Stony Brook University **Geometric Structures**
Mathematics Department MAT 360
Oleg Viro March 18, 2010

Lectures 15-17.

Lines and Planes.

Agreement on Notation.

- A capital letter denotes a point.
- A small latin letter denotes line.
- A Greek letter denotes a plane.

Basic proberties of the plane. Axioms.

(1) If $A, B \in l$, $A \neq B$, and $A, B \in \alpha$, then $l \subset \alpha$.

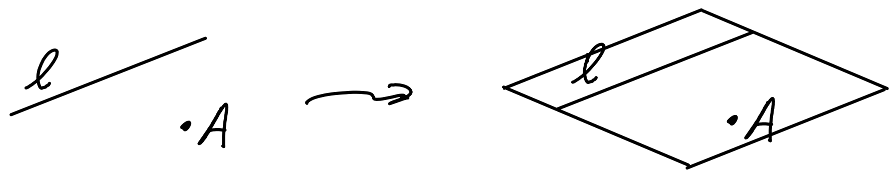
In words: a line that meets a plane in two distinct points, is contained in the plane

(2) If $\alpha \cap \beta \neq \varnothing$, $\alpha \neq \beta$, then $\alpha \cap \beta$ is a line.

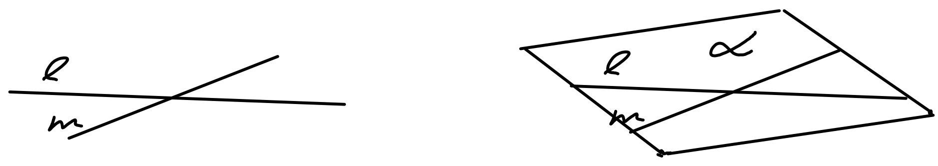
In words: Two planes are either disjoint, or coincide of their intersection is a line.

(3) Through every three points not lying in the same line, one can draw a plane, and such a plane is unique.

Corollaries. (1) For any $A \notin l$, there exists a unique plane $\alpha \supset A, l$.



(2) For any l, m such that $l \neq m$, $l \cap m \neq \varnothing$, there exists a unique plane $\alpha \supset l, m$.



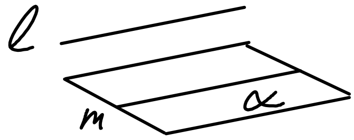
Definition. Two lines are called *parallel*, if they lie in a plane and are either disjoint or coincide.

(3) Any two distinct parallel lines lie in a *unique* plane.

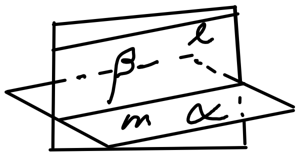
Definition. Two lines which are not contained in a plane are said to be *skew*.

Definition. Line l and plane α are said to be *parallel*, if they are either disjoint (i.e., $l \cap \alpha = \varnothing$) or $l \subset \alpha$.

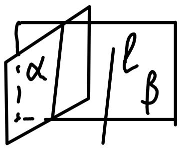
Theorem 1. If $m \subset \alpha$ and $l \parallel m$, then $l \parallel \alpha$.



Theorem 2. If $l \parallel \alpha$, $\beta \supset l$, $\beta \cap \alpha = m$, then $m \parallel l$.

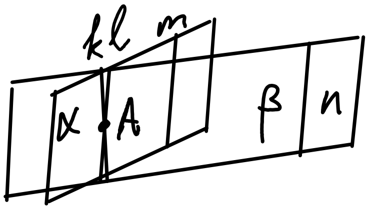


Corllalry 1. $\alpha \parallel l \parallel \beta \implies l \parallel (\alpha \cap \beta)$.



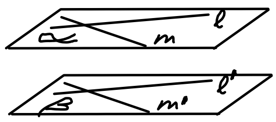
Corllary 2. $l \parallel m, m \parallel n \implies l \parallel n$.

Proof. Draw plane $\alpha \supset l, m$. Choose $A \in l$ and draw plane $\beta \supset n, A$. Let $k = \alpha \cap \beta$.
Since $n \parallel m$, then $m \parallel \beta$ by Theorem 1.
Since $m \parallel \beta$, then $m \parallel (\beta \cap \alpha) = k$ by Theorem 2.
Since $k \parallel m$ and $l \parallel m$ and $A \in l, k$, then $k = l$.
Since $n \parallel m$, then $n \parallel \alpha$ by Theorem 1.
Since $n \parallel \alpha$, then $n \parallel (\alpha \cap \beta) = k$ by Theorem 2. Since $k = l$ and $n \parallel k$, it follows $n \parallel l$. □



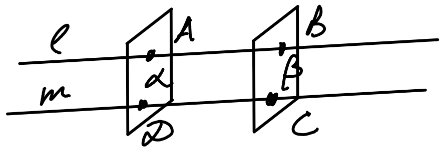
Parallel planes. Definition. $\alpha \parallel \beta$ if either $\alpha \cap \beta = \varnothing$ or $\alpha = \beta$.

Theorem 1. If $l, m \subset \alpha$, $\;l', m' \subset \beta$, $\;l \parallel l'$ and $m \parallel m'$, then $\alpha \parallel \beta$.

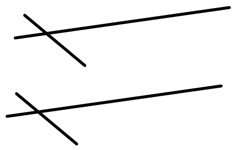


Theorem 2. $\alpha \parallel \beta \implies (\gamma \cap \alpha) \parallel (\gamma \cap \beta)$.

Theorem 3. If $\alpha \parallel \beta$, $l \parallel m$, $A = l \cap \alpha$, $B = l \cap \beta$, $C = m \cap \beta$, $D = m \cap \alpha$, then $|AB| = |CD|$.



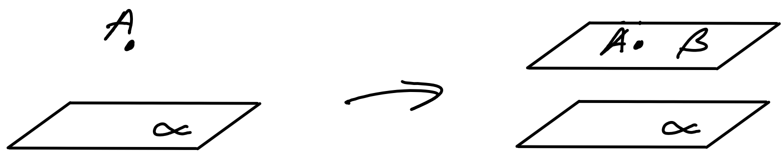
Theorem 4. Angles with parallel and equidirected sides are congruent and lie in parallel planes.



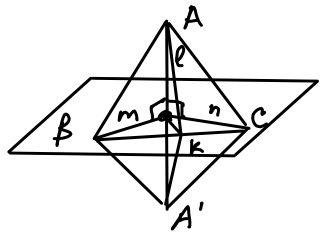
Problem 1. Given skew lines l, m and a point $A \notin l, m$, draw a line n through A meeting l and m.

When such line does not exist?

Problem 2. Given $A \notin \alpha$, find $\beta \ni A$, $\beta \parallel \alpha$.
Is such β *unique*?

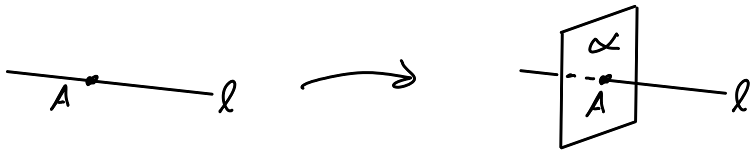


Perpendiculars and slants. Theorem 1. If $l \perp m, n$, $m, n \subset \alpha$, m is not parallel to n and $k \subset \alpha$, then $k \perp l$.



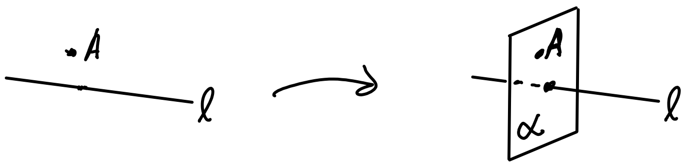
Definition. $l \perp \alpha$ if $m \perp l$ for any $m \subset \alpha$, $m \ni (l \cap \alpha)$.

Theorem 2. For any $A \in l$, there exists a unique α such that $\alpha \perp l$, $\alpha \ni A$.

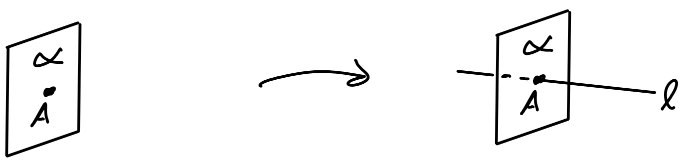


Corollary 1. All lines m such that $m \perp l$ and $m \ni A \in l$ lie on a plane.

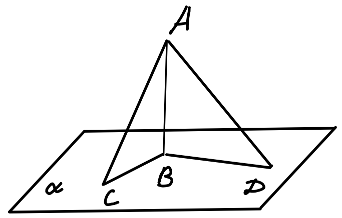
Corollary 2. For given l and A there exists a unique α such that $\alpha \perp l$ and $A \in \alpha$.



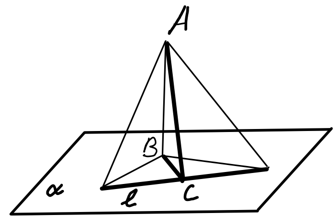
Corollary 3. For given $A \in \alpha$, there exists a unique l such that $A \in l$ and $l \perp \alpha$.



Theorem. Comparing perpendiculars and slants.
For $A \notin \alpha$ and $B, C, D \in \alpha$ such that $AB \perp \alpha$,
$|BC| = |BD|$ if and only if $|AC| = |AD|$;
if $|BC| > |BD$ if and only if $|AC| > |AD|$.

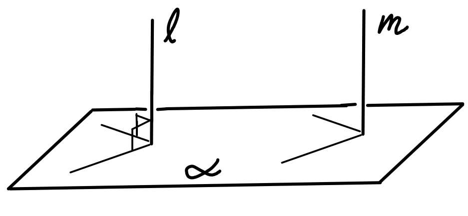


Theorem of three perpendiculars.
Given $A \notin \alpha$, $B, C \in \alpha$, $AB \perp \alpha$, $C \in l \subset \alpha$,
if $l \perp BC$, then $l \perp AC$.



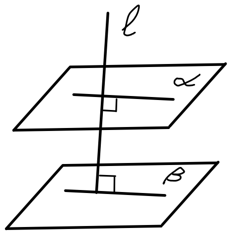
Converse theorem. Given $A \notin \alpha$, $B, C \in \alpha$, $AB \perp \alpha$, $C \in l \subset \alpha$,
if $l \perp AC$, then $l \perp BC$.

Theorem. If $l \perp \alpha$ and $l \parallel m$, then $m \perp \alpha$.



Converse theorem. If $l \perp \alpha$ and $m \perp \alpha$, then $l \parallel m$.

Theorem. If $l \perp \alpha$ and $\alpha \parallel \beta$, then $l \perp \beta$.



Converse theorem. If $\alpha, \beta \perp l$, then $\alpha \parallel \beta$.

Corollary. For any A and α, there exists a unique l such that $l \ni A$ and $l \perp \alpha$.