Topological manifolds

Manifolds

47. Locally Euclidean Spaces

⌈47′1⌋ Definition of Locally Euclidean Space

Let n be a non-negative integer. A topological space X is called a *locally Euclidean space of dimension* n if each point of X has a neighborhood homeomorphic either to $\mathbb{R}^n$ or $\mathbb{R}^n_+$. Recall that $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_1 \geq 0\}$, it is defined for $n \geq 1$.

47.A The notion of 0-dimensional locally Euclidean space coincides with the notion of discrete topological space.

Proof. Each point in a 0-dimensional locally Euclidean space has a neighborhood homeomorphic to $\mathbb{R}^0$ and hence consisting of a single point. Therefore each point is open. □

47.B Prove that the following spaces are locally Euclidean:

(1) $\mathbb{R}^n$,

(2) any open subset of $\mathbb{R}^n$,

(3) the n-sphere $S^n = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$,

(4) real projective space $\mathbb{R}P^n = S^n/_{x \sim -x}$,

(5) complex projective space $\mathbb{C}P^n = \mathbb{C}^{n+1} \smallsetminus 0/_{x \sim y \text{ if } \exists \zeta \in \mathbb{C} : y = \zeta x}$,

(6) $\mathbb{R}^n_+$,

(7) any open subset of $\mathbb{R}^n_+$,

(8) the n-ball $D^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$,

(9) torus $S^1 \times S^1$,

(10) a handle (a torus with a hole),

(11) a sphere with holes,

(12) a sphere with handles,

(13) the Klein bottle $S^1 \times I/(z, 0) \sim (\bar{z}, 1)$,

(14) the n-cube I^n,

Proof. (1) Obvious

(2) In any open subset of $\mathbb{R}^n$ each point has a ball neighborhood. An open ball in $\mathbb{R}^n$ is homeomorphic to the whole $\mathbb{R}^n$.

(3) The complement of a point in S^n is homeomorphic to $\mathbb{R}^n$.

(4) An Euclidean neighborhood of a point is the image of the complement of a hyperplane which does not contain the corresponding pair of antipodal points. The complement of a hyperplane in S^n consists of two hemispheres. Each of them is homeomorphic to $\mathbb{R}^n$.

(5) At least one of the homogeneuous coordinates z_i of a point $[z_0{:}Z_1{:}\dots{:}z_n]$ is not zero. The set of points for which the same z_i does not vanish is a neighborhood of this point. Indeed, it is open in $\mathbb{C}P^n$, because it has the preimage in S^{2n+1} that is the complement of the hyperplane $z_i = 0$. This neighborhood is homeomorphic to $\mathbb{C}^n = \mathbb{R}^{2n}$, the homeomorphism is defined by formula $[z_0{:}z_1{:}\dots{:}z_n] \mapsto (z_0/z_i, z_1/z_0, \dots z_n/z_i)$.

(6) Obvious.

(7) Each point in an open subset of $\mathbb{R}^n_+$ has a ball neighborhood. If the point lies on the boundary hyperplane and the radius is sufficiently small, then the ball is an open half-ball and is homeomorphic to $\mathbb{R}^n_+$. Otherwise, by a choice of sufficiently small radius it can be made the entire ball of $\mathbb{R}^n$.

(8) Hint: an inversion centered at boundary point turns D^n into $\mathbb{R}^n_+$. In the 2-dimensional examples, please, draw Euclidean neighbohoods.

□

47.1 Prove that an open subspace of a locally Euclidean space of dimension n is a locally Euclidean space of dimension n.

47.2 Prove that a bouquet of two circles is not locally Euclidean.

47.C If X is a locally Euclidean space of dimension p and Y is a locally Euclidean space of dimension q then $X \times Y$ is a locally Euclidean space of dimension $p + q$.

Proof. Let $(a, b) \in X \times Y$. Then $a \in X$ has a neighborhood U homeomorphic either to $\mathbb{R}^p$ or $\mathbb{R}^p_+$ and $b \in Y$ has a neighborhood V homeomorphic either to $\mathbb{R}^q$ or to $\mathbb{R}^q_+$. Then (a, b) has a neighborhood $U \times V$ in $X \times Y$. Let us check if it is homeomorphic either to $\mathbb{R}^{p+q}$ or to $\mathbb{R}^{p+q}_+$.

For this, it would suffice to prove that

- $\mathbb{R}^p \times \mathbb{R}^q$ is homeomorphic to $\mathbb{R}^{p+q}$,
- $\mathbb{R}^p \times \mathbb{R}^q_+$ is homeomorphic to $\mathbb{R}^{p+q}_+$,

- $\mathbb{R}^p_+ \times \mathbb{R}^q$ is homeomorphic to $\mathbb{R}^{p+q}_+$,
- $\mathbb{R}^p_+ \times \mathbb{R}^q_+$ is homeomorphic to $\mathbb{R}^{p+q}_+$.

The homeomorphisms come from homeomorphisms establishing associativity and commutativity of the cartesian product, because $\mathbb{R}^n = (\mathbb{R})^n$ and $\mathbb{R}^n_+ = (\mathbb{R})^{n-1} \times \mathbb{R}_+$. The existence of last homeomorphism is reduced by the same arguments to existence of a homeomorphism $\mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}^2_+$. The product $\mathbb{R}_+ \times \mathbb{R}_+$ is identified with the quadrant $\{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0\}$. The quadrant is mapped homeomorphically to the half-plane $\mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$ by the restriction of map $\mathbb{C} \to \mathbb{C} : x + iy \mapsto (x + iy)^2$. □

⌈47′2⌋ Dimension

47.D Can a topological space be simultaneously a locally Euclidean space of dimension both 0 and $n > 0$?

Proof. No. In a locally Euclidean space of dimension 0 each point is open, see 47.A. In a locally Euclidean space of dimension n there are points which have a neighborhood homeomorphic to $\mathbb{R}^n$, and in $\mathbb{R}^n$ with $n > 0$ points are not open. □

47.E Can a topological space be simultaneously a locally Euclidean space of dimension both 1 and $n > 1$?

Proof. No. Assume there exists a locally Euclidean space X of dimensions 1 and $n > 1$. Let $a \in X$. It has a neighborhood homeomorphic to $\mathbb{R}^1$ or $\mathbb{R}^1_+$. In $\mathbb{R}^1_+$ each point except 0 has a neighborhood homeomorphic to $\mathbb{R}^1$. Therefore without loss of generality we may assume that a has a neighborhood, say U, homeomorphic to $\mathbb{R}^1$. Notice that for any point of U, U is a neighborhood, therefore any point in U has a neighborhood homeomorphic to $\mathbb{R}^1$.

Since X is locally Euclidean of dimension $n > 0$, there exists a neighborhood of a homeomorphic either $\mathbb{R}^n$ or $\mathbb{R}^n_+$. In $\mathbb{R}^n_+$ any point which does not belong to the boundary hyperplane has a neighborhood $\{x \in \mathbb{R}^n \mid x_1 > 0\}$ homeomorphic to $\mathbb{R}^n$. Therefore without loss of generality we may assume that a has a neighborhood homeomorphic to $\mathbb{R}^n$.

In R^n open balls form a base of neighborhoods. Therefore any neighborhood of a contains a neighborhood homeomorphic to $\mathbb{R}^n$. Hence there exists a neighborhood $V \subset U$ of a homeomorphic to $\mathbb{R}^n$. In turn, in U which is homeomorphic to $\mathbb{R}^1$ there is a base of neighborhoods homeomorphic to $\mathbb{R}^1$. There exists an element W of this base which is contained in V.

Now consider $U \smallsetminus a \supset V \smallsetminus a \supset W \smallsetminus a$. The set $U \smallsetminus a$ has two connected components, $V \smallsetminus a$ is connected. Therefore $V \smallsetminus a$ is contained in one of the connected components of $U \smallsetminus a$. On the other hand, $W \smallsetminus a$ has two connected components, and a homeomorphism $U \to \mathbb{R}^1$ maps them to open intervals adjacent to the image of a from opposite sides. The component of $W \smallsetminus a$ do not fit to a single connected component of $U \smallsetminus a$. However, being subsets of a connected $V \smallsetminus a \subset U \smallsetminus a$, they must fit in a singleconnected component of $U \smallsetminus a$. Contradiction.

Another proof isindicated in problems 47.3 and 47.4. □

47.3 Prove that any nonempty open connected subset of a locally Euclidean space of dimension 1 can be made disconnected by removing two points.

47.4 Prove that any nonempty locally Euclidean space of dimension $n > 1$ contains a nonempty open set, which cannot be made disconnected by removing any two points.

47.F Can a topological space be simultaneously a locally Euclidean space of dimension both 2 and $n > 2$?

47.F.1 Let U be an open subset of $\mathbb{R}^2$ and a $p \in U$. Prove that $\pi_1(U \setminus \{p\})$ admits an epimorphism onto $\mathbb{Z}$.

47.F.2 Deduce from 47.F.1 that a topological space cannot be simultaneously a locally Euclidean space of dimension both 2 and $n > 2$.

We see that dimension of locally Euclidean topological space is a topological invariant at least for the cases when it is not greater than 2. In fact, this holds true without that restriction. However, one needs some technique to prove this. One possibility is provided by dimension theory, see, e.g., W. Hurewicz and H. Wallman, *Dimension Theory* Princeton, NJ, 1941. Other possibility is to generalize the arguments used in 47.F.2 to higher dimensions. However, this demands a knowledge of high-dimensional homotopy groups.

47.5 Deduce that a topological space cannot be simultaneously a locally Euclidean space of dimension both n and $p > n$ from the fact that $\pi_{n-1}(S^{n-1}) = \mathbb{Z}$. Cf. 47.F.2

⌈47′3⌋ Interior and Boundary

A point a of a locally Euclidean space X is said to be an *interior* point of X if a has a neighborhood (in X) homeomorphic to $\mathbb{R}^n$. A point $a \in X$, which is not interior, is called a *boundary* point of X.

47.6 Which points of $\mathbb{R}^n_+$ have a neighborhood homeomorphic to $\mathbb{R}^n_+$?

47.G Formulate a definition of boundary point independent of a definition for interior point.

Proof. A point of a locally Euclidean space of dimension n is a boundary point if it has no neighborhood homeomorphic to $\mathbb{R}^n$.

(A usual mistake is to say that a point is boundary if it has a neighborhood homeomorphic to $\mathbb{R}^n_+$. Why this is not correct can be seen already in the case of $\mathbb{R}^n_+$: in this space each point has a neighborhood homeomorphic to $\mathbb{R}^n_+$, the whole space.) □

Let X be a locally Euclidean space of dimension n. The set of all interior points of X is called the *interior* of X and denoted by $\operatorname{int} X$. The set of all boundary points of X is called the *boundary* of X and denoted by ∂X.

These terms (interior and boundary) are used also with different meaning. The notions of boundary and interior points of a set in a topological space and the interior part and boundary of a set in a topological space are introduced in general topology, see, e.g., Section 6. They have almost nothing to do with the notions discussed here. In both senses the terminology is classical, which is impossible to change. This does not create usually a danger of confusion.

Notations are not as commonly accepted as words. We take an easy opportunity to select unambiguous notations: we denote the interior part of a set A in a topological space X by $\operatorname{Int}_X A$ or $\operatorname{Int} A$, while the interior of a locally Euclidean space X is denoted by $\operatorname{int} X$; the boundary of a set in a topological space is denoted by symbol Fr, while the boundary of locally Euclidean space is denoted by symbol ∂.

47.H For a locally Euclidean space X the interior $\operatorname{int} X$ is an open dense subset of X, the boundary ∂X is a closed nowhere dense subset of X.

47.I The interior of a locally Euclidean space of dimension n is a locally Euclidean space of dimension n without boundary (i.e., with empty boundary; in symbols: $\partial(\operatorname{int} X) = \varnothing$).

47.J The boundary of a locally Euclidean space of dimension n is a locally Euclidean space of dimension $n-1$ without boundary (i.e., with empty boundary; in symbols: $\partial(\partial X) = \varnothing$).

47.K $\operatorname{int} \mathbb{R}^n_+ \supset \{x \in \mathbb{R}^n : x_1 > 0\}$ and

$$\partial \mathbb{R}^n_+ \subset \{x \in \mathbb{R}^n : x_1 = 0\}.$$

47.7 For any $x, y \in \{x \in \mathbb{R}^n : x_1 = 0\}$, there exists a homeomorphism $f : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ with $f(x) = y$.

47.L Either $\partial \mathbb{R}^n_+ = \varnothing$ (and then $\partial X = \varnothing$ for any locally Euclidean space X of dimension n), or $\partial \mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_1 = 0\}$.

In fact, the second alternative holds true. However, this is not easy to prove for all dimensions. Let us start with the lowest ones.

47.M Prove that $\partial \mathbb{R}^1_+ = \{0\}$.

Proof. We have to prove that 0 has no neighborhood homeomorphic to $\mathbb{R}$ in $\mathbb{R}^1_+$. Assume, it has. Let U be such a neighborhood. Since $[0, \varepsilon)$ is a base of neighborhoods of 0 in $\mathbb{R}^1_+$, there exists neighborhood V from this base contained in U. In turn, there is a neighborhood $W \subset V$ which is homeomorphic to $\mathbb{R}$, since in $\mathbb{R}$ $(a-\varepsilon, a+\varepsilon)$ constitute a base of neighborhoods of $a \in \mathbb{R}$.

Consider inclusions $W \smallsetminus a \subset V \smallsetminus a \subset U \smallsetminus a$. The middle set is connected, hence it is contained in one of the connected components of $U \smallsetminus a$. Hence, both connected components of $W \smallsetminus a$ are contained in one connected component of $U \smallsetminus a$. However, a homeomorphism $U \to \mathbb{R}$ maps W onto an open interval and connected components of $W \smallsetminus a$ are contained in different connected components of $U \smallsetminus a$. Cf. proof of 47.E. □

47.N Prove that $\partial\mathbb{R}^2_+ = \{x \in \mathbb{R}^2 : x_1 = 0\}$. (Cf. 47.F.1.)

47.8 Deduce that a $\partial\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_1 = 0\}$ from $\pi_{n-1}(S^{n-1}) = \mathbb{Z}$. (Cf. 47.N, 47.5)

47.O Deduce from $\partial\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_1 = 0\}$ for all $n \geq 1$ that

$$\operatorname{int}(X \times Y) = \operatorname{int} X \times \operatorname{int} Y$$

and

$$\partial(X \times Y) = (\partial(X) \times Y) \cup (X \times \partial Y).$$

The last formula resembles Leibniz formula for derivative of a product.

47.P Riddle. *Can this be just a coincidence?*

47.Q Prove that

(1) $\partial(I \times I) = (\partial I \times I) \cup (I \times \partial I)$,
(2) $\partial D^n = S^{n-1}$,
(3) $\partial(S^1 \times I) = S^1 \times \partial I = S^1 \amalg S^1$,
(4) the boundary of Möbius strip is homeomorphic to circle.

47.R Corollary. Möbius strip is not homeomorphic to cylinder $S^1 \times I$.

48. Manifolds

⌈48′1⌋ Definition of Manifold

A topological space is called a *manifold* of dimension n if it is

- locally Euclidean of dimension n,
- second countable,
- Hausdorff.

A manifold of dimension d is called also a *d-manifold*.

48.A Prove that the three conditions of the definition are independent.

What does 48.A mean? In what sense the conditions may be independent? The strongest independence here means that for any of the conditions there exists a space not satisfying this condition, but satisfying the other two.

For two of the conditions, one can easily find lots of required examples. For the Hausdorff property it is not that obvious.

48.A.1 Prove that $\mathbb{R} \cup_i \mathbb{R}$, where $i : \{x \in \mathbb{R} : x < 0\} \to \mathbb{R}$ is the inclusion, is a non-Hausdorff second countable locally Euclidean space of dimension one.

48.B Check whether the spaces listed in Problem 47.B are manifolds.

A compact manifold without boundary is said to be ***closed***.

As in the case of interior and boundary, this term coincides with one of the basic terms of general topology. Of course, the image of a closed manifold under embedding into a Hausdorff space is a closed subset of this Hausdorff space (as any compact subset of a Hausdorff space). However absence of boundary does not work here, and even non-compact manifolds may be closed subsets. They are closed in themselves, as any space. Here we meet again an ambiguity of classical terminology. In the context of manifolds the term closed relates rather to the idea of a closed surface.

⌈48′2⌋ Components of Manifold

48.C *A connected component of a manifold is a manifold.*

Proof. Any subspace of a manifold is second countable and Hausdorff, because these properties are hereditary. Property of being locally Euclidean is not hereditary. However, the neighborhoods of a point that are homeomorphic $\mathbb{R}^n$ or $\mathbb{R}^n_+$ are connected and are contained in the connected component. □

48.D *A connected component of a manifold is path-connected.*

Proof. In a manifold each point has a path-connected neighborhood (namely, a neighborhood homeomorphic to $\mathbb{R}^n$ or $\mathbb{R}^n_+$). Therefore by Theorem 14.T connected components of a manifold are path-connected. □

48.E *A connected component of a manifold is open in the manifold.*

Proof. In a manifold each point has a neighborhood homeomorphic to $\mathbb{R}^n$ or $\mathbb{R}^n_+$, this neighborhood is connected and therefore it is contained in a connected component. Hence each point is interior for the connected component containing it. □

48.F *A manifold is the sum of its connected components.*

Proof. Any connected component is closed by Theorem 12.K and open by Theorem 48.E. A space partitioned into sets each of which is both open and closed is the sum of the elements of the partition. □

48.G The set of connected components of any manifold is countable. If the manifold is compact, then the number of the components is finite.

Proof. By Theorem 48.E connected components form an open cover of the manifold. Since a manifold is second countable, by the Lindelöf Theorem this cover contains a countable subcover. If the manifold is compact, it contains a finite subcover. □

48.1 Prove that a manifold is connected, iff its interior is connected.

48.2 The fundamental group of a manifold is countable.

⌈48′3⌋ Making New Manifolds out of Old Ones

48.H Open subset. Prove that an open subspace of a manifold of dimension n is a manifold of dimension n.

48.I Interior. The interior of a manifold of dimension n is a manifold of dimension n without boundary.

48.J Boundary. The boundary of a manifold of dimension n is a manifold of dimension $n-1$ without boundary.

48.3 Boundary of Compact Manifold. The boundary of a compact manifold of dimension n is a closed manifold of dimension $n-1$.

48.K Product. If X is a manifold of dimension p and Y is a manifold of dimension q then $X \times Y$ is a manifold of dimension $p+q$.

48.L Covering Space. Prove that a covering space (in narrow sense) of a manifold is a manifold of the same dimension.

48.M Covered Space. Prove that if the total space of a covering is a manifold then the base is a manifold of the same dimension.

48.N Gluing along Components of Boundary. Let X and Y be manifolds of dimension n, A and B components of ∂X and ∂Y respectively. Then for any homeomorphism $h : B \to A$ the space $X \cup_h Y$ is a manifold of dimension n.

48.N.1 Prove that the result of gluing of two copies of $\mathbb{R}^n_+$ by the identity map of the boundary hyperplane is homeomorphic to $\mathbb{R}^n$.

48.O Riddle. Can a manifold be embedded into a manifold of the same dimension without boundary?

Let X be a manifold. Denote by DX the space $X \cup_{\mathrm{id}_{\partial X}} X$ obtained by gluing of two copies of X by the identity mapping $\mathrm{id}_{\partial X} : \partial X \to \partial X$ of the boundary. DX is called the *double* of X.

48.P Double. Prove that DX is a manifold without boundary of the same dimension as X.

48.Q Double of a Compact Manifold. Prove that a double of a manifold is compact, iff the original manifold is compact.

⌈48′4⌋ Collars and Bites

Let X be a manifold. An embedding $c : \partial X \times I \to X$ such that $c(x, 0) = x$ for each $x \in \partial X$ is called a *collar* of X. A collar can be thought of as a neighborhood of the boundary presented as a cylinder over boundary.

48.4 Existence Collar Theorem. *Every manifold has a collar.*

Let U be an open set in the boundary of a manifold X. For a continuous function $\varphi : \partial X \to \mathbb{R}_+$ with $\varphi^{-1}(0, \infty) = U$ set

$$B_\varphi = \{(x, t) \in \partial X \times \mathbb{R}_+ \,:\, t \le \varphi(x)\}.$$

A *bite* on X at U is an embedding $b : B_\varphi \to X$ with some $\varphi : \partial X \to \mathbb{R}_+$ such that $b(x, 0) = x$ for each $x \in \partial X$.

This is a generalization of collar. Indeed, a collar is a bite at $U = \partial X$ with $\varphi = 1$.

48.4.1 Prove that if $U \subset \partial X$ is contained in an open subset of X homeomorphic to $\mathbb{R}^n_+$, then there exists a bite of X at U.

48.4.2 Prove that for any bite $b : B \to X$ of a manifold X the closure of $X \smallsetminus b(B)$ is a manifold.

48.4.3 Let $b_1 : B_1 \to X$ be a bite of X and $b_2 : B_2 \to \mathrm{Cl}(X \smallsetminus b_1(B_1))$ be a bite of $\mathrm{Cl}(X \smallsetminus b_1(B_1))$. Construct a bite $b : B \to X$ of X with $b(B) = b_1(B_1) \cup b_2(B_2)$.

48.4.4 Prove that if there exists a bite of X at ∂X then there exists a collar of X.

48.5 Uniqueness Collar Theorem. *For any two collars $c_1, c_2 : \partial X \times I \to X$ there exists a homeomorphism $h : X \to X$ with $h(x) = x$ for $x \in \partial X$ such that $h \circ c_1 = c_2$.*

This means that a collar is unique up to homeomorphism.

48.5.1 For any collar $c : \partial X \times I \to X$ there exists a collar $c' : \partial X \times I \to X$ such that $c(x, t) = c'(x, t/2)$.

48.5.2 For any collar $c : \partial X \times I \to X$ there exists a homeomorphism

$$h : X \to X \cup_{x \mapsto (x,1)} \partial X \times I$$

with $h(c(x, t)) = (x, t)$.

48.R Gluing along a Manifold with Boundary. Let X and Y be manifolds of dimension n, A and B closed subsets of ∂X and ∂Y respectively. If A and B are manifolds of dimension $n - 1$ then for any homeomorphism $h : B \to A$ the space $X \cup_h Y$ is a manifold of dimension n.

49. Isotopy

⌈49′1⌋ Isotopy of Homeomorphisms

Let X and Y be topological spaces, $h, h' : X \to Y$ homeomorphisms.

A homotopy $h_t : X \to Y$, $t \in [0,1]$ connecting h and h' is called an *isotopy* between h and h' if h_t is a homeomorphism for each $t \in [0,1]$.

h, h' are said to be *isotopic* if there exists an isotopy between h and h'.

49.A Being isotopic is an equivalence relation on the set of homeomorphisms $X \to Y$.

49.B Isotopy vs. Homotopy. Find a topological space X such that homotopy between homeomorphisms $X \to X$ does not imply isotopy.

Existence of such X means that the isotopy classification of homeomorphisms can be more refined than the homotopy classification of them.

49.1 Classify homeomorphisms of circle S^1 to itself up to isotopy.

49.2 Classify homeomorphisms of line $\mathbb{R}^1$ to itself up to isotopy.

The set of isotopy classes of homeomorphisms $X \to X$ (i.e. the quotient of the set of self-homeomorphisms of X by isotopy relation) is called the *mapping class group* or *homeotopy group* of X.

49.C For any topological space X, the mapping class group of X is a group under the operation induced by composition of homeomorphisms.

49.3 Find the mapping class group of the union of the coordinate lines in the plane.

49.4 Find the mapping class group of the union of bouquet of two circles.

⌈49′2⌋ Isotopy of Embeddings

Homeomorphisms are topological embeddings of special kind. The notion of isotopy of homeomorphism is extended in an obvious way to the case of embeddings.

Let X and Y be topological spaces, $h, h' : X \to Y$ topological embeddings.

A homotopy $h_t : X \to Y$, $t \in [0,1]$ connecting h and h' is called an *(embedding) isotopy* between h and h' if h_t is an embedding for each t.

Embeddings h, h' are said to be *isotopic* if there exists an isotopy between h and h'.

49.D Being isotopic is an equivalence relation on the set of embeddings $X \to Y$.

⌈49′3⌋ Isotopy of Sets

A family A_t, $t \in I$ of subsets of a topological space X is called an *isotopy of the set* $A = A_0$, if the graph $\Gamma = \{(x,t) \in X \times I \,|\, x \in A_t\}$ of the family is fibrewise homeomorphic to the cylinder $A \times I$, i. e. there exists a homeomorphism $A \times I \to \Gamma$ mapping $A \times \{t\}$ to $\Gamma \cap X \times \{t\}$ for any $t \in I$.

Such a homeomorphism gives rise to an isotopy of embeddings $\Phi_t : A \to X$, $t \in I$ with $\Phi_0 = \text{in}$, $\Phi_t(A) = A_t$. An isotopy of a subset is also called a *subset isotopy.* Subsets A and A' of the same topological space X are said to be *isotopic in* X, if there exists a subset isotopy A_t of A with $A' = A_1$.

49.E This is an equivalence relation on the set of subsets of X.

As it follows immediately from the definitions, any embedding isotopy determines an isotopy of the image of the initial embedding and any subset isotopy is accompanied with an embedding isotopy. However the relation between the notions of subset isotopy and embedding isotopy is not too close because of the following two reasons:

(1) an isotopy Φ_t accompanying a subset isotopy A_t starts with the inclusion of A_0 (while arbitrary isotopy may start with any embedding);

(2) an isotopy accompanying a subset isotopy is determined by the subset isotopy only up to composition with an isotopy of the identity homeomorphism $A \to A$ (an isotopy of a homeomorphism is a special case of embedding isotopies, since homeomorphisms can be considered as a sort of embeddings).

⌈49′4⌋ Ambient Isotopy

An isotopy of a subset A in X is said to be *ambient*, if it may be accompanied with an embedding isotopy $\Phi_t : A \to X$ extendible to an isotopy $\tilde{\Phi}_t : X \to X$ of the identity homeomorphism of the space X. The isotopy $\tilde{\Phi}_t$ is said to be *ambient* for Φ_t. This gives rise to obvious refinements of the equivalence relations for subsets and embeddings introduced above.

49.F Find isotopic, but not ambiently isotopic sets in $[0, 1]$.

49.G If sets $A_1, A_2 \subset X$ are ambiently isotopic, then the complements $X \smallsetminus A_1$ and $X \smallsetminus A_2$ are homeomorphic and hence homotopy equivalent.

49.5 Find isotopic, but not ambiently isotopic sets in $\mathbb{R}$.

49.6 Prove that any isotopic compact subsets of $\mathbb{R}$ are ambiently isotopic.

49.7 Find isotopic, but not ambiently isotopic compact sets in R^3.

49.8 Prove that any two embeddings $S^1 \to \mathbb{R}^3$ are isotopic. Find embeddings $S^1 \to \mathbb{R}^3$ that are not ambiently isotopic.

⌈49′5⌋ Isotopies and Attaching

49.9 Any isotopy $h_t : \partial X \to \partial X$ extends to an isotopy $H_t : X \to X$.

49.10 Let X and Y be manifolds of dimension n, A and B components of ∂X and ∂Y respectively. Then for any isotopic homeomorphisms $f, g : B \to A$ the manifolds $X \cup_f Y$ and $X \cup_g Y$ are homeomorphic.

49.11 Let X and Y be manifolds of dimension n, let B be a compact subset of ∂Y. If B is a manifold of dimension $n-1$ then for any embeddings $f, g : B \to \partial X$ ambiently isotopic in ∂X the manifolds $X \cup_f Y$ and $X \cup_g Y$ are homeomorphic.

⌈49′6⌋ Connected Sums

49.H *Let X and Y be manifolds of dimension n, and $\varphi : \mathbb{R}^n \to X$, $\psi : \mathbb{R}^n \to Y$ be embeddings. Then*

$$X \smallsetminus \varphi(\operatorname{Int} D^n) \cup_{\psi(S^n) \to X \smallsetminus \varphi(\operatorname{Int} D^n) : \psi(a) \mapsto \varphi(a)} Y \smallsetminus \psi(\operatorname{Int} D^n)$$

is a manifold of dimension n.

This manifold is called a ***connected sum*** of X and Y.

49.12 Find pairs of manifolds connected sums of which are homeomorphic to

(1) S^1,
(2) Klein bottle,
(3) sphere with three crosscaps.

The term *connected sum* somehow alludes on its connectedness and distinction from *disjoint sum.* However, a connected sum is not necessarily connected. It is not connected if at least one of the summands is not.

49.13 Find a disconnected connected sum of connected manifolds. Describe, under what circumstances this can happen.

49.I Show that the topological type of the connected sum of X and Y depends not only on the topological types of X and Y.

HINT. Consider connected sums of $\mathbb{R}^1_+$ with itself. □

49.J Let X and Y be manifolds of dimension n, and $\varphi : \mathbb{R}^n \to X$, $\psi : \mathbb{R}^n \to Y$ be embeddings. Let $h : X \to X$ be a homeomorphism. Then the connected sums of X and Y defined via ψ and φ, on one hand, and via ψ and $h \circ \varphi$, on the other hand, are homeomorphic.

Chapter XI

Manifolds of Low Dimensions

In different geometric subjects there are different ideas which dimensions are low and which high. In topology of manifolds low dimension means at most 4. However, in this chapter only dimensions up to 2 will be considered, and even most of two-dimensional topology will not be touched. Manifolds of dimension 4 are the most mysterious objects in the field. Dimensions higher than 4 are easier: there is enough room for most of the constructions that topology needs.

50. One-Dimensional Manifolds

⌈50′1⌋ Zero-Dimensional Manifolds

This section is devoted to topological classification of manifolds of dimension one. We could skip the case of 0-manifolds due to triviality of the problem.

50.A Topological Classification of 0-Manifolds. *Two 0-dimensional manifolds are homeomorphic iff they have the same number of points.*

Proof. Indeed, any 0-manifold is just a countable discrete topological space, and the only topological invariant needed for topological classification of 0-manifolds is the number of points. □

The case of 1-manifolds is also simple, but requires more detailed considerations. Surprisingly, many textbooks manage to ignore 1-manifolds entirely.

⌈50′2⌋ Reduction to Connected Manifolds

50.B *Two manifolds are homeomorphic iff there exists a one-to-one correspondence between their components such that the corresponding components are homeomorphic.*

Proof. Each manifold is the sum of its connected components. □

Thus, for topological classification of n-dimensional manifolds it suffices to classify only *connected* n-dimensional manifolds.

⌈50′3⌋ Examples

50.C *What connected 1-manifolds do you know?*

(1) *Do you know any closed connected 1-manifold?*

(2) *Do you know a connected compact 1-manifold, which is not closed?*

(3) *What non-compact connected 1-manifolds do you know?*

(4) *Is there a non-compact connected 1-manifolds with boundary?*

⌈50′4⌋ How to Distinguish Them From Each Other?

50.D Fill the following table with pluses and minuses.

Manifold X	Is X compact?	Is ∂X empty?
S^1		
$\mathbb{R}^1$		
I		
$\mathbb{R}^1_+$		

⌈50′5⌋ Statements of Main Theorems

50.E *Any connected 1-manifold is homeomorphic to one of the following:*

- *circle* S^1,
- *line* $\mathbb{R}^1$,
- *interval* I,
- *half-line* $\mathbb{R}^1_+$.

This theorem splits into the following four theorems:

50.F *Any closed connected 1-manifold is homeomorphic to S^1.*

50.G *Any non-compact connected 1-manifold without boundary is homeomorphic to $\mathbb{R}^1$.*

50.H *Any compact connected 1-manifold with nonempty boundary is homeomorphic to I.*

50.I *Any non-compact connected 1-manifold with nonempty boundary is homeomorphic to $\mathbb{R}^1_+$.*

⌈50′6⌋ Lemma on 1-Manifold Covered with Two Lines

50.J Lemma. *Any connected manifold of dimension 1 covered with two open sets homeomorphic to $\mathbb{R}^1$ is homeomorphic either to $\mathbb{R}^1$, or S^1.*

Let X be a connected manifold of dimension 1 and $U, V \subset X$ be its open subsets homeomorphic to $\mathbb{R}$. Denote by W the intersection $U \cap V$. Let $\varphi : U \to \mathbb{R}$ and $\psi : V \to \mathbb{R}$ be homeomorphisms.

50.J.1 Prove that each connected component of $\varphi(W)$ is either an open interval, or an open ray, or the whole $\mathbb{R}$.

Proof. This is exactly what a connected open subset of the line can be. See 12.U. □

50.J.2 Prove that a homeomorphism between two open connected subsets of $\mathbb{R}$ is a (strictly) monotone continuous function.

Proof. See 11.M, 11.N and 11.4. □

50.J.3 Prove that if a sequence x_n of points of W converges to a point $a \in U \smallsetminus W$ then it does not converge in V.

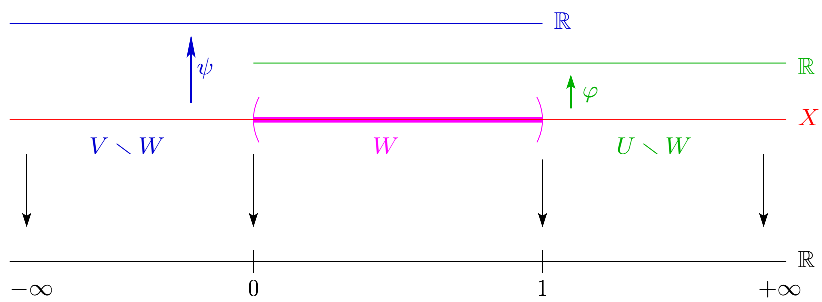
Proof. Because a sequence of points in a Hausdorff space has at most one limit, see 15.D. □

50.J.4 Prove that if there exists a bounded connected component C of $\varphi(W)$, then $C = \varphi(W)$, $V = W$, $X = U$ and hence X is homeomorphic to $\mathbb{R}$.

Proof. A bounded component C of $\varphi(W)$ is an open interval. Both of its end points are limits of monotone sequences of its points. By Lemma 50.J.3 the corresponding monotone sequences of points in $\psi(W)$ have no limits. Hence the component $\psi\varphi^{-1}(C)$ of $\psi(W)$ is unbounded both from above and below, and coincides with the whole line. □

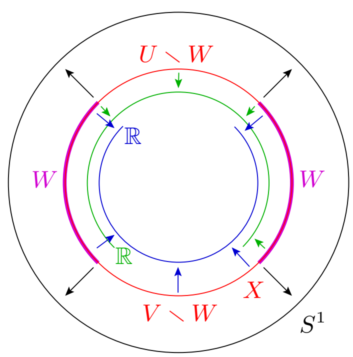
50.J.5 In the case of connected W and $U \neq V$, construct a homeomorphism $X \to \mathbb{R}$ which takes:

- W to $(0, 1)$,
- U to $(0, +\infty)$, and
- V to $(-\infty, 1)$.



50.J.6 In the case of W consisting of two connected components, construct a homeomorphism $X \to S^1$, which takes:

- W to $\{z \in S^1 : -1/\sqrt{2} < \operatorname{Im}(z) < 1/\sqrt{2}\}$,
- U to $\{z \in S^1 : -1/\sqrt{2} < \operatorname{Im}(z)\}$, and
- V to $\{z \in S^1 : \operatorname{Im}(z) < 1/\sqrt{2}\}$.



⌈50′7⌋ Without Boundary

50.F.1 Deduce Theorem 50.F from Lemma 50.I.

Proof. Consider a cover of X by open sets homeomorphic to $\mathbb{R}$. Since X is compact, it contains a finite subcover $U_1, U_2, \dots U_n$. The number n of its elements is greater than 1, because if $n = 1$ then $X = U_1$ would be homeomorphic to $\mathbb{R}$ and hence non-compact, which would contradict the assumption that X is closed. If $n = 2$, then by Lemma 50.I X is homeomorphic either to $\mathbb{R}$ or to S^1, and by the assumption the former cannot happen.

Assume that the statement has been proven for $n \le k$ and prove it for $n = k+1$. Since X is connected, $U_{k+1} \cap (U_1 \cup \dots \cup U_k) \neq \varnothing$. By changing the numeration, we may assume that $U_{k+1} \cap U_k \neq \varnothing$. Then $V = U_{k+1} \cup U_k$ satisfies the conditions of Lemma

50.I. By Lemma 50.I, V is homeomorphic either to S^1 or $\mathbb{R}$. In the former case V is a closed subset of X (as a compact subset in Hausdorff X). It is also open, and hence $V = X$, by connectedness of X. Therefore in this case X is homeomorphic to S^1. In the case if V is homeomorphic to $\mathbb{R}$, the sets $U_1, U_2, \dots, U_{k-1}, V$ form a cover of X consisting of k open sets homeomorphic to $\mathbb{R}$. Then, by the inductive assumption, X is homeomorphic to S^1. □

50.G.1 Deduce Theorem 50.G from Lemma 50.I.

Proof. Consider a cover of X by open sets homeomorphic to $\mathbb{R}$. Due to second countability of X, by the Lindelöf theorem this cover contains a countable subcover. Let $U_1, U_2, \dots, U_n \dots$ be such subcover. By connectedness of X, we can renumerate the elements of this subcover in such a way that $U_n \cap (U_1 \cup \dots \cup U_{n-1}) \neq \varnothing$ for any n. Then all $V_n = U_1 \cup \dots \cup U_n$ are open connected sets.

Choose a homeomorphism of $V_1 = U_1$ onto a unit open interval on $\mathbb{R}$. Assume that we have proved that V_k is homeomorphic to $\mathbb{R}$, and moreover, have constructed a homeomorphism of V_k onto an open interval of length at most k on the line. By Lemma 50.I the union $V_{k+1} = V_k \cup U_{k+1}$ is homeomorphic either to S^1 or to $\mathbb{R}$. In the former case, V_{k+1} would be closed and open in X, hence it would coincide with X, which would contradict the assumption that X is not compact. Thus V_{k+1} is homeomorphic to $\mathbb{R}$. The set V_k is an open connected subset in V_{k+1}, and any homeomorphism $V_{k+1} \to \mathbb{R}$ maps it to a convex open set, that is either onto an open interval, or an open ray or the whole $\mathbb{R}$. In any of these cases, the homeomorphism of V_k onto an open interval of length at most k can be extended to a homeomorphism of V_{k+1} onto an open interval of length at most $k+1$.

This inductive construction gives a homeomorphism of $X = \cup_n V_n$ onto a union of increasing sequence of open intervals. This union is an open connected set on $\mathbb{R}$ and hence it is homeomorphic to $\mathbb{R}$. □

⌈50′8⌋ With Boundary

50.H.1 Prove that any compact connected 1-manifold with boundary can be embedded into S^1.

Proof. Any compact connected 1-manifold X with boundary is embedded into its double DX. The double of a compact manifold is a closed manifold, the double of a connected manifold with non-empty boundary is connected. Thus DX is a closed connected 1-manifold. By Theorem 50.F, DX is homeomorphic to S^1. □

50.H.2 List all connected compact subsets of S^1.

Proof. A connected compact subset A of S^1 is either S^1, or a closed arc, or a point. Indeed, if $A \neq S^1$, then by a stereographic projection from any point of its complement, A is mapped homeomorphically to a compact connected subset of $\mathbb{R}$. Connectedness in $\mathbb{R}$ implies convexity, a subsets of $\mathbb{R}$ is compact iff it is closed and bounded. This gives either a closed interval or a point. □

50.H.3 Deduce Theorem 50.H from 50.H.2, and 50.H.1.

50.I.1 Prove that any non-compact connected 1-manifold with non-empty boundary can be embedded into $\mathbb{R}^1$.

Proof. Any non-compact connected 1-manifold X is embedded into its double DX. The double of non-compact manifold is a non-compact manifold without boundary, the double of a connected manifold with non-empty boundary is connected. Thus DX is a non-compact connected 1-manifold without boundary. By Theorem 50.G, DX is homeomorphic to $\mathbb{R}$. □

50.I.2 Deduce Theorem 50.I from 50.I.1.

⌈50′9⌋ Corollaries of Classification

50.K Prove that connected sum of closed 1-manifolds is defined up homeomorphism by topological types of summands.

50.L Which 0-manifolds bound a compact 1-manifold?

Proof. A 0-manifold X bounds a compact 1-manifold iff the number of point of X is even. Indeed, a compact 1-manifold has a finite number of connected components, and each of them is homeomorphic either to S^1 or I. □

⌈50′10⌋ Orientations of 1-manifolds

Orientation of a *connected non-closed* 1-manifold is a linear order on the set of its points such that the corresponding interval topology (see, e.g., 7.P.) coincides with the topology of this manifold.

Orientation of a *connected closed* 1-manifold is a cyclic order on the set of its points such that the topology of this cyclic order (see 8′3) coincides with the topology of the 1-manifold.

Orientation of an *arbitrary* 1-manifold is a collection of orientations of its connected components (each component is equipped with an orientation).

50.M Orientability. *Any* 1*-manifold admits an orientation.*

50.N An orientation of 1-manifold induces an orientation (i.e., a linear ordering of points) on each subspace homeomorphic to $\mathbb{R}$ or $\mathbb{R}_+$. Vice versa, an orientation of a 1-manifold is determined by a collection of orientations of its open subspaces homeomorphic to $\mathbb{R}$ or $\mathbb{R}_+$, if the subspaces cover the manifold and the orientations agree with each other: the orientations of any two subspaces define the same orientation on each connected component of their intersection.

50.O Let X be a cyclicly ordered set, $a \in X$ and $B \subset X \smallsetminus \{a\}$. Define in $X \smallsetminus \{a\}$ a linear order induced, as in 8.8, by the cyclic order on $X \smallsetminus \{a\}$, and equip B with the linear order induced by this linear order on $X \smallsetminus \{a\}$.

Prove that if B admits a bijective monotone map onto $\mathbb{R}$, or $[0;1]$, or $[0;1)$, or $(0;1]$, then this linear order on B does not depend on a.

The construction of 50.O allows one to define an orientation on any 1-manifold which is a subspace of an *oriented closed* 1-manifold. A 1-manifold, which is a subspace of an *oriented non-closed* 1-manifold X, inherits from X an orientation as a linear order. Thus, any 1-manifold, which is a subspace of an *oriented* 1-manifold X, inherits from X an orientation. This orientation is said to be ***induced*** by the orientation of X.

A topological embedding $X \to Y$ of an oriented 1-manifold to another one is said to ***preserve*** the orientation if it maps the orientation of X to the orientation induced on the image by the orientation of Y.

50.P *Any two orientation preserving embeddings of an oriented connected 1-manifold X to an oriented connected 1-manifold Y are isotopic.*

50.Q *If two embeddings of an oriented 1-manifold X to an oriented 1-manifold Y are isotopic and one of the embeddings preserves the orientation, then the other one also preserves the orientation*

50.R Corollary. *Orientation of a closed segment is determined by the ordering of its end points.*

An orientation of a segment is shown by an arrow directed from the initial end point to the final one. In order to show an orientation of a 1-manifold, one usually equips each connected component with an arrow, as an arrow shows an oriented closed segment embedded into the component.

50.S *A connected 1-manifold admits two orientations. A 1-manifold consisting of n connected components admits 2^n orientations.*

⌈50′11⌋ Mapping Class Groups

50.T Find the mapping class groups of

$$\begin{array}{llllll} (1) & S^1, & (2) & \mathbb{R}^1, & (3) & \mathbb{R}^1_+, \\ (4) & [0,1], & (5) & S^1 \amalg S^1, & (6) & \mathbb{R}^1_+ \amalg \mathbb{R}^1_+. \end{array}$$

50.1 Find the mapping class group of an arbitrary 1-manifold with finite number of components.

⌈50′12⌋ Involutions

Recall that a non-identity continuous map $f : X \to X$ is called an ***involution*** if $f^2 = \mathrm{id}_X$.

50.U *A continuous involution of a topological space is a homeomorphism.*

50.2 *Prove that an involution of a non-closed connected 1-manifold reverses orientation.*

50.3 Riddle. *Relate the preceding problem with the fact that any 1-manifold is orientable.*

50.4 Does Theorem 50.2 generalize to any periodic homeomorphism of a non-closed connected manifold?

50.5 Does a non-closed connected 1-manifold admit a homeomorphism $f \neq \mathrm{id}$ with $f^9 = \mathrm{id}$?

50.6 Prove that an orientation preserving involution of a 1-manifold has no fixed points.

Involutions $f, g : X \to X$ are said to be equivalent if there exists a homeomorphism $h : X \to X$ such that $hg = fh$. In other words, equivalence of involutions is conjugacy in the group of all homeomorphisms of X.

An involution is said to be ***trivial*** if it is the identity map.

50.V Classification of involutions on connected 1-manifolds.

(1) Any non-trivial involution of S^1 is equivalent either to the antipodal symmetry $z \mapsto -z$, or symmetry against a line $z \mapsto \overline{z}$.

(2) Any non-trivial involution of $\mathbb{R}$ is equivalent to the symmetry with respect to the origin $x \mapsto -x$.

(3) Any non-trivial involution of I is equivalent to the symmetry with respect to the midpoint $x \mapsto \frac{1}{2} - x$.

(4) Half-line $\mathbb{R}_+$ admits no non-trivial involution.

50.7 Classify involutions up to equivalence on an arbitrary 1-manifold.

51. Two-Dimensional Manifolds: General Picture

Examples of 2-manifolds and various information about 2-manifolds are scattered throughout the whole book. Here we repeat the most important of this information and then formulate the main results, postponing technical proofs to forthcoming sections.

⌈51′1⌋ Examples

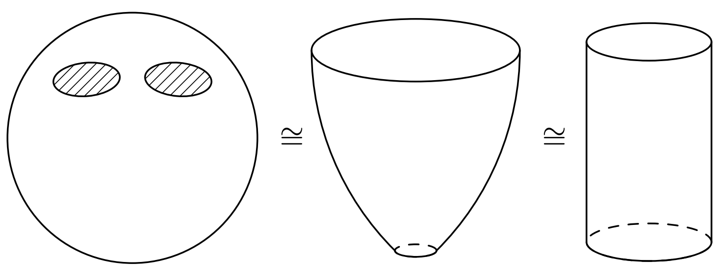
Recall (see Section 22'14) that by deleting from the torus $S^1 \times S^1$ the interior of an embedded closed disk, we obtain a space called *handle*. Similarly, by deleting from the 2-sphere the interior of n disjoint embedded disks, we obtain a *sphere with n holes*.

51.A *A sphere with a hole is homeomorphic to the disk D^2.*

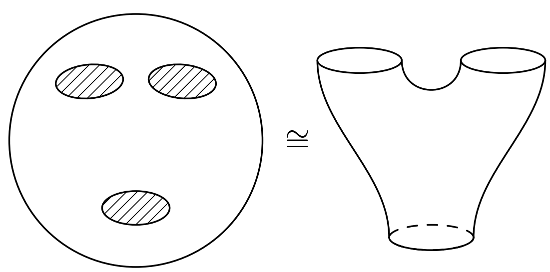
Proof. For example, the stereographic projection from an inner point of the hole maps the sphere with a hole onto a disk homeomorphically. □

51.B *A sphere with two holes is homeomorphic to the cylinder $S^1 \times I$.*

Proof. The stereographic projection from an inner point of one of the holes homeomorphically maps the sphere with two holes onto a *disk with a hole*. Prove that the latter is homeomorphic to a cylinder. (Another option: if we take the center of the projection in the hole in an appropriate way, then the projection maps the sphere with two holes onto a circular ring, which is obviously homeomorphic to a cylinder.) □



A sphere with three holes has a special name. It is called a *pair of pants* .

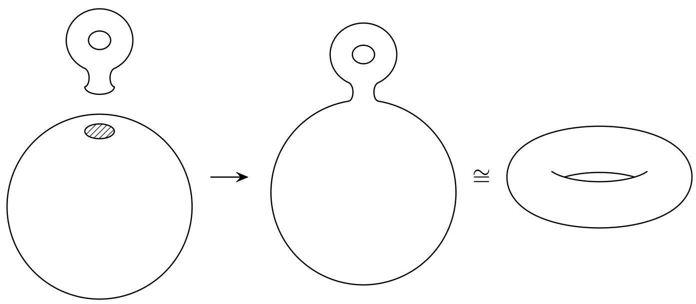


The result of attaching p copies of a handle to a sphere with p holes via embeddings homeomorphically mapping the boundary circles of the handles onto those of the holes is a *sphere with p handles*, or, in a more ceremonial way (and less understandable, for a while), an *orientable connected closed surface of genus p*.

51.1 Prove that a sphere with p handles is well-defined up to homeomorphism (i.e., the topological type of the result of gluing does not depend on the attaching embeddings).

51.C *A sphere with one handle is homeomorphic to the torus $S^1 \times S^1$.*

Proof. By definition, the handle is homeomorphic to a torus with a hole, while the sphere with a hole is homeomorphic to a disk, which precisely fills in the hole. □



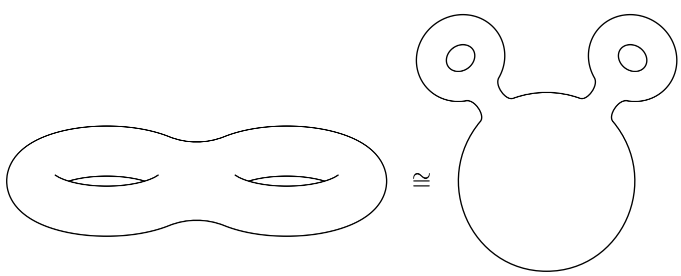
51.D *A sphere with two handles is homeomorphic to the result of gluing together two copies of a handle via the identity map of the boundary circle.*

Proof. Cut a sphere with two handles into two symmetric parts each of which is homeomorphic to a handle.

Another, more formal proof: a sphere with two holes is homeomorphic to the cylinder $S^1 \times I$, see 51.B. Attaching a cylinder by a homeomorphism of one of its boundary circles to a component C of a boundary of any 2-manifold M results a 2-manifold homeomorphic to the same M. Indeed, C has a collar $C \times I \subset M$ (see Section 48'4) and the result $S^1 \times I \cup_{h:S^1\times\{1\}\to C\times\{0\}} C \times I$ of attaching a cylinder $S^{\times} I$ to $C \times I$ is homeomorphic to $C \times I$ via the homeomorphism.

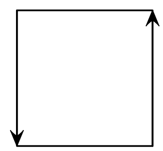
$$S^1 \times I \cup_{h:S^1\times\{1\}\to C\times\{0\}} C \times I \to C \times I$$

which is defined on $S^1 \times I$ by $(x, t) \mapsto (h(x), \frac{1}{2}t)$ and on $C \times I$ by $(x, t) \mapsto (x, \frac{1}{2}(t + 1))$. Observe that the restriction of this homeomorphism to $C \times \{1\}$ is the identity. Therefore we can extend it to a homeomorphism $S^1 \times I \cup_h M \to M$ by the identity on $M \smallsetminus (C \times I)$. □



A sphere with two handles is called a ***pretzel***. Sometimes, this word also denotes a sphere with more handles.

The ***Möbius strip*** or ***Möbius band*** is defined as $I^2/[(0,t) \sim (1,1-t)]$. In other words, this is the quotient space of the square I^2 by the partition into centrally symmetric pairs of points on the vertical edges of I^2, and singletons that do not lie on the vertical edges. The Möbius strip is obtained, so to speak, by identifying the vertical sides of a square in such a way that the directions shown on them by arrows are superimposed:

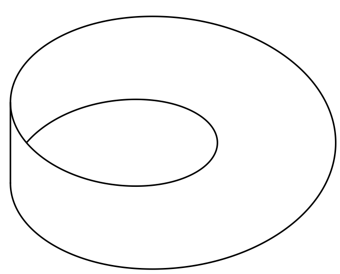


51.E *Prove that the Möbius strip is homeomorphic to the surface that is swept in $\mathbb{R}^3$ by a segment rotating in a half-plane around the midpoint, while the half-plane rotates around its boundary line. The ratio of the angular velocities of these rotations is such that the rotation of the half-plane through $360°$ takes the same time as the rotation of the segment through $180°$. See Figure.*

Proof. To simplify the formulas, we replace the square I^2 by a rectangle. Here is a formal argument: consider the map

$$\varphi : [0, 2\pi] \times [-\tfrac{1}{2}, \tfrac{1}{2}] \to \mathbb{R}^3 : (x, y) \mapsto \big((1 + y \sin \tfrac{x}{2}) \cos x, (1 + y \sin \tfrac{x}{2}) \sin x, y \cos \tfrac{x}{2}\big).$$

Check that φ really maps the square onto the Möbius strip and that $S(\varphi)$ is the given partition. Certainly, the starting point of the argument is not a specific formula. First of all, you should imagine the required map. We map the horizontal midline of the unit square onto the mid-circle of the Möbius strip, and we map each of the vertical segments of the square onto a segment of the strip orthogonal to the mid-circle. This mapping maps the vertical sides of the square to one and the same segment, but here the opposite vertices of the square are identified with each other (check this). □



The space obtained from a sphere with q holes by attaching q copies of the Möbius strip via embeddings of the boundary circles of the Möbius strips onto the boundary circles of the holes (the boundaries of the holes) is a *sphere with q cross-caps*, or a *non-orientable connected closed surface of genus q*.

51.2 Prove that a sphere with q cross-caps is well defined up to homeomorphism (i.e., the topological type of the result of gluing does not depend on the attaching embeddings).

51.F *A sphere with a cross-cap is homeomorphic to the projective plane.*

Proof. Combine the results of 51.A and 22.J. □

Klein bottle is $I^2/[(t,0) \sim (t,1),\ (0,t) \sim (1,1-t)]$. In other words, this is the quotient space of square I^2 by the partition into

- singletons in its interior,
- pairs of points $(t,0), (t,1)$ on horizontal edges that lie on the same vertical line,
- pairs of points $(0,t), (1,1-t)$ symmetric with respect to the center of the square that lie on the vertical edges, and
- the quadruple of vertices.

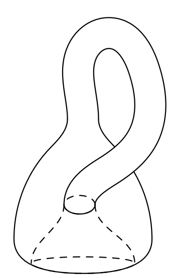
51.3 Present the Klein bottle as a quotient space of

(1) a cylinder;
(2) the Möbius strip.

51.4 Prove that $S^1 \times S^1/[(z,w) \sim (-z,\bar{w})]$ is homeomorphic to the Klein bottle. (Here $\bar{w}$ denotes the complex number conjugate to w.)

51.5 Embed the Klein bottle in $\mathbb{R}^4$ (cf. 51.E and 51.3).

51.6 Embed the Klein bottle in $\mathbb{R}^4$ so that the image of this embedding under the orthogonal projection $\mathbb{R}^4 \to \mathbb{R}^3$ would look as follows:



51.G *A sphere with two cross-caps is homeomorphic to the Klein bottle.*

Proof. Consider the Klein bottle as a quotient space of a square and cut the square into 5 horizontal (rectangular) strips of equal width. Then the quotient space of the middle strip is a Möbius band, the quotient space of the union of the two extreme strips is one more Möbius band, and the quotient space of the remaining two strips is a ring, i.e., precisely a sphere with two holes. (Here is another, maybe more visual, description. Look at the picture of the Klein bottle: it has a horizontal plane of symmetry. Two horizontal planes close to the plane of symmetry cut the Klein bottle into two Möbius bands and a ring.) □

A sphere, spheres with handles, and spheres with cross-caps are *basic surfaces*.

51.H *Prove that a sphere with p handles and q cross-caps is homeomorphic to a sphere with $2p+q$ cross-caps (here $q>0$).*

Proof. The most visual approach here is as follows: single out one of the handles and one of the cross-caps. Replace the handle by a "tube" whose boundary circles are attached to those of two holes on the sphere, which should be sufficiently small and close to each other. After that, start moving one of the holes. (The topological type of the quotient space does not change in the course of such a motion.) First, bring the hole to the boundary of the Möbius strip, then shift it onto the Möbius strip, drag it once along the Möbius strip, shift it from the Möbius strip, and, finally, return the hole to the initial spot. As a result, we transform the initial handle (a torus with a hole) into a Klein bottle with a hole, which splits into two Möbius strips (see Problem 22.U). □

51.7 Classify up to homeomorphism those spaces which are obtained by attaching p copies of $S^1 \times I$ to a sphere with $2p$ holes via embeddings of the boundary circles of the cylinders onto the boundary circles of the sphere with holes.

51.I What connected 2-manifolds do you know?

(1) List *closed* connected 2-manifold that you know.

(2) Do you know a connected *compact* 2-manifold, which is not closed?

(3) What *non-compact* connected 2-manifolds do you know?

(4) Is there a *non-compact* connected 2-manifolds with non-empty boundary?

51.8 Construct non-homeomorphic non-compact connected manifolds of dimension two without boundary and with isomorphic infinitely generated fundamental group.

For notions relevant to this problem see what follows.

⌈51′2⌋ Ends and Odds

Let X be a non-compact Hausdorff topological space, which is a union of an increasing sequence of its compact subspaces

$$C_1 \subset C_2 \subset \dots \subset C_n \subset \dots \subset X.$$

Each connected component U of $X \setminus C_n$ is contained in some connected component of $X \setminus C_{n-1}$. A decreasing sequence $U_1 \supset U_2 \supset \dots \supset U_n \supset \dots$ of connected components of

$$(X \setminus C_1) \supset (X \setminus C_2) \supset \dots \supset (X \setminus C_n) \supset \dots$$

respectively is called an *end of X with respect to $C_1 \subset \dots \subset C_n \subset \dots$*.

51.9 Let X and C_n be as above, D be a compact set in X and V a connected component of $X \setminus D$. Prove that there exists n such that $D \subset C_n$.

51.10 Let X and C_n be as above, D_n be an increasing sequence of compact sets of X with $X = \cup_{n=1}^{\infty} D_n$. Prove that for any end $U_1 \supset \dots \supset U_n \supset \dots$ of X with respect to C_n there exists a unique end $V_1 \supset \dots \supset V_n \supset \dots$ of X with respect to D_n such that for any p there exists q such that $V_q \subset U_p$.

51.11 Let X, C_n and D_n be as above. Then the map of the set of ends of X with respect to C_n to the set of ends of X with respect to D_n defined by the statement of **??** is a bijection.

Theorem 51.11 allows one to speak about *ends* of X without specifying a system of compact sets

$$C_1 \subset C_2 \subset \dots \subset C_n \subset \dots \subset X$$

with $X = \cup_{n=1}^{\infty} C_n$. Indeed, 51.10 and 51.11 establish a canonical one-to-one correspondence between ends of X with respect to any two systems of this kind.

51.12 Prove that $\mathbb{R}^1$ has two ends, $\mathbb{R}^n$ with $n > 1$ has only one end.

51.13 Find the number of ends for the universal covering space of the bouquet of two circles.

51.14 Does there exist a 2-manifold with a finite number of ends which cannot be embedded into a compact 2-manifold?

Proof. Yes, for example, a plane with infinite number of handles. □

51.15 Prove that for any compact set $K \subset S^2$ with connected complement $S^2 \setminus K$ there is a natural map of the set of ends of $S^2 \setminus K$ to the set of connected components of K.

Let W be an open set of X. The set of ends $U_1 \supset \dots \supset U_n \supset \dots$ of X such that $U_n \subset W$ for sufficiently large n is said to be *open.*

51.16 Prove that this defines a topological structure in the set of ends of X.

The set of ends of X equipped with this topological structure is called the *space of ends* of X. Denote this space by $\mathcal{E}(X)$.

51.8.1 Construct non-compact connected manifolds of dimension two without boundary and with isomorphic infinitely generated fundamental group, but with non-homeomorphic spaces of ends.

51.8.2 Construct non-compact connected manifolds of dimension two without boundary and with isomorphic infinitely generated fundamental group, but with different number of ends.

51.8.3 Construct non-compact connected manifolds of dimension two without boundary with isomorphic infinitely generated fundamental group and the same number of ends, but with different topology in the space of ends.

51.8.4 Let K be a completely disconnected closed set in S^2. Prove that the map $\mathcal{E}(S^2 \setminus K) \to K$ defined in 51.15 is continuous.

51.8.5 Construct a completely disconnected closed set $K \subset S^2$ such that this map is a homeomorphism.

51.17 Prove that there exists an uncountable family of pairwise nonhomeomorphic connected 2-manifolds without boundary.

The examples of non-compact manifolds dimension 2 presented above show that there are too many non-compact connected 2-manifolds. This makes impossible any really useful topological classification of non-compact 2-manifolds.

Theorems reducing the homeomorphism problem for 2-manifolds of this type to the homeomorphism problem for their spaces of ends do not seem to be useful: spaces of ends look not much simpler than the surfaces themselves.

However, there is a special class of non-compact 2-manifolds, which admits a simple and useful classification theorem. This is the class of *simply connected* non-compact 2-manifolds without boundary. We postpone its consideration to section 54′4. Now we turn to the case, which is the simplest and most useful for applications.

⌈51′3⌋ Homeomorphism and Homotopy Classifications of Basic Surfaces

51.J *The fundamental group of a sphere with g handles admits the following presentation:*

$$\langle a_1, b_1, a_2, b_2, \dots a_g, b_g \mid a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \dots a_g b_g a_g^{-1} b_g^{-1} = 1 \rangle.$$

Proof. See, for example, Section 46. □

51.K *The fundamental group of a sphere with g cross-caps admits the following presentation:*

$$\langle a_1, a_2, \dots a_g \mid a_1^2 a_2^2 \dots a_g^2 = 1 \rangle.$$

Proof. See, for example, Section 46. □

51.L *Spheres with different numbers of handles have non-isomorphic fundamental groups.*

When we want to prove that two finitely presented groups are not isomorphic, one of the first natural moves is to abelianize the groups. (Recall that to *abelianize* a group G means to quotient G out by the commutator subgroup. The commutator subgroup $[G, G]$ is the normal subgroup generated by the commutators $a^{-1}b^{-1}ab$ for all $a, b \in G$. Abelianization means adding relations $ab = ba$ for any $a, b \in G$.)

Abelian finitely generated groups are well known. Any finitely generated Abelian group is isomorphic to a product of a finite number of cyclic groups. If the abelianized groups are not isomorphic, then the original groups are not isomorphic as well.

51.L.1 *The abelianized fundamental group of a sphere with g handles is a free Abelian group of rank $2g$ (i.e., is isomorphic to $\mathbb{Z}^{2g}$).*

Proof. Indeed, the single relation in the fundamental group of the sphere with g handles means that the product of g commutators of the generators a_i and b_i equals 1, and so it "vanishes" after the abelianization. □

51.M *Fundamental groups of spheres with different numbers of cross-caps are not isomorphic.*

51.M.1 *The abelianized fundamental group of a sphere with g cross-caps is isomorphic to $\mathbb{Z}^{g-1} \times \mathbb{Z}_2$.*

Proof. Taking the elements $a_1, \dots, a_{g-1}$, and $b = a_1 a_2 \dots a_g$ as generators in the commuted group, we obtain an Abelian group with g generators and a single relation $b^2 = 1$. □

51.N Homotopy Classification of Basic Surfaces.

Spheres with different numbers of handles are not homotopy equivalent.

Spheres with different numbers of cross-caps are not homotopy equivalent.

A sphere with handles is not homotopy equivalent to a sphere with cross-caps.

Proof. The first statement follows from 51.L.1, the second from 51.M.1 and the third one, from 51.L.1 and 51.M.1. □

If X is a path-connected space, then the abelianized fundamental group of X is the 1-*dimensional* (or *first*) *homology group* of X and denoted by $H_1(X)$. If X is not path-connected, then $H_1(X)$ is the direct sum of the first homology groups of all path-connected components of X. Thus 51.L.1 can be rephrased as follows: if F_g is a sphere with g handles, then $H_1(F_g) = \mathbb{Z}^{2g}$.

⌈51′4⌋ Closed Surfaces

51.O *Any connected closed manifold of dimension two is homeomorphic either to sphere S^2, or sphere with handles, or sphere with crosscaps.*

Recall that according to Theorem 51.N the basic surfaces represent pairwise distinct topological (and even homotopy) types. Therefore, 51.N and 51.O together give topological and homotopy classifications of closed two-dimensional manifolds.

We do not recommend to have a try at proving Theorem 51.O immediately and, especially, in the form given above. All known proofs of 51.O can be decomposed into two main stages: firstly, a manifold under consideration is equipped with some additional structure (like triangulation or smooth structure); then using this structure a required homeomorphism is constructed. Although the first stage appears in the proof necessarily and is rather difficult, it is not useful outside the proof. Indeed, any closed 2-manifold, which we meet in a concrete mathematical context, is either equipped, or can be easily equipped with the additional structure. The methods of imposing the additional structure are much easier, than a general proof of existence for such a structure in an arbitrary 2-manifold.

Therefore, we suggest for the first case to restrict ourselves to the second stage of the proof of Theorem 51.O, prefacing it with general notions related to the most classical additional structure, which can be used for this purpose.

⌈51′5⌋ Compact Surfaces with Boundary

As in the case of one-dimensional manifolds, classification of compact two-dimensional manifolds with boundary can be easily reduced to the classification of closed manifolds. In the case of one-dimensional manifolds it was very useful to double a manifold. In two-dimensional case there is a construction providing a closed manifold related to a compact manifold with boundary even closer than the double.

51.P *Contracting to a point each connected component of the boundary of a two-dimensional compact manifold with boundary gives rise to a closed two-dimensional manifold.*

51.18 A space homeomorphic to the quotient space of 51.P can be constructed by attaching copies of D^2 one to each connected component of the boundary.

51.Q *Any connected compact manifold of dimension 2 with nonempty boundary is homeomorphic either to sphere with holes, or sphere with handles and holes, or sphere with crosscaps and holes.*

52. Triangulations

⌈52′1⌋ Triangulations of Surfaces

By an *Euclidean triangle* we mean the convex hall of three non-collinear points of Euclidean space. Of course, it is homeomorphic to disk D^2, but it is not solely the topological structure that is relevant now. The boundary of a triangle contains three distinguished points, its *vertices*, which divide the boundary into three pieces, its *edges*. A *topological triangle* in a topological space X is an embedding of an Euclidean triangle into X. A *vertex* (respectively, *edge*) of a topological triangle $T \to X$ is the image of a vertex (respectively, an edge) of T in X.

A set of topological triangles in a 2-manifold X is a *triangulation* of X provided the images of these triangles form a fundamental cover of X and any two of the images either are disjoint or intersect in a common side or in a common vertex.

52.A Prove that in the case of compact X the former condition (about fundamental cover) means that the number of triangles is finite.

52.B Prove that the condition about fundamental cover means that the cover is locally finite.

52.C Triangulation as a Cellular Decomposition. *A triangulation of a 2-manifold turns it into a cellular space, 0-cells of which are the vertices of all triangles of the triangulation, 1-cells are the sides of the triangles, and 2-cells are the interiors of the triangles.* □

This statement allows us to apply all the terms introduced above for cellular spaces. In particular, we can speak about skeletons, cellular subspaces and cells. However, in the latter two cases we rather use terms *triangulated subspace* and *simplex*. Triangulations and terminology related to them appeared long before cellular spaces. Therefore in this context the adjective *cellular* is replaced usually by adjectives *triangulated* or *simplicial*.

⌈52′2⌋ Two Properties of Triangulations of Surfaces

52.D Unramified. *Let E be a side of a triangle involved into a triangulation of a 2-manifold X. Then there exist at most two triangles of this triangulation for which E is a side.*

HINT. Cf. 47.F.1, 47.F.2 and 47.N. □

52.E Local strong connectedness. *Let V be a vertex of a triangle involved into a triangulation of a 2-manifold X and T, T' be two triangles of the triangulation adjacent to V. Then there exists a sequence $T = T_1, T_2, \dots, T_n = T'$ of triangles of the triangulation such that V is a vertex of each of them and triangles T_i, T_{i+1} have common side for each $i = 1, \dots, n-1$.*

Proof. Consider the union S of all the triangles in the triangulation adjacent to V. The statement that we are going to prove means that $S \smallsetminus \{V\}$ is connected. Assume the contrary. The interior $\operatorname{Int} S$ of S is a neighborhood of V. Removing V makes it non-connected. Moreover, in this neighborhood there is a base of neighborhoods of V with the same property. It consists of the results of contracting $\operatorname{Int} S$ towards V using the geometry of the triangles.

On the other hand, since X is a locally Euclidean space of dimension 2, V has a base of neighborhoods each of which is not separated by V. The arguments used in the proofs of 47.E and 47.M lead to a contradiction. □

⌈52′3⌋ Homotopy Type of Compact Surface with Non-Empty Boundary

52.F *Any compact connected triangulated 2-manifold with non-empty boundary collapses*[1] *to a one-dimensional simplicial subspace.*

Proof. A triangle adjacent to the boundary can be collapsed through its edge which lies on the boundary. If after some chain of elementary collapses of pairs edge-triangle the triangulation would still contain triangles, then the union of these triangles would be a 2-manifold without boundary, because each edge in the union would belong to two triangles - more than two is prohibited by 52.D, less (one triangle) would allow a further collapse. This union would be a closed 2-manifold. Hence it would be closed subset of the original manifold. By 52.D and 52.E it would be also open. By connectedness of the original 2-manifold, it would coincide with it, but this would contradict to the assumption of non-empty boundary. □

52.G Corollary. *Any compact connected triangulated 2-manifold with non-empty boundary is homotopy equivalent to a bouquet of circles.*

Proof. By Theorem 43.A, any connected finite 1-dimensional cellular space is homotopy equivalent to a bouquet of circles. □

⌈52′4⌋ The Euler Characteristic of a Compact Surface

52.H *The Euler characteristic of a triangulated compact connected 2-manifold X with non-empty boundary does not depend on triangulation. It is equal to $1 - r$, where r is the rank of the one-dimensional homology group of the 2-manifold.*

Proof. Collapsing does not change the Euler characteristic. Therefore, by Theorem 52.F it is equal to the Euler characteristic of its one-dimensional simplicial subspace. By Theorem 45.B, the fundamental group of this subspace (and the whole 2-manifold as they are homotopy equivalent) is free of rank $r = 1 - \chi(X)$. Thus $\chi(X) = 1 - r$. □

52.I Corollary. *The Euler characteristic of a triangulated compact connected 2-manifold with non-empty boundary is not greater than 1.* □

52.J Corollary. *The Euler characteristic of a triangulated closed connected 2-manifold with non-empty boundary is not greater than 2.* □

53. Topological Classification of Compact Surfaces

In this section we consider a classical solution of the topological classification problem for compact surfaces. We classify compact triangulated 2-manifolds

[1]For a definition of collapse see Section 44'2.

in a way which provides also an algorithm for building a homeomorphism between a given surface and one of the standard surfaces. Another proof is outlined in the last Section.

⌈53′1⌋ Families of Polygons

Triangulations provide a combinatorial description of 2-dimensional manifolds, but this description is usually too bulky. Here we will study other, more practical way to present 2-dimensional manifolds combinatorially. The main idea is to use larger building blocks.

Let $\mathcal{F}$ be a collection of convex polygons $P_1, P_2, \dots$. Let the sides of these polygons be oriented and paired off. Then we say that this is a *family of polygons*. There is a natural quotient space of the sum of polygons involved in a family: one identifies each side with its pair-mate by a homeomorphism, which respects the orientations of the sides. This quotient space is called just the *quotient of the family*.

53.A *The quotient of a family of polygons is a 2-manifold without boundary.* □

53.B *The topological type of the quotient of a family does not change when the homeomorphism between the sides of a distinguished pair is replaced by other homeomorphism which respects the orientations.*

Hint. See 49'5. □

53.C *Any triangulation of a 2-manifold gives rise to a family of polygons whose quotient is homeomorphic to the 2-manifold.*

Hint. Use Theorems 52.D and 52.E. □

A family of polygons can be described combinatorially: Assign a letter to each distinguished pair of sides. Go around the polygons writing down the letters assigned to the sides and equipping a letter with exponent -1 if the side is oriented against the direction in which we go around the polygon. At each polygon we write a word. The word depends on the side from which we started and on the direction of going around the polygon. Therefore it is defined up to cyclic permutation and inversion. The collection of words assigned to all the polygons of the family is called a *phrase associated with the family of polygons*. It describes the family to the extend sufficient to recovering the topological type of the quotient.

53.1 Prove that the quotient of the family of polygons associated with phrase $aba^{-1}b^{-1}$ is homeomorphic to $S^1 \times S^1$.

53.2 Identify the topological type of the quotient of the family of polygons associated with phrases

(1) aa^{-1};
(2) ab, ab;
(3) aa;
(4) $abab^{-1}$;
(5) $abab$;
(6) $abcabc$;
(7) $aabb$;
(8) $a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}\dots a_gb_ga_g^{-1}b_g^{-1}$;
(9) $a_1a_1a_2a_2\dots a_ga_g$.

53.D A collection of words is a phrase associated with a family of polygons, iff each letter appears twice in the words.

A family of polygons is called *irreducible* if the quotient is connected.

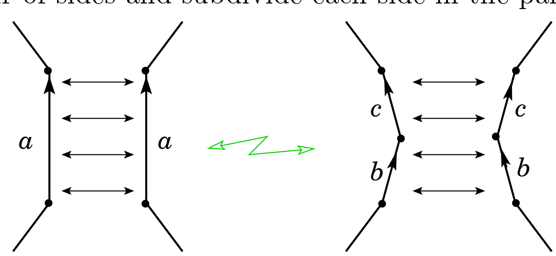
53.E A family of polygons is irreducible, iff a phrase associated with it does not admit a division into two collections of words such that there is no letter involved in both collections.

⌈53′2⌋ Operations on Family of Polygons

Although any family of polygons defines a 2-manifold, there are many families defining the same 2-manifold. There are simple operations which change a family, but do not change the topological type of the quotient of the family. Here are the most obvious and elementary of these operations.

(1) Simultaneous reversing orientations of sides belonging to one of the pairs.

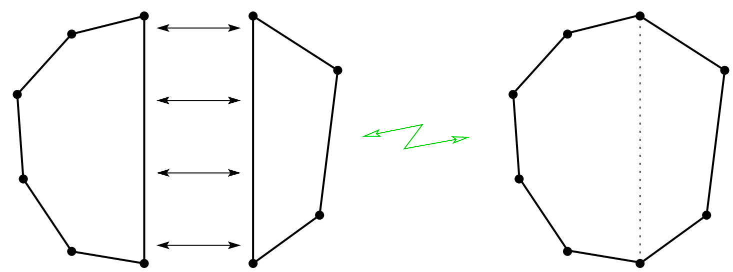
(2) Select a pair of sides and subdivide each side in the pair into two



sides. The orientations of the original sides define the orderings of the halves. Unite the first halves into one new pair of sides, and the second halves into the other new pair. The orientations of the original sides define in an obvious way orientations of their halves. This operation is called 1-*subdivision.* In the quotient it effects in subdivision of a 1-cell (which is the image of the selected pair of sides) into two 1-cells. This 1-cells is replaced by two 1-cells and one 0-cell.

(3) The inverse operation to 1-subdivision. It is called 1-*consolidation.*

(4) Cut one of the polygons along its diagonal into two polygons.



The sides of the cut constitute a new pair. They are equipped with an orientation such that gluing the polygons by a homeomorphism respecting these orientations recovers the original polygon. This operation is called 2-*subdivision.* In the quotient it effects in subdivision of a 2-cell into two new 2-cells along an arc whose end-points are 0-cells (may be coinciding). The original 2-cell is replaced by two 2-cells and one 1-cell.

(5) The inverse operation to 2-subdivision. It is called 2-*consolidation.*

⌈53′3⌋ Topological and Homotopy Classification of Closed Surfaces

53.F Reduction Theorem. *Any finite irreducible family of polygons can be reduced by the five elementary operations to one of the following standard families:*

(1) aa^{-1}

(2) $a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}\dots a_gb_ga_g^{-1}b_g^{-1}$

(3) $a_1a_1a_2a_2\dots a_ga_g$ *for some natural* g.

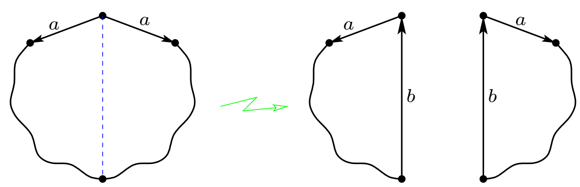
53.G Corollary, see 51.O. *Any triangulated closed connected manifold of dimension 2 is homeomorphic to either sphere, or sphere with handles, or sphere with crosscaps.*

Theorems 53.G and 51.N provide classifications of triangulated closed connected 2-manifolds up to homeomorphisms and homotopy equivalence.

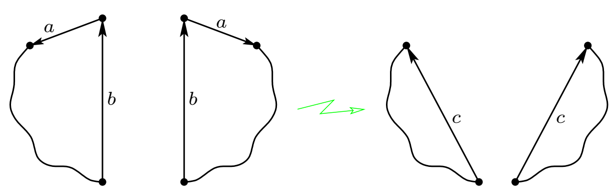
53.F.1 Reduction to Single Polygon. *Any finite irreducible family of n polygons can be reduced by a sequence of $n-1$ 2-consolidations to a family consisting of a single polygon.* □

53.F.2 Cancellation. *A family of polygons consisting of a single polygon that has at least 4 sides and corresponds to a word containing a fragment aa^{-1} or $a^{-1}a$, where a is any letter, can be transformed by a 2-subdivision followed by 1-consolidation and 2-consolidation to a a family corresponding to the word obtained from the original one by erasing this fragment.*

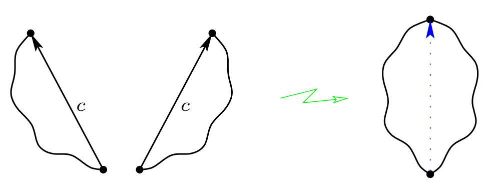
Proof. Since the number of sides is at least 4, the polygon has a diagonal which connects the vertex separating the sides a and a^{-1} and a vertex which is not adjacent to any of these sides. Cut the polygon along this diagonal (i.e., perform a 2-subdivision along it).



Now make 1-consolidation uniting the new sides which came from the diagonal with the sides that we want to eliminate:



Finally, make 2-consolidation:



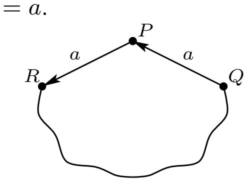
□

53.F.3 Reduction to Single Vertex. *An irreducible family of polygons consisting of a single polygon with $r \geq 4$ sides to which the cancellation procedure of Lemma 53.F.2 cannot be applied any more can be turned by elementary transformations to a polygon such that all its vertices are projected to a single point of the quotient.*

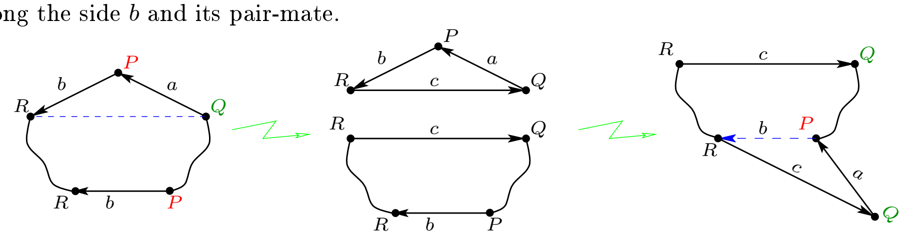
Proof. Assign different letters to the images of vertices in the quotient space, and assign the same letters to the vertices. So, the vertices of the polygon projected to the same point are assigned with the same letter, and vertices projected to different points are assigned with different letters.

If there are vertices projected to different points, then there is a side with vertices which are assigned with different letters, say P and Q. Below we describe a procedure which decreases by 1 the number of vertices with letter P and increases by 1 the number of vertices with letter Q. Denote the side PQ by a. Let $b = PR$ be the other side adjacent to P.

The sides a and b are not pair-mates. Indeed, b cannot be a^{-1}, because by the assumption the procedure of Lemma 53.F.2 cannot be applied. On the other hand, b cannot be identified under factorization with a, because in such a case Q being an initial point of the side $QP = a$ were projected to the same point as P which is the initial point of the next side $PR = a$.



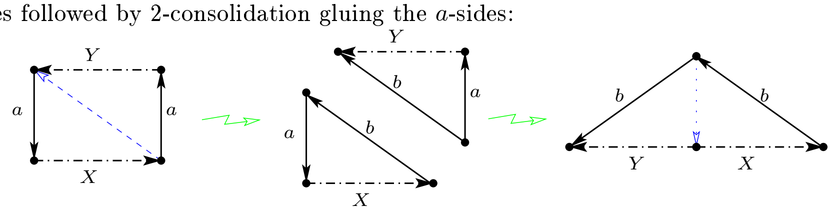
Draw the diagonal $d = QR$ and make 2-subdivision along it. Then make 2-consolidation along the side b and its pair-mate.



As promised, the number of P-vertices decreased by 1, while the number of Q-vertices increased by 1. By applying this procedure, and, under each opportunity, applying the procedure of Lemma 53.F.2, one can make all the vertices projecting to the same point. □

53.F.4 Separation of a Crosscap. A family corresponding to a phrase consisting of a single word $XaYa$, where X and Y are words and a is a letter, can be transformed to the family corresponding to the word $bbY^{-1}X$.

Proof. It is done by a 2-subdivision along the diagonal connecting the initial points of the a-sides followed by 2-consolidation gluing the a-sides:

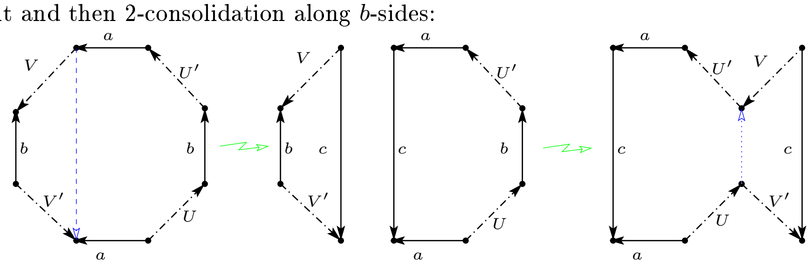


□

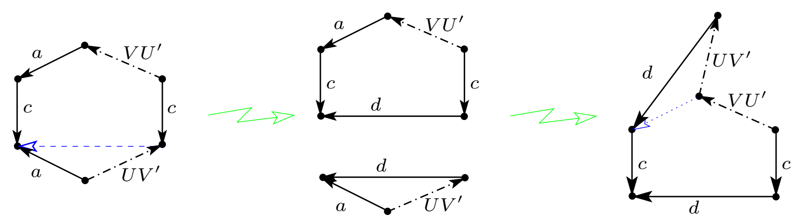
53.F.5 If a family corresponds to a phrase consisting of a word $XaYa^{-1}$, where X and Y are nonempty words and a is a letter, the quotient of this family has a single vertex in the natural cell decomposition, and there is no possibility for applying the procedure of Lemma 53.F.4, then $X = UbU'$ and $Y = Vb^{-1}V'$.

53.F.6 Separation of Handle. A family corresponding to a phrase consisting of a word $UbU'aVb^{-1}V'a^{-1}$, where U, U', V, and V' are words and a, b are letters, can be transformed to the family presented by phrase $dcd^{-1}c^{-1}UV'VU'$.

Proof. First, draw the diagonal connecting the end points of the a-sides, make 2-subdivision along it and then 2-consolidation along b-sides:



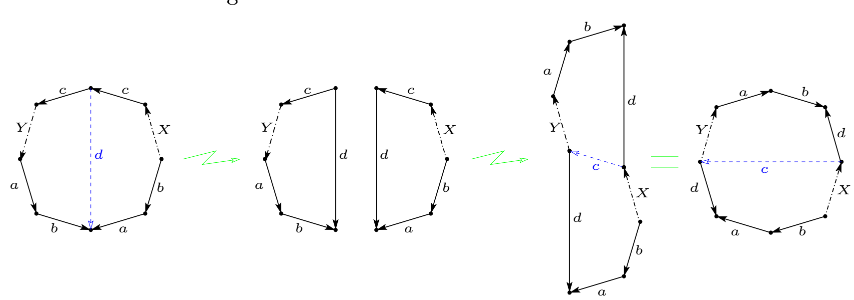
In the result draw the diagonal connecting the end points of the new pair of sides, make 2-subdivision along the diagonal and 2-consolidation along the a-sides.



□

53.F.7 Handle plus Crosscap Equals 3 Crosscaps. A family corresponding to phrase $aba^{-1}b^{-1}XccY$ can be transformed by elementary transformations to the family corresponding to phrase $uuvvwwY^{-1}X^{-1}$.

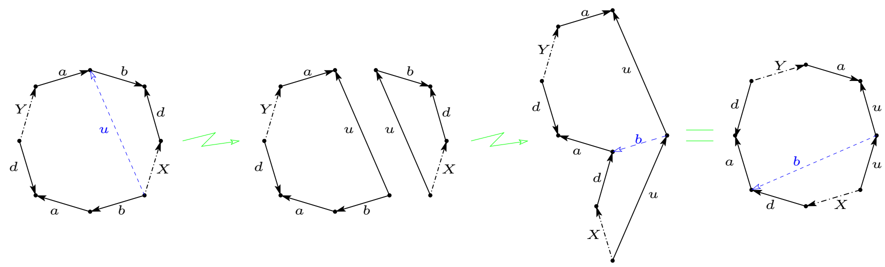
Proof. Eight elementary transformations are needed. We present them in four pairs. Each pair is a 2-subdivision followed by a 2-consolidation. We conclude each pair with reshaping of the resulted non-convex octagon into a nice convex form. First we make 2-subdivision along the diagonal connecting the middle vertices of the part cc representing the Möbius band and the part $aba^{-1}b^{-1}$ representing the handle. Then we make 2-consolidation along c.



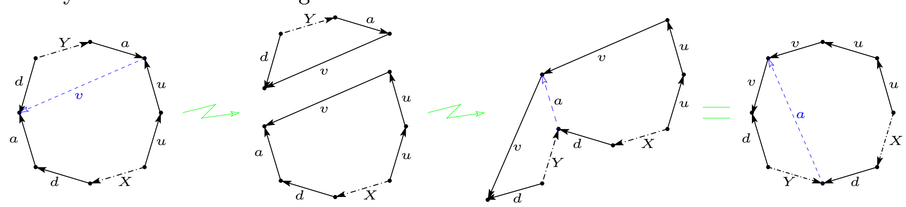
We have obtained three pairs of sides, a, b and d, oriented coherently. The rest of the proof consists of three applications of the crosscap separation from the proof of

Lemma 53.F.4. We do need to separate three crosscaps. However, we cannot just to refer to Lemma 53.F.4, because its application could destroy pairs of sides needed for the next application. It requires at least special proof that one can avoid this. Instead, we just separate 3 crosscaps thoughtlessly.

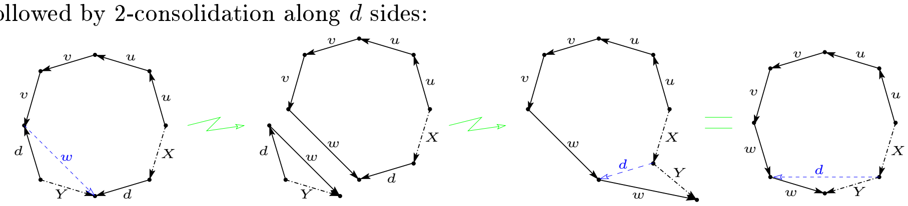
First, we draw the diagonal connecting the initial points of b sides, make 2-subdivision along it, and then 2-consolidation along b sides.



Connect the arrowheads of a sides with diagonal and make 2-subdivision along it followed by 2-consolidation along a sides.



Finally, make 2-subdivision along the diagonal connecting the arrowheads of d sides followed by 2-consolidation along d sides:



□

54. Recognizing Closed Surfaces

54.A What is the topological type of the 2-manifold, which can be obtained as follows: Take two disjoint copies of disk. Attach three parallel strips connecting the disks and twisted by π. The resulting surface S has a connected boundary. Attach a copy of disk along its boundary by a homeomorphism onto the boundary of the S. This is the space to recognize.

54.B *Euler characteristic of the cellular space obtained as quotient of a family of polygons is invariant under homotopy equivalences.*

54.1 How can 54.B help to solve 54.A?

54.2 Let X be a closed connected surface. What values of $\chi(X)$ allow to recover the topological type of X? What ambiguity is left for other values of $\chi(X)$?

⌈54′1⌋ Orientations

By an *orientation of a polygon* one means orientation of all its sides such that each vertex is the final end point for one of the adjacent sides and initial for the other one. Thus an orientation of a polygon includes orientation of all its sides. Each segment can be oriented in two ways, and each polygon can be oriented in two ways.

An orientation of a family of polygons is a collection of orientations of all the polygons comprising the family such that for each pair of sides one of the pair-mates has the orientation inherited from the orientation of the polygon containing it while the other pair-mate has the orientation opposite to the inherited orientation. A family of polygons is said to be *orientable* if it admits an orientation.

54.3 Which of the families of polygons from Problem 53.2 are orientable?

54.4 Prove that a family of polygons associated with a word is orientable iff each letter appear in the word once with exponent -1 and once with exponent 1.

54.C *Orientability of a family of polygons is preserved by the elementary operations.*

A surface is said to be *orientable* if it can be presented as the quotient of an orientable family of polygons.

54.D A surface S is orientable, iff any family of polygons whose quotient is homeomorphic to S is orientable.

54.E Spheres with handles are orientable. Spheres with crosscaps are not.

⌈54′2⌋ More About Recognizing Closed Surfaces

54.5 How can the notion of orientability and 54.C help to solve 54.A?

54.F *Two closed connected manifolds of dimension two are homeomorphic iff they have the same Euler characteristic and either are both orientable or both non-orientable.*

⌈54′3⌋ Recognizing Compact Surfaces with Boundary

54.G Riddle. Generalize orientabilty to the case of nonclosed manifolds of dimension two. (Give as many generalization as you can and prove that they are equivalent. The main criterium of success is that the generalized orientability should help to recognize the topological type.)

54.H *Two compact connected manifolds of dimension two are homeomorphic iff they have the same Euler characteristic, are both orientable or both nonorientable and their boundaries have the same number of connected components.*

⌈54′4⌋ Simply Connected Surfaces

54.6 Theorem*. *Any simply connected non-compact manifold of dimension two without boundary is homeomorphic to* $\mathbb{R}^2$.

54′4.1 Any simply connected triangulated non-compact manifold without boundary can be presented as the union of an increasing sequence of compact simplicial subspaces $C_0 \supset C_1 \supset C_2 \supset \dots \supset C_n \supset \dots$ such that each of them is a 2-manifold with boundary and $\operatorname{Int} C_n \subset C_{n+1}$ for each n.

54′4.2 Under conditions of 54′4.1 the sequence C_n can be modified in such a way that each C_n becomes simply connected.

54.7 Corollary. *The universal covering of any surface with empty boundary and infinite fundamental group is homeomorphic to* R^2.

55. Combinatorics and Subdivisions of Triangulations

⌈55′1⌋ Scheme of Triangulation

Triangulations allow to describe a surface by a simple combinatorial object.

Let X be a 2-manifold and $\mathcal{T}$ a triangulation of X. Denote the set of vertices of $\mathcal{T}$ by V. Denote by Σ_2 the set of triples of vertices, which are vertices of a triangle of $\mathcal{T}$. Denote by Σ_1 the set of pairs of vertices, which are vertices of a side of $\mathcal{T}$. Put $\Sigma_0 = S$. This is the set of vertices of $\mathcal{T}$. Put $\Sigma = \Sigma_2 \cup \Sigma_1 \cup \Sigma_0$. The pair (V, Σ) is called the *(combinatorial) scheme* of $\mathcal{T}$.

55.1 Prove that the combinatorial scheme (V, Σ) of a triangulation of a 2-manifold has the following properties:

(1) Σ is a set consisting of subsets of V,
(2) each element of Σ consists of at most 3 elements of V,
(3) three-element elements of Σ cover V,
(4) any subset of an element of Σ belongs to Σ,
(5) intersection of any collection of elements of Σ belongs to Σ,

(6) for any two-element element of Σ there exist exactly two three-element elements of Σ containing it.

Recall that objects of this kind appeared above, in Section 24′3. Let V be a set and Σ is a set of finite subsets of V. The pair (V, Σ) is called a *triangulation scheme* if

- any subset of an element of Σ belongs to Σ,
- intersection of any collection of elements of Σ belongs to Σ,
- any one element subset of V belongs to Σ.

For any simplicial scheme (V, Σ) in 24′3 a topological space $S(V, \Sigma)$ was constructed. This is, in fact, a cellular space, see 42.14.

55.2 Prove that if (V, Σ) is the combinatorial scheme of a triangulation of a 2-manifold X then $S(V, \Sigma)$ is homeomorphic to X.

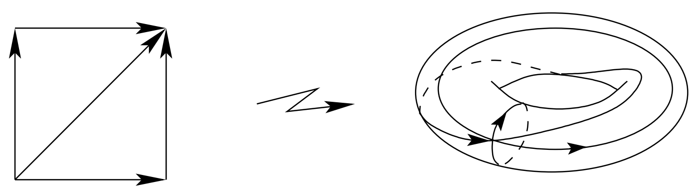
55.3 Let (V, Σ) be a triangulation scheme such that

(1) V is countable,
(2) each element of Σ consists of at most 3 elements of V,
(3) three-element elements of Σ cover V,
(4) for any two-element element of Σ there exist exactly two three-element elements of Σ containing it

Prove that (V, Σ) is a combinatorial scheme of a triangulation of a 2-manifold.

⌈55′2⌋ Examples

55.4 Consider the cover of torus obtained in the obvious way from the cover of the square by its halves separated by a diagonal of the square.



Is it a triangulation of torus? Why not?

55.5 Prove that the simplest triangulation of S^2 consists of 4 triangles.

55.6* *Prove that a triangulation of torus $S^1 \times S^1$ contains at least 14 triangles, and a triangulation of the projective plane contains at least 10 triangles.*

⌈55′3⌋ Subdivision of a Triangulation

A triangulation $\mathcal{S}$ of a 2-manifold X is said to be a ***subdivision*** of a triangulation $\mathcal{T}$, if each triangle of $\mathcal{S}$ is contained in some triangle[2] of $\mathcal{T}$. Then $\mathcal{S}$ is also called a ***refinement*** of $\mathcal{T}$.

[2] Although triangles which form a triangulation of X have been defined as topological embeddings, we hope that a reader guess that when one of such triangles is said to be contained in another one this means that the image of the embedding which is the former triangle is contained in the image of the other embedding which is the latter.

There are several standard ways to subdivide a triangulation. Here is one of the simplest of them. Choose a point inside a triangle τ, call it a new vertex, connect it by disjoint arcs with vertices of τ and call these arcs new edges. These arcs divide τ to three new triangles. In the original triangulation replace τ by these three new triangles. This operation is called a ***star subdivision centered at*** τ. See Figure 1.

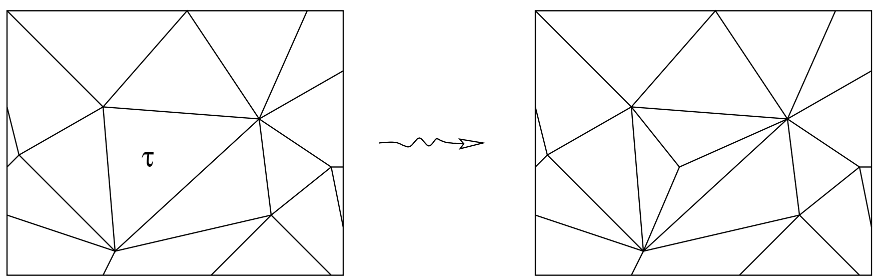


Figure 1. Star subdivision centered at triangle τ

55.7 Give a formal description of a star subdivision centered at a triangle τ. I.e., present it as a change of a triangulation thought of as a collection of topological triangles. What three embeddings of Euclidean triangles are to replace τ? Show that the replacement gives rise to a triangulation. Describe the corresponding operation on the combinatorial scheme.

Here is another subdivision defined locally. One adds a new vertex taken on an edge ε of a given triangulation. One connects the new vertex by two new edges to the vertices of the two tringles adjacent to ε. The new edges divide these triangles, each to two new triangles. The rest of triangles of the original triangulation are not affected. This operation is called a ***star subdivision centered at*** ε. See Figure 2.

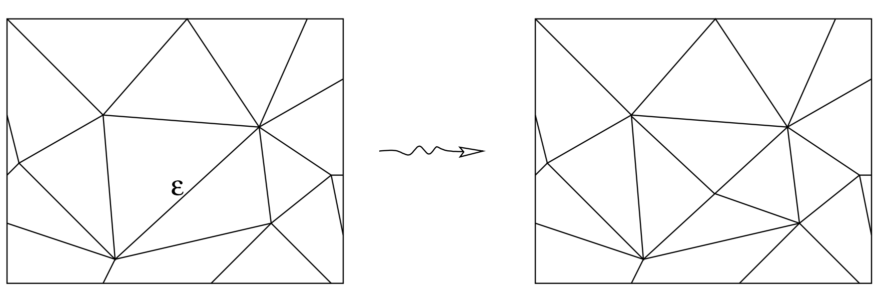


Figure 2. Star subdivision centered at edge ε.

55.8 Give a formal description of a star subdivision centered at edge ε. What four embeddings of Euclidean triangles are to replace the topological triangles with edge ε? Show that the replacement gives rise to a triangulation. Describe the corresponding operation on the combinatorial scheme.

55.9 Find a triangulation and its subdivision, which cannot be presented as a composition of star subdivisions at edges or triangles.

55.10* *Prove that any subdivision of a triangulation of a compact surface can be presented as a composition of a finite sequences of star subdivisions centered at edges or triangles and operations inverse to such subdivisions.*

By a *baricentric subdivision* of a triangle we call a composition of a star subdivision centered at this tringle followed by star subdivisions at each of its edges. See Figure 3.

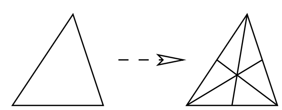


Figure 3. Baricentric subdivision of a triangle.

Baricentric subdivision of a triangulation of 2-manifold is a subdivision which is a simultaneous baricentric subdivision of all triangles of this triangulation. See Figure 4.

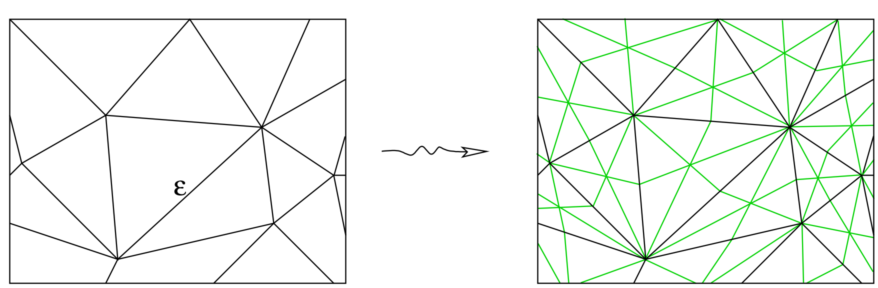


Figure 4. Baricentric subdivision of a triangulation.

55.A Establish a natural one-to-one correspondence between vertices of a baricentric subdivision a simplices (i.e., vertices, edges and triangles) of the original tringulation.

55.B Establish a natural one-to-one correspondence between triangles of a baricentric subdivision and triples each of which is formed of a triangle of the original triangulation, an edge of this triangle and a vertex of this edge.

The expression *baricentric subdivision* has appeared in a diiferent context, see Section 21. Let us relate the two notions sharing this name .

55.11 Baricentric subdivision of a triangulation and its scheme. Prove that the combinatorial scheme of the baricentric subdivision of a triangulation of a 2-manifold coincides with the baricentric subdivision of the scheme of the original triangulation (see 24′4).

⌈55′4⌋ Triangulations in dimension one

By an *Euclidean segment* we mean the convex hall of two different points of a Euclidean space. It is homeomorphic to I. A *topological segment* or *topological edge* in a topological space X is a topological embedding of an Euclidean segment into X. A set of topological segments in a 1-manifold X is a *triangulation* of X if the images of these topological segments constitute a fundamental cover of X and any two of the images either are disjoint or intersect in one common end point.

Traingulations of 1-manifolds are similar to triangulations of 2-manifolds considered above.

55.C Find counter-parts for theorems above. Which of them have no counter-parts? What is a counter-part for the property 52.D? What are counter-parts for star and baricentric subdivisions?

55.D Find homotopy classification of triangulated compact 1-manifolds using arguments similar to the ones from Section 52′3. Compare with the topological classification of 1-manifolds obtained in Section 50.

55.E What values take the Euler characteristic on compact 1-manifolds?

Proof. All non-negative inetegers. □

55.F What is relation of the Euler characteristic of a compact triangulated 1-manifold X and the number of points in ∂X?

Proof. $\chi(X) = \frac{1}{2}\chi(\partial X) = \frac{1}{2}\sharp(\partial X)$. To prove this, consider double DX of X, and observe that $\chi(DX) = 2\chi(X) - \chi(\partial X)$, while $\chi(DX) = 0$, since DX is a closed 1-manifold. □

55.G *Triangulation of a 2-manifold X gives rise to a triangulation of its boundary ∂X. Namely, the edges of the triangualtion of ∂X are the sides of triangles of the original triangulation which lie in ∂X.*

⌈55′5⌋ Triangualtions in higher dimensions

55.H Generalize everything presented above in this section to the case of manifolds of higher dimensions.

56. Handle Decompositions

⌈56′1⌋ Handles and Their Anatomy

Together with triangulations, it is useful to consider representations of a manifold as a union of balls of the same dimension, but adjacent to each other as if they were thickening of cells of a cellular space

A space $D^p \times D^{n-p}$ is called a *(standard) handle of dimension n and index p*. Its subset $D^p \times \{0\} \subset D^p \times D^{n-p}$ is called the *core* of handle $D^p \times D^{n-p}$, and a subset $\{0\} \times D^{n-p} \subset D^p \times D^{n-p}$ is called its cocore. The boundary $\partial(D^p \times D^{n-p}) =$ of the handle $D^p \times D^{n-p}$ can be presented as union of its *base* $D^p \times S^{n-p-1}$ and *cobase* $S^{p-1} \times D^{n-p}$.

56.A Draw all standard handles of dimensions ≤ 3.

A topological embedding h of the standard handle $D^p \times D^{n-p}$ of dimension n and index p into a manifold of the same dimension n is called a *handle of dimension n and index p*. The image under h of $\operatorname{Int} D^p \times \operatorname{Int} D^{n-p}$ is called the *interior* of h, the image of the core $h(D^p \times \{0\})$ of the standard handle is called the *core* of h, the image $h(\{0\} \times D^{n-p})$ of cocore, the *cocore*, etc.

⌈56′2⌋ Handle Decomposition of a Manifold

Let X be a manifold of dimension n. A collecton of n-dimensional handles in X is called a *handle decomposition of X*, if

(1) the images of these handles constitute a locally finite cover of X,

(2) the interiors of these handles are pairwise disjoint,

(3) the base of each of the handles is contained in the union of cobases of the handles of smaller indices.

Let X be a manifold of dimension n with boundary. A collection of n-dimensional handles in X is called a *handle decomposion of X modulo boundary*, if

(1) the images of these handles constitute a locally finite cover of X,

(2) the interiors of these handles are pairwise disjoint,

(3) the base of each of the handles is contained in the union of ∂X and cobases of the handles of smaller indices.

A composition of a handle $h : D^p \times D^{n-p} \to X$ with the homeomorphism of transposition of the factors $D^p \times D^{n-p} \to D^{n-p} \times D^p$ turns the handle h of index p into a handle of the same dimension n, but of the complementary index $n-p$. The core of the handle turns into the cocore, while the base, to cobase.

56.B Composing each handle with the homeomorphism transposing the factors turns a handle decomposition of manifold into a handle decomposition modulo boundary of the same manifold. Vice versa, a handle decomposition modulo boundary turns into a handle decomposition of the same manifold.

Handle decompositions obtained from each other in this way are said to be *dual* to each other.

56.C Riddle. For an n-manifold whose boundary splits into two $(n-1)$-manifolds with disjoint closures, define handle decomposition modulo one of these $(n-1)$-manifolds so that the dual handle decomposition would be modulo the complementary part of the boundary.

56.1 Find handle decompositions with a minimal number of handles for the following manifolds:

(a) circle S^1;	(b) sphere S^n;	(c) ball D^n
(d) torus $S^1 \times S^1$;	(e) handle;	(f) cylinder $S^1 \times I$;
(g) Möbius band;	(h) projective plane $\mathbb{R}P^2$;	(i) projective space $\mathbb{R}P^n$;
(j) sphere with p handles;	(k) sphere with p cross-caps;	(l) sphere with n holes.

⌈56′3⌋ Handle Decomposition and Triangulation

Let X be a 2-manifold, τ its triangulation, τ' its baricentric subdivision, and τ'' the baricentric subdivision of τ'. For each simplex S of τ denote by H_S the union of all simplices of τ'' which contain the unique vertex of τ' that lies in $\operatorname{int} S$. Thus, if S is a vertex then H_S is the union of all triangles of τ'' containing this vertex, if S is an edge then H_S is the union all of the triangles of τ'' which intersect with S but do not contain any of its vertices, and, finally, if S is a triangle of τ then H_S is the union of all triangles of τ'' which lie in S but do not intersect its boundary.

56.D Handle Decomposition out of a Triangulation. *Sets H_S constitute a handle decomposition of X. The index of H_S equals the dimension of S.*

56.2 Can every handle decomposition of a 2-manifold be constructed from a triangulation as indicated in 56.D?

56.3 How to triangulate a 2-manifold which is equipped with a handle decomposition?

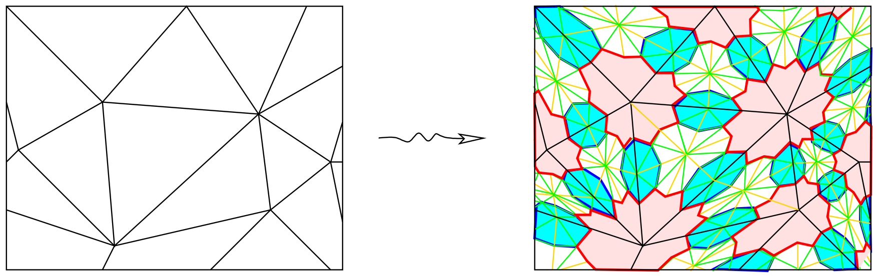


Figure 5. Construction of a handle decomposition from a triangulation.

⌈56′4⌋ Regular Neighborhoods

Let X be a 2-manifold, τ its triangulation, and A be a simplicial subspace of X. The union of all those simplices of the double baricentric subdivision τ'' of τ which intersect A is called the *regular* or *second baricentric neighborhood* of A (with respect to τ).

Of course, usually regular neighborhood is not open in X, since it is the union of simplices, which are closed. So, it is a neighborhood of A only in the broad sense (its interior contains A).

56.E *A regular neighborhood of A in X is a 2-manifold. It coincides with the union of handles corresponding to the simplices contained in A. These handles constitute a handle decomposition of the regular neighborhood.* □

56.F Collapse Induces Homemorphism. *Let X be a triangulated 2-manifold and $A \subset X$ be its triangulated subspace. If $X \searrow A$, then X is homeomorphic to a regular neighborhood of A.*

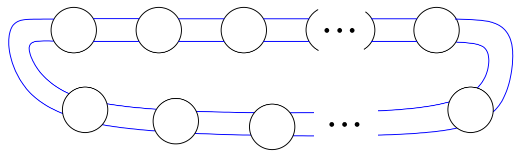
56.G Corollary. *In a triangulated 2-manifold, any triangulated subspace, which is a tree, has regular neighborhood homeomorphic to disk.* □

56.H Corollary. *Any triangulated compact connected 2-manifold with non-empty boundary is homeomorphic to a regular neighborhood of some of its 1-dimensional triangulated subspaces.*

Proof. This follows from Theorems 52.F and 56.F. □

56.I In a triangulated 2-manifold, any triangulated subspace homeomorphic to circle has a regular neighborhood homeomorphic either to the Möbius band or cylinder $S^1 \times I$.

Proof. The regular neighborhood is the union of a circular sequence of handles of indices 0 and 1. In this sequence the indices of handles alternate.



□

In the case of Möbius band the circle is said to be ***one-sided***, in the case of cylinder, ***two-sided***.

⌈56′5⌋ Cutting 2-Manifold Along a Curve

56.J Cut Along a Curve. Let F be a triangulated surface and $C \subset F$ be a compact one-dimensional manifold contained in the 1-skeleton of F and satisfying condition $\partial C = \partial F \cap C$. Prove that there exists a 2-manifold T and surjective map $p : T \to F$ such that:

(1) $p| : T \smallsetminus p^{-1}(C) \to F \smallsetminus C$ is a homeomorphism,

(2) $p| : p^{-1}(C) \to C$ is a two-fold covering.

56.K Uniqueness of Cut. The 2-manifold T and map p which exist according to Theorem 56.J, are unique up to homeomorphism: if $\tilde{T}$ and $\tilde{p}$ are other 2-manifold and map satisfying the same hypothesis then there exists a homeomorphism $h : \tilde{T} \to T$ such that $p \circ h = \tilde{p}$.

The 2-manifold T described in 56.J is called the result of ***cutting of*** F ***along*** C. It is denoted by F ✂ C. This is not at all the complement $F \smallsetminus C$, although a copy of $F \smallsetminus C$ is contained in F ✂ C as a dense subset homotopy equivalent to the whole F ✂ C.

56.L Triangulation of Cut Result. F ✂ C possesses a unique triangulation such that the natural map F ✂ $C \to F$ maps homeomorphically edges and triangles of this triangulation onto edges and, respectivly, triangles of the original triangulation of F.

56.M Let X be a triangulated 2-manifold, C be its triangulated subspace homeomorphic to circle, and let F be a regular neighborhood of C in X. Prove

(1) F ✂ C consists of two connected components, if C is two-sided on X, it is connected if C is one-sided;

(2) the inverse image of C under the natural map X ✂ $C \to X$ consists of two connected components if C is two-sided on X, it is connected if C is one-sided on X.

This proposition discloses the meaning of words *one-sided and two-sided* circle on a 2-manifold. Indeed, both connected components of the result of cutting of a regular neighborhood, and connected components of the inverse image of the circle can claim its right to be called a *side* of the circle or a *side of the cut.*

56.4 Describe the topological type of $F \between C$ for the following F and C:

(1) F is sphere S^2, and C is its equator;
(2) F is a Möbius strip, and C is its middle circle (deformation retract);
(3) $F = S^1 \times S^1$, $C = S^1 \times 1$;
(4) F is torus $S^1 \times S^1$ standardly embedded into $\mathbb{R}^3$, and C is the trefoil knot lying on F, that is $\{(z, w) \in S^1 \times S^1 \mid z^2 = w^3\}$;
(5) F is a Möbius strip, C is a segment: findtwo topologically different position of C on F and describe $F \between C$ for each of them;
(6) $F = \mathbb{R}P^2$, $C = \mathbb{R}P^1$.
(7) $F = \mathbb{R}P^2$, C is homeomorphic to circle: find two topologically different position C on F and describe $F \between C$ for each of them.

56.N Euler Characteristic and Cut. *Let F be a triangulated compact 2-manifold and $C \subset \operatorname{int} F$ be a closed one-dimensional contained in the 1-skeleton*

of the triangulation of F. Then $\chi(F \between C) = \chi F$.

56.O Find the Euler characteristic of $F \between C$, if $\partial C \neq \varnothing$.

56.P Generalized Cut (Incise). Let F be a triangulated 2-manifold and $C \subset F$ be a compact 1-dimensional manifold contained in 1-skeleton of F and satisfying condition $\partial F \cap C \subset \partial C$. Let $D = C \setminus (\partial C \setminus \partial F)$. Prove that there exist a 2-manifold T and sujective continuous map $p : T \to F$ such that:

(1) $p| : T \setminus p^{-1}(D) \to F \setminus D$ is a homeomorphism,

(2) $p| : p^{-1}(D) \to D$ is a two-fold covering.

56.Q Uniqueness of Cut. The 2-manifold T and map p, which exist according to Theorem 56.P, ae unique up to homeomorphism: if $\tilde{T}$ and $\tilde{p}$ are other 2-manifold and map satisfying the same hypothesis then there exists a homeomorphism $h : \tilde{T} \to T$ such that $p \circ h = \tilde{p}$.

The 2-Manifold T described in 56.P is also called the result of *cutting of F along C* and denoted by $F \between C$.

56.5 Show that if C is a segment contained in the interior of a 2-manifold F then $F \between C$ is homeomorphic to $F \setminus \operatorname{Int} B$, where B is the subset of $\operatorname{int} F$ homeomorphic to disk.

56.6 Show that if C is a segment such that one of its end points is in $\operatorname{int} F$ and the other one is on ∂F then $F \mathbin{\text{✄}} C$ is homeomorphic to F.

⌈56′6⌋ Orientations

Recall that an *orientation of a segment* is a linear order of the set of its points. It is determined by its restriction to the set of its end points, see 50.R. To describe an orientation of a segment it suffices to say which of its end points is initial and which is final.

Similarly, orientation of a triangle can be described in a number of ways, each of which can be chosen as the definition. By an *orientation of a triangle* one means a collection of orientations of its edges such that each vertex of the triangle is the final point for one of the edges adjacent to it and initial point for the other edge. Thus, an orientation of a triangle defines an orientation on each of its sides.

A segment admits two orientations. A triangle also admits two orientations: one is obtained from another one by change of the orientation on each side of the triangle. Therefore an orientation of any side of a triangle defines an orientation of the triangle.

Vertices of an oriented triangle are cyclicly ordered: a vertex A follows immediately the vertex B which is the initial vertex of the edge which finishes at A. Similarly the edges of an oriented triangle are cyclicly ordered: a side a follows immediately the side b which final end point is the initial point of a.

Vice versa, each of these cyclic orders defines an orientation of the triangle.

An *orientation of a triangulation* of a 2-manifold is a collection of orientations of all triangles constituting the triangulation such that for each edge the orientations defined on it by the orientations of the two adjacent triangles are opposite to each other. A triangulation is said to be *orientable*, if it admits an orientation.

56.R Number of Orientations. *A triangulation of a connected 2-manifold is either non-orientable or admits exactly two orientations. These two orientations are opposite to each other. Each of them can be recovered from the orientation of any triangle involved in the triangulation.*

56.S Lifting of Triangulation. Let B be a triangulated surface and $p : X \to B$ be a covering. Can you equip X with a triangulation?

56.T Lifting of Orientation. Let B be an oriented triangulated surface and $p : X \to B$ be a covering. Equip X with a triangulation such that p maps each simplex of this triangulation homeomorphically onto a simplex of the original triangulation of B. Is this triangulation orientable?

Proof. Yes, it is orientable. An orientation can be obtain by taking on each triangle of X the orientation which is mapped by p to the orientation of its image. □

56.U Let X be a triangulated surface, $C \subset X$ be a 1-dimensional manifold contained in 1-skeleton of X. If the triangulation of X is orientable, then C is two-sided.

57. Another Topological Classification of Compact Triangulated 2-Manifolds

Topological classification of compact 2-manifolds given in Section 53 above is classical. In Section 53 we followed the classical textbook of topology by Seifert and Threlfall. There are many other proofs of this fundamental result. In particular there have been attempts to provide a shorter proof. John H. Conway in several lectures proposed his ZIP proof (Zero Irrelevance Proof) of this theorem, it was published in 1999 in American Mathematical Monthly by George K. Francis and Jeffrey R. Weeks.

Below we outline yet another proof. Its distinctive feature is reliance to the notions that have proved their usefulness in higher dimensions.

⌈57′1⌋ Spines and Their Regular Neighborhoods

Let X be a triangulated compact connected 2-manifold with non-empty boundary. A simplicial subspace S of the 1-skeleton of X is a *spine* of X if X collapses to S.

57.A *Let X be a triangulated compact connected 2-manifold with non-empty boundary. Then a regular neighborhood of its spine is homeomorphic to X.*

57.B Corollary. *A triangulated compact connected 2-manifold with non-empty boundary admits a handle decomposition without handles of index 2.*

A *spine* of a *closed* connected 2-manifold is a spine of this manifold with an interior of a triangle from the triangulation removed.

57.C *A triangulated closed connected 2-manifold admits a handle decomposition with exactly one handle of index 2.*

57.D *A spine of a triangulated closed connected 2-manifold is connected.*

57.E Corollary. *The Euler characteristic of a closed connected triangulated 2-manifold is not greater than 2. If it is equal to 2, then the 2-manifold is homeomorphic to* S^2.

57.F Corollary: Extremal Case. *Let* X *be a closed connected triangulated 2-manifold* X. *If* $\chi(X) = 2$, *then* X *is homeomorphic to* S^2.

⌈57′2⌋ Simply connected compact 2-manifolds

57.G *A simply connected compact triangulated 2-manifold with non-empty boundary collapses to a point.*

57.H Corollary. *A simply connected compact triangulated 2-manifold with non-empty boundary is homeomorphic to disk* D^2.

57.I Corollary. *Let* X *be a compact connected triangulated 2-manifold* X *with* $\partial X \neq \varnothing$. *If* $\chi(X) = 1$, *then* X *is homeomorphic to* D^2.

⌈57′3⌋ Splitting off crosscaps and handles

57.J *A non-orientable triangulated 2-manifold* X *is a connected sum of* $\mathbb{R}P^2$ *and a triangulated 2-manifold* Y. *If* X *is connected, then* Y *is also connected.*

57.K *Under conditions of Theorem 57.J, if* X *is compact then* Y *is compact and* $\chi(Y) = \chi(X) + 1$.

57.L *If on an orientable connected triangulated 2-manifold* X *there is a simple closed curve* C *contained in the 1-skeleton of* X *such that* $X \smallsetminus C$ *is connected, then* C *is contained in a simplicial subspace* H *of* X *homeomorphic to torus with a hole and* X *is a connected sum of a torus and a triangulated connected orientable 2-manifold* Y.

If X *is compact, then* Y *is compact and* $\chi(Y) = \chi(X) + 2$.

57.M *A compact connected triangulated 2-manifold with non-empty connected boundary is a connected sum of a disk and some number of copies of the projective plane and/or torus.*

57.N Corollary. *A simply connected closed triangulated 2-manifold is homeomorphic to* S^2.

57.O *A compact connected triangulated 2-manifold with non-empty boundary is a connected sum of a sphere with holes and some number of copies of the projective plane and/or torus.*

57.P *A closed connected triangulated 2-manifold is a connected sum of some number of copies of the projective plane and/or torus.*

⌈57′4⌋ Splitting of a Handle on a Non-Orientable 2-Manifold

57.Q *A connected sum of torus and projective plane is homeomorphic to a connected sum of three copies of the projective plane.*

57.Q.1 *On torus there are 3 simple closed curves which meet at a single point transversal to each other.*

Proof. Represent the torus as the quotient space of the unit square. Take the images of a diagonal of the square and the two segments connecting the midpoints of the opposite sides of the square. □

57.Q.2 *A connected sum of a surface S with $\mathbb{R}P^2$ can be obtained by deleting an open disk from S and identifying antipodal points on the boundary of the hole.*

57.Q.3 *On a connected sum of torus and projective plane there exist three disjoint one-sided simple closed curves.*