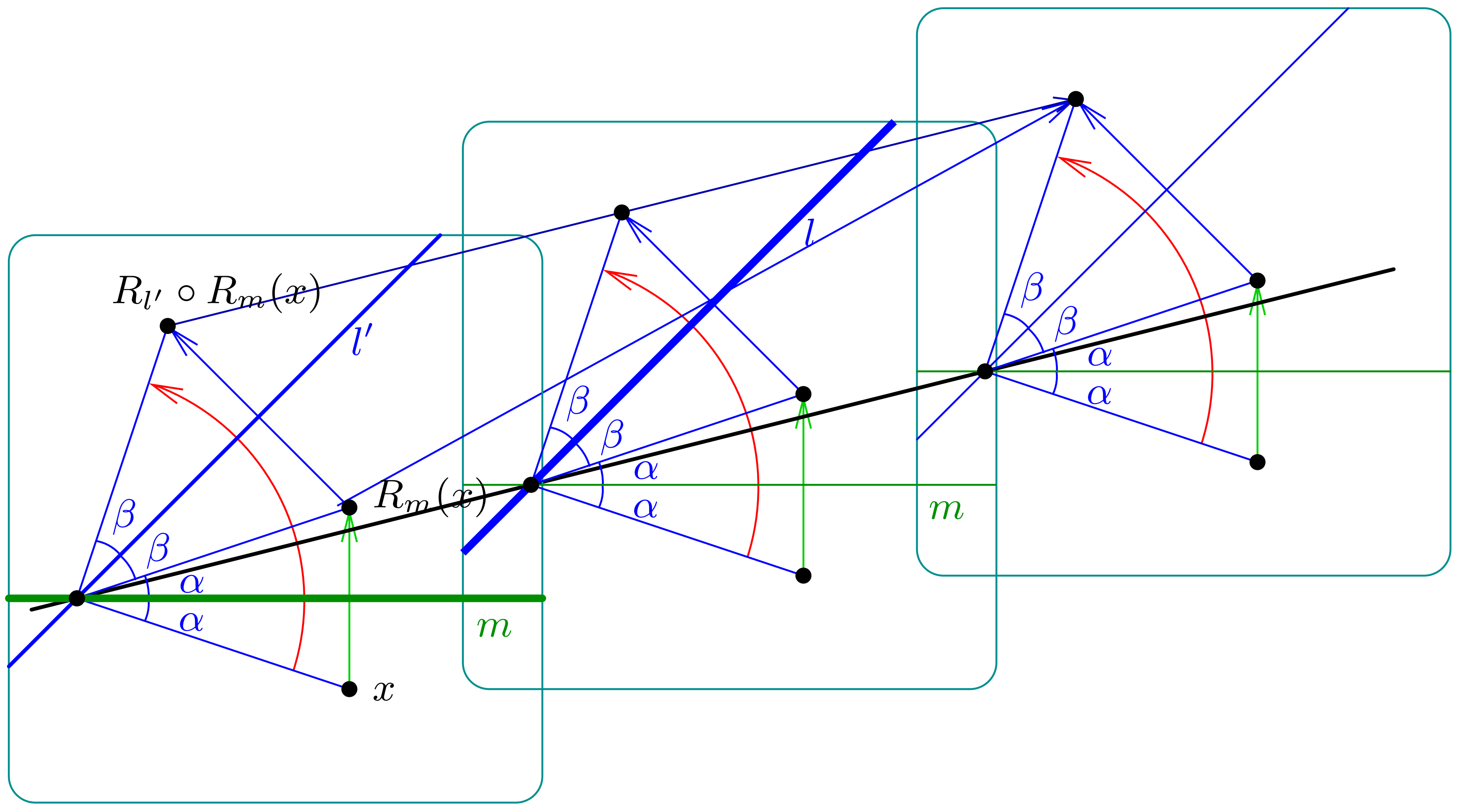
Screw displacement



Screw displacement

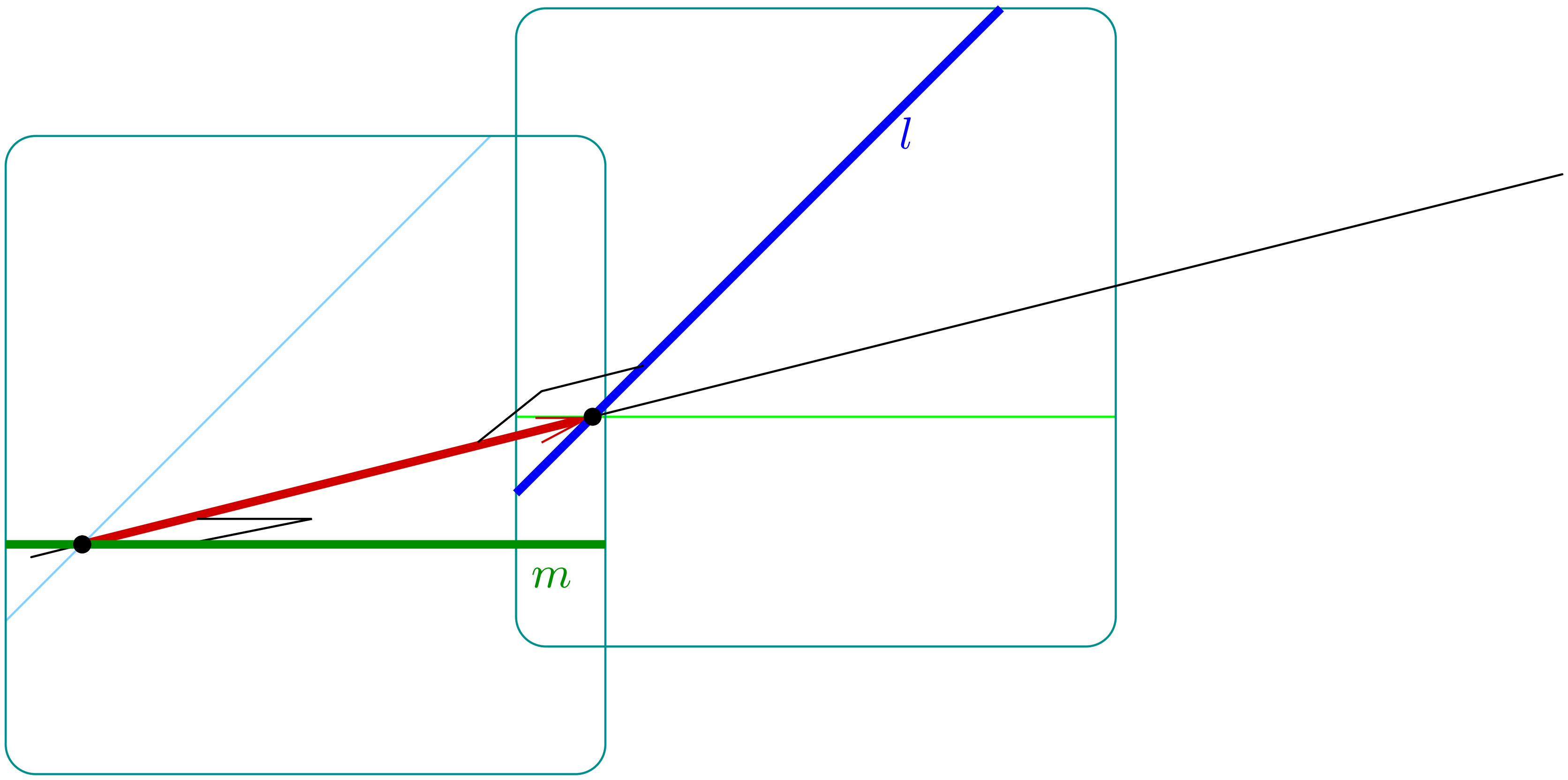


Screw displacement



A biflipper presenting a screw displacement is an arrow with two perpendicular lines at the end points skew to each other.

Screw displacement



Head to tail for screws

Given two screw displacement, present them by biflippers.

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By gliding the biflippers along their axes and rotating about the axes, make the arrowhead of the first biflipper coinciding with the tail of the second biflipper.

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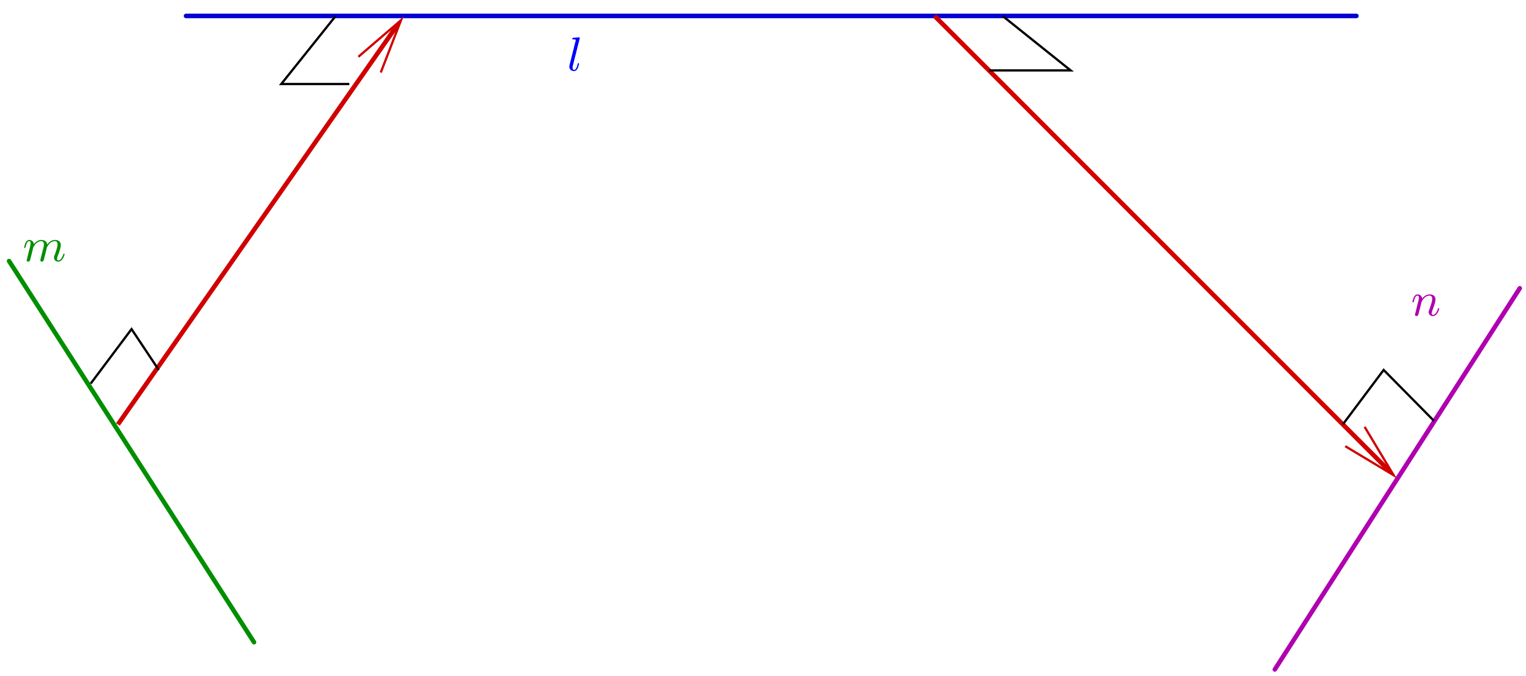
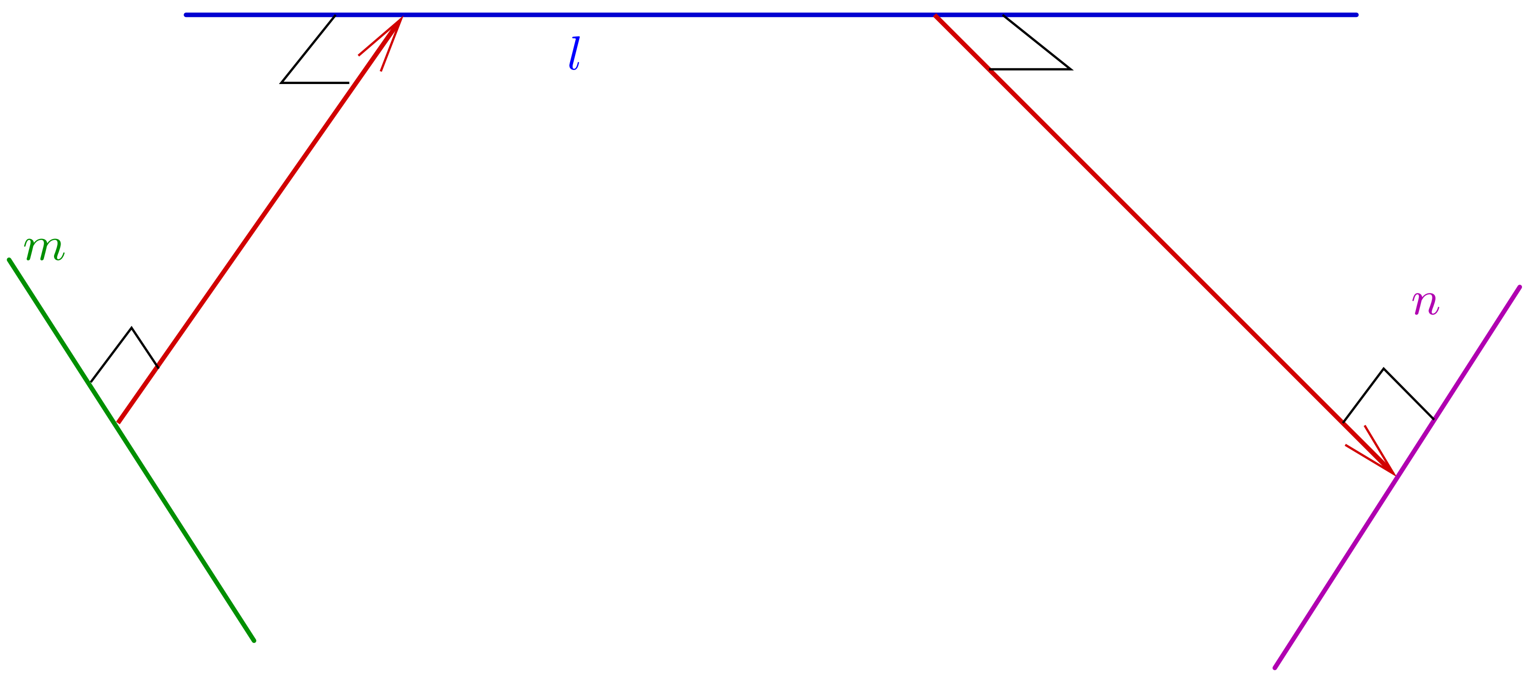


Head to tail for screws

Given two screw displacement, present them by biflippers.

Find the common perpendicular for the axes of the biflippers.

Find common perpendicular for the tail of the first biflipper and head of the second biflipper. Draw an arrow along it connecting the flippers which are left.



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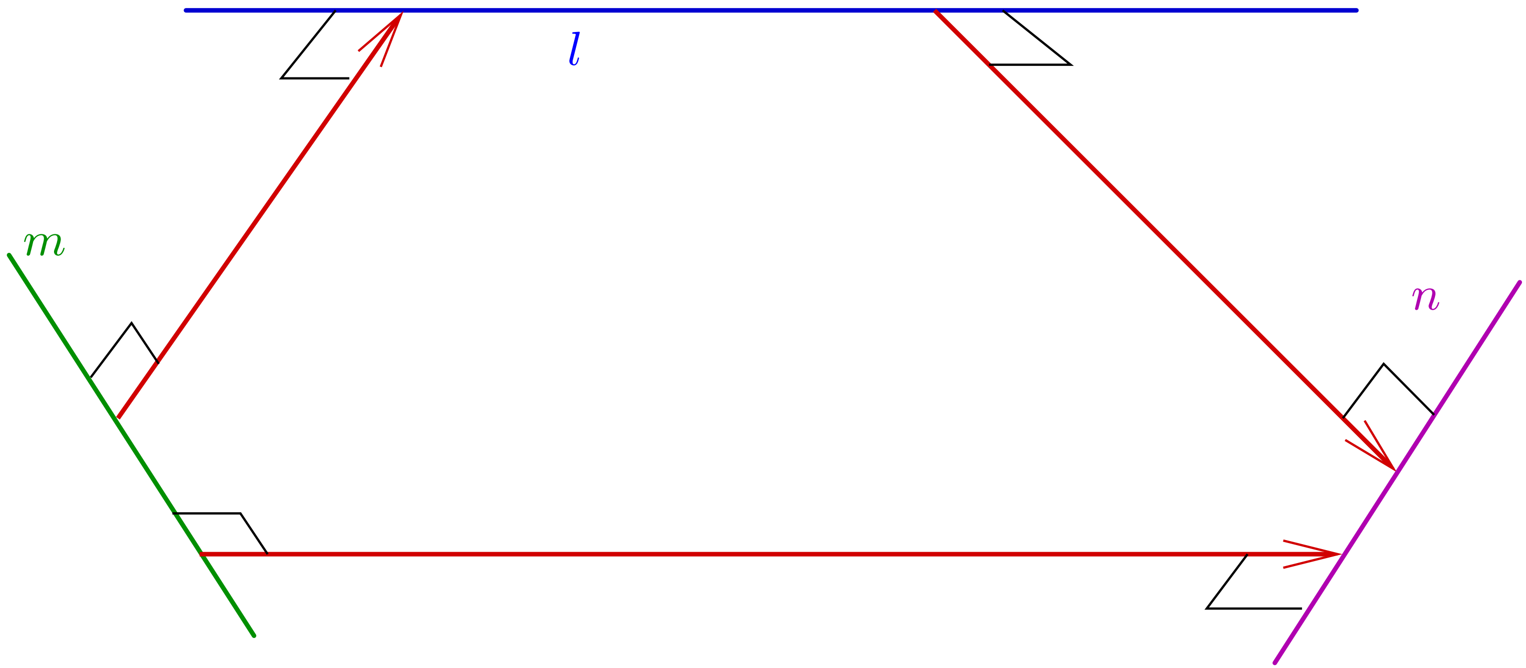
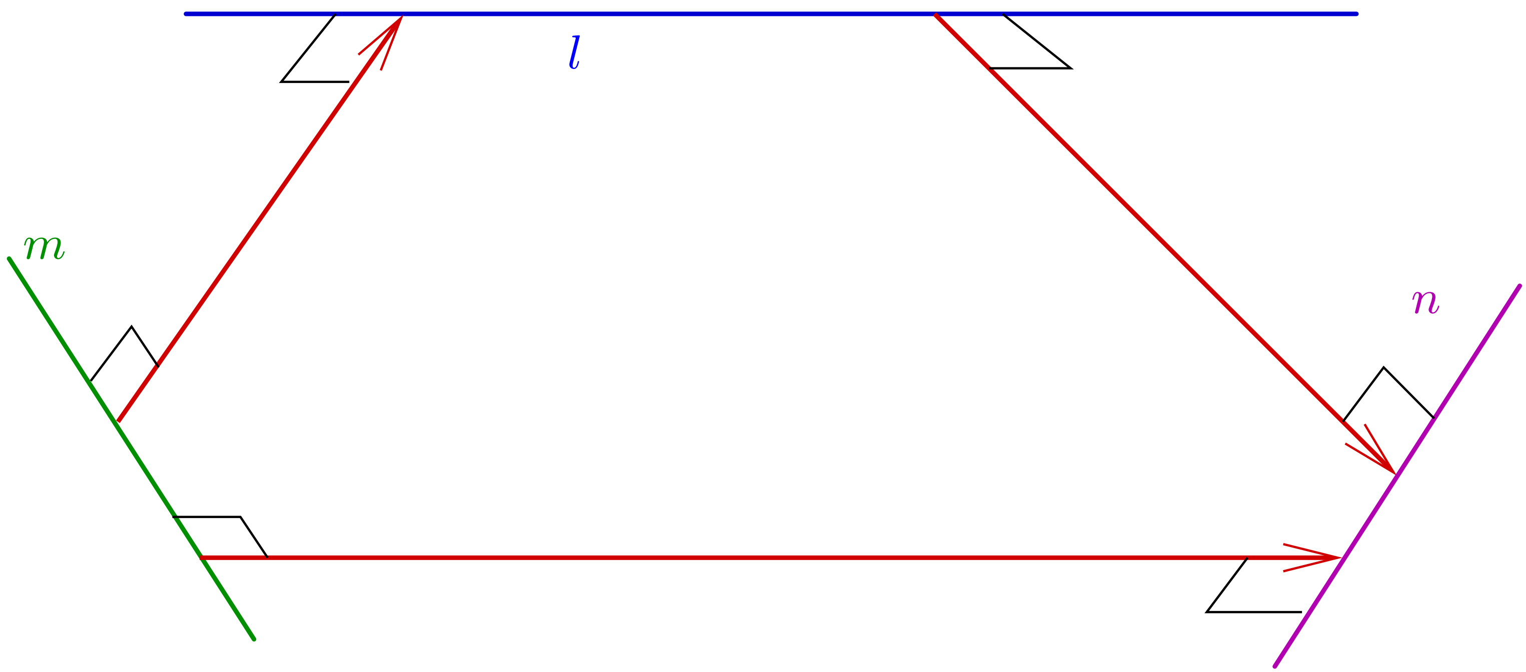


Head to tail for screws

Given two screw displacement, present them by biflippers.

Find the common perpendicular for the axes of the biflippers.

Find common perpendicular for the tail of the first biflipper and head of the second biflipper. Draw an arrow along it connecting the flippers which are left. Erase old arrows and their common flippers.



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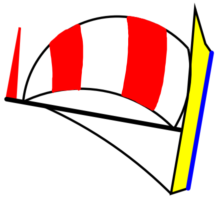

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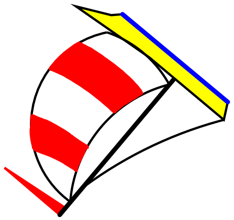

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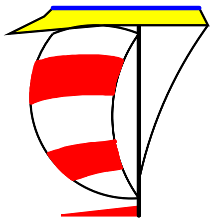

Thank you for your attention!

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