Strangeness

Talk at the conference in honor of
Jean Jacques Risler

”Around Hilbert’s 16th Problem”

Oleg Viro

August 7, 2008

Introduction

Choice of curves

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To make life easier, stay with majority.

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say, rational curves of a given degree

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Can the philosophy of general position help here?

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For rational curves: only double singular points.

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What is a counter-part of the maximality for
generic curves of a fixed degree and genus?

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Extend the 16th Hilbert problem to
generic curves of a fixed degree and genus?

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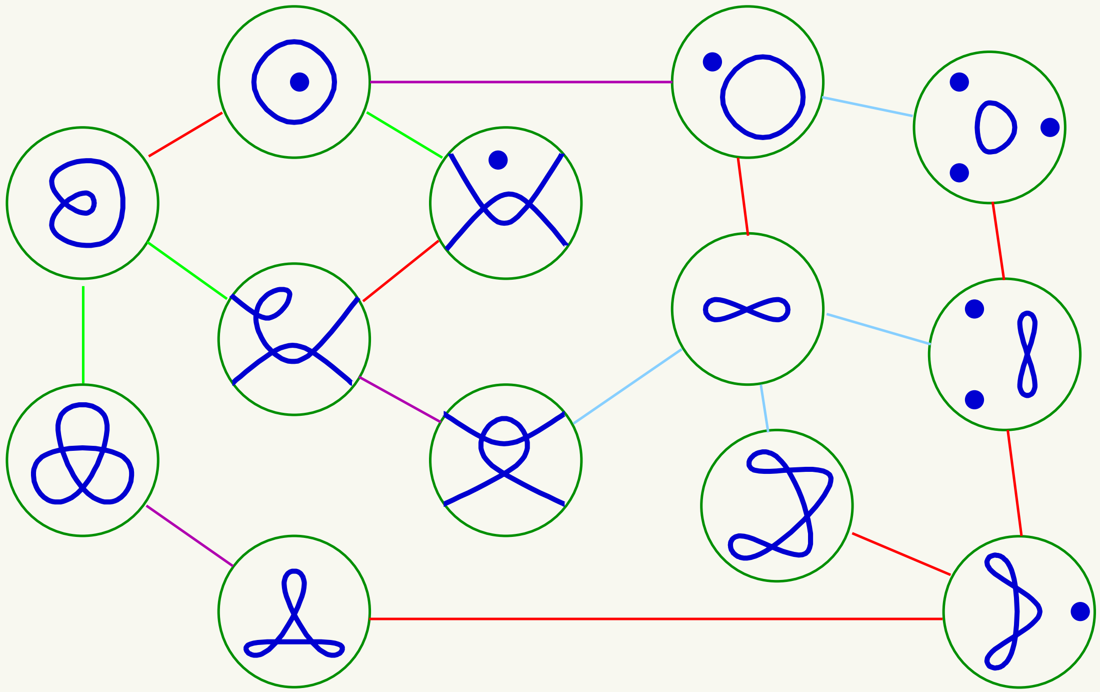
Simple rational quartics



Topological schemes of rational real plane projective curves with infinitely many real points and only double singular points.

Affine curves

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$=$ rotation number of the velocity vector
$=$ degree of the Gauss map $S^1 \to S^1 : z \mapsto \frac{C'(z)}{|C'(z)|}$.

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Arnold’s invariants

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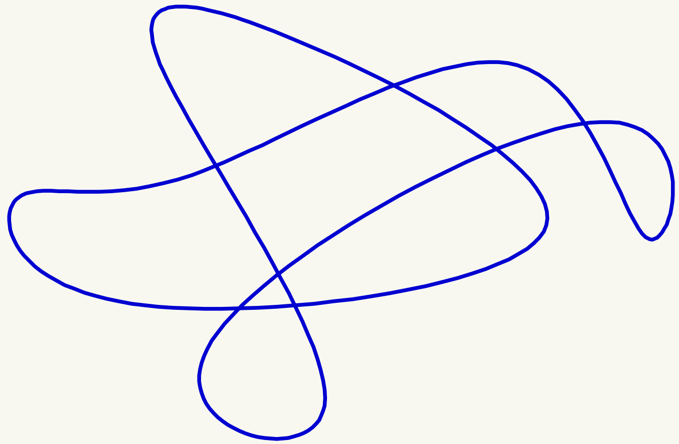

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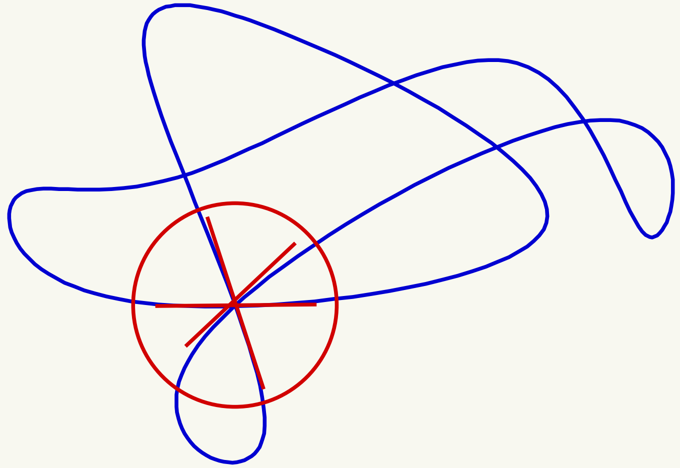

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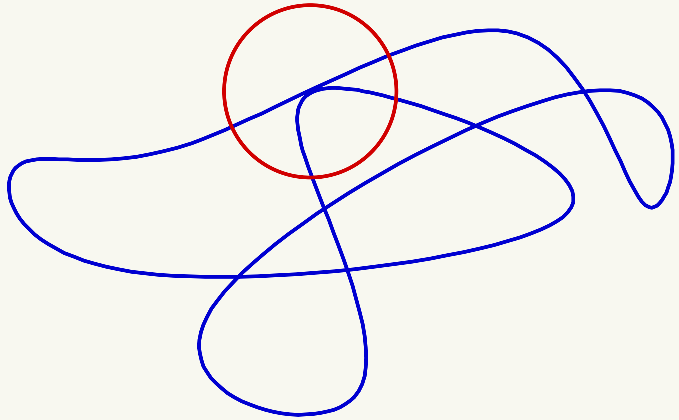

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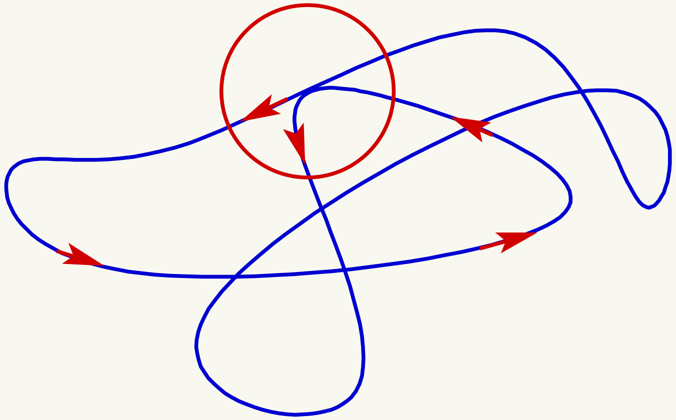

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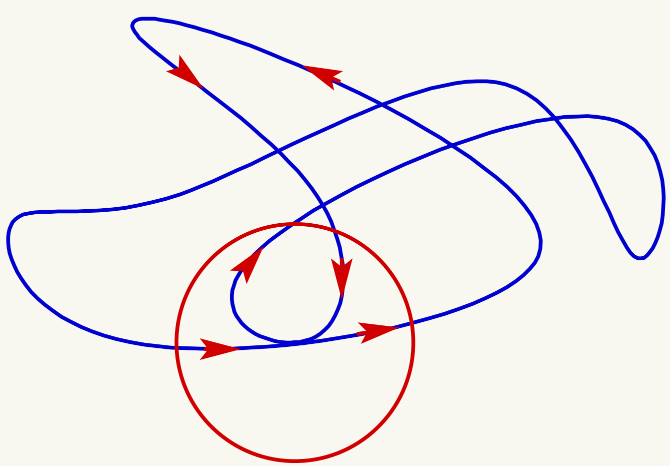

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The discriminant is stratified.

Main strata of discriminant

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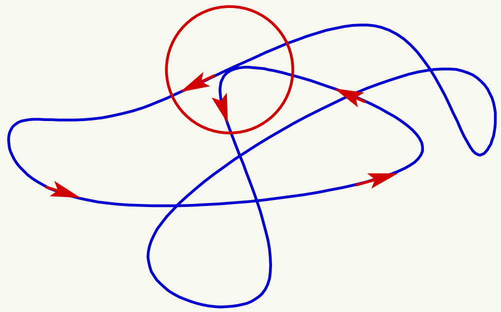

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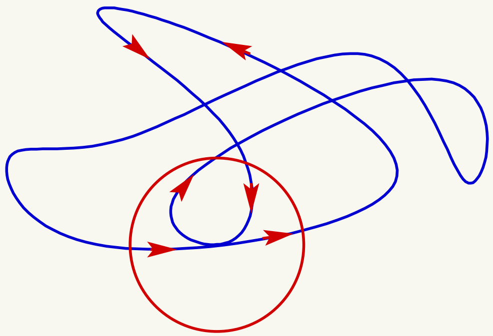

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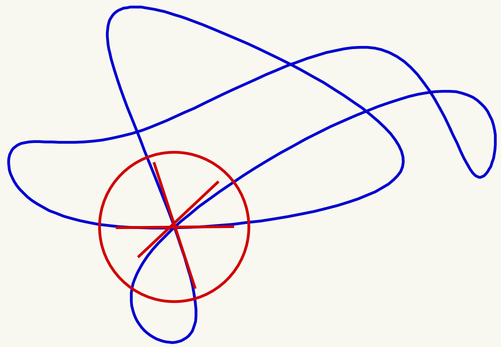
• the set Dst of all immersions which have only one triple point, this point is ordinary, and besides this point there are only ordinary double points.



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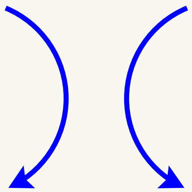

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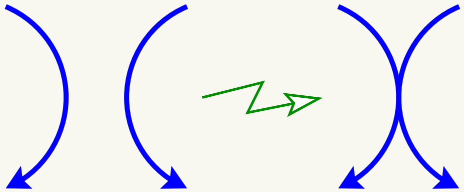

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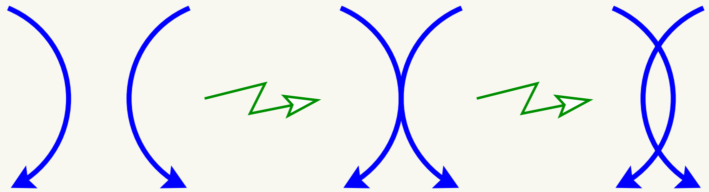

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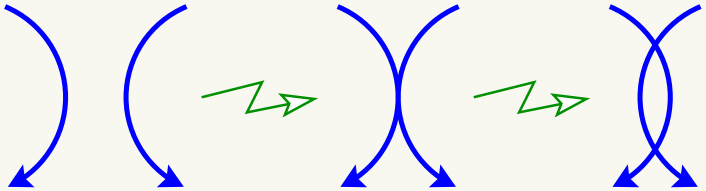

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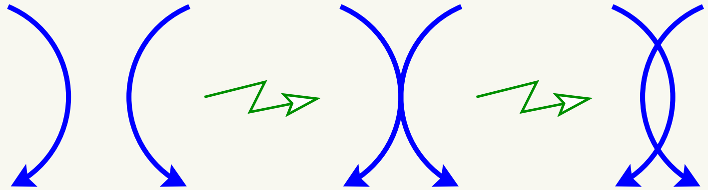

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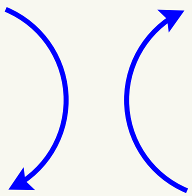

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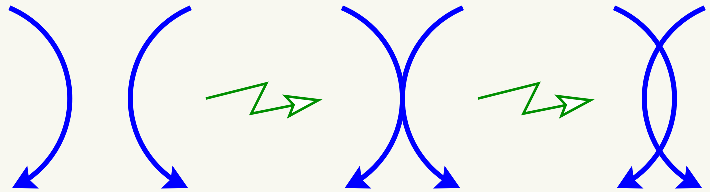

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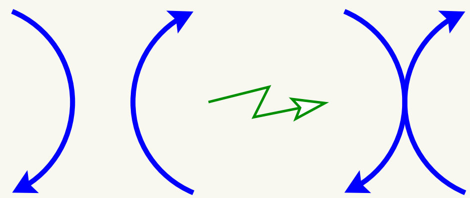

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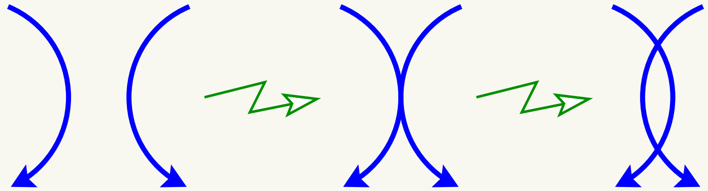

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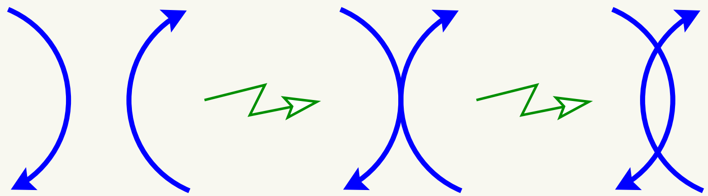

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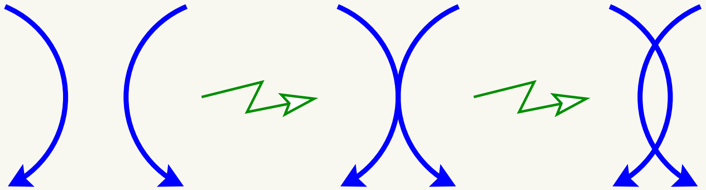
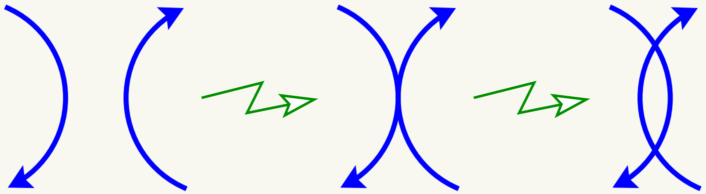


Direct self-tangency perestrojka. Passing through Dj_+



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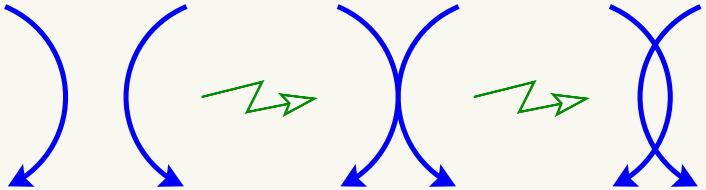

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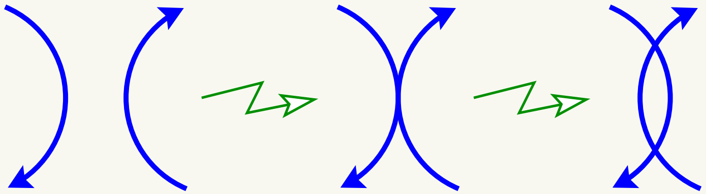

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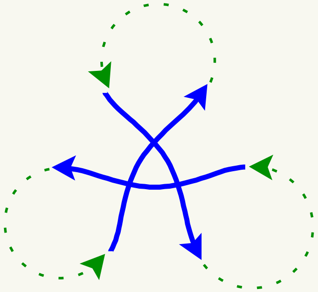

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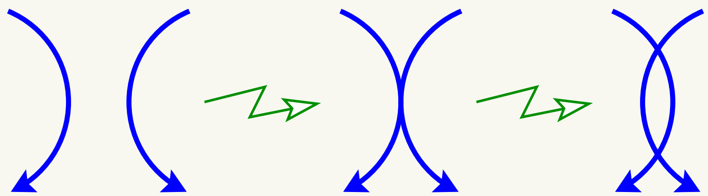

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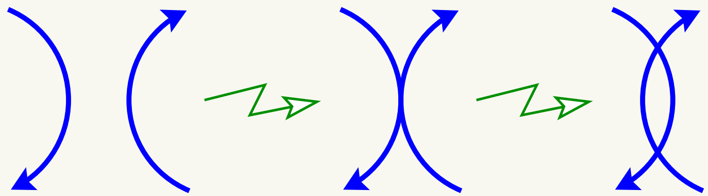

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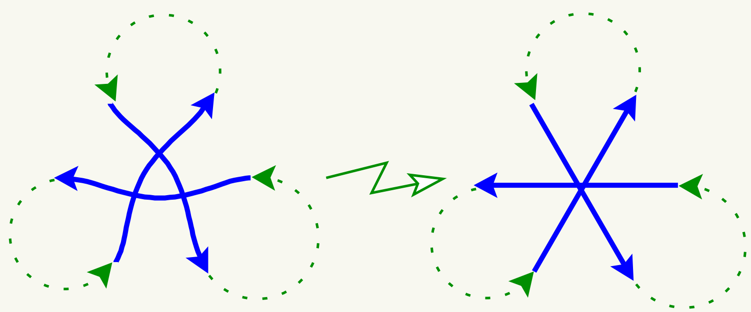

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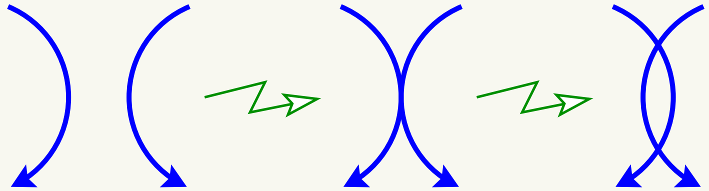

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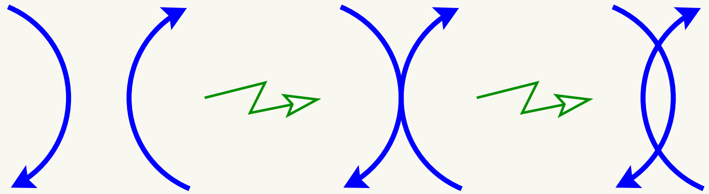

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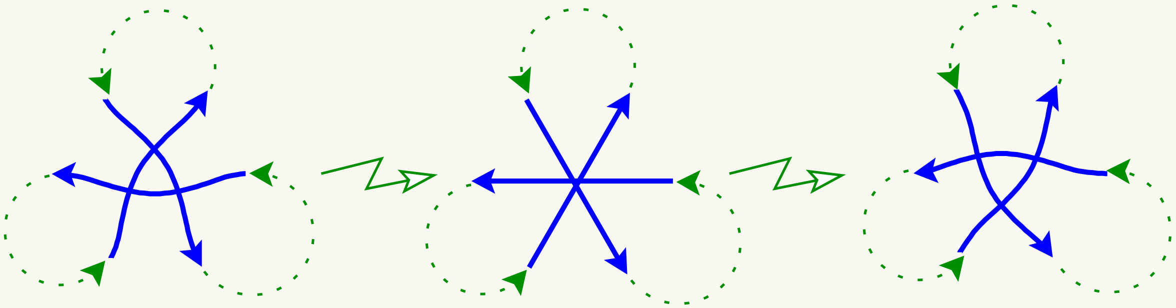

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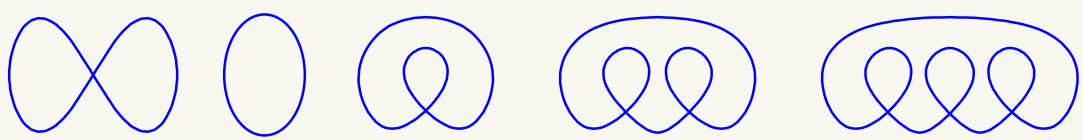
- For curves  $\dots$:

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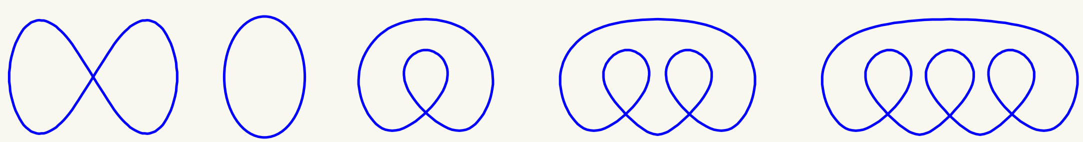
• For curves  $\dots$:

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$$J^+(K_0) = 0, \quad J^+(K_{i+1}) = -2i \quad (i = 0, 1, \dots);$$
$$J^-(K_0) = -1, \quad J^-(K_{i+1}) = -3i \quad (i = 0, 1, \dots);$$
$$St(K_0) = 0, \quad St(K_{i+1}) = i \quad (i = 0, 1, \dots).$$

Co-orientations of the strata

To make sense of the increments

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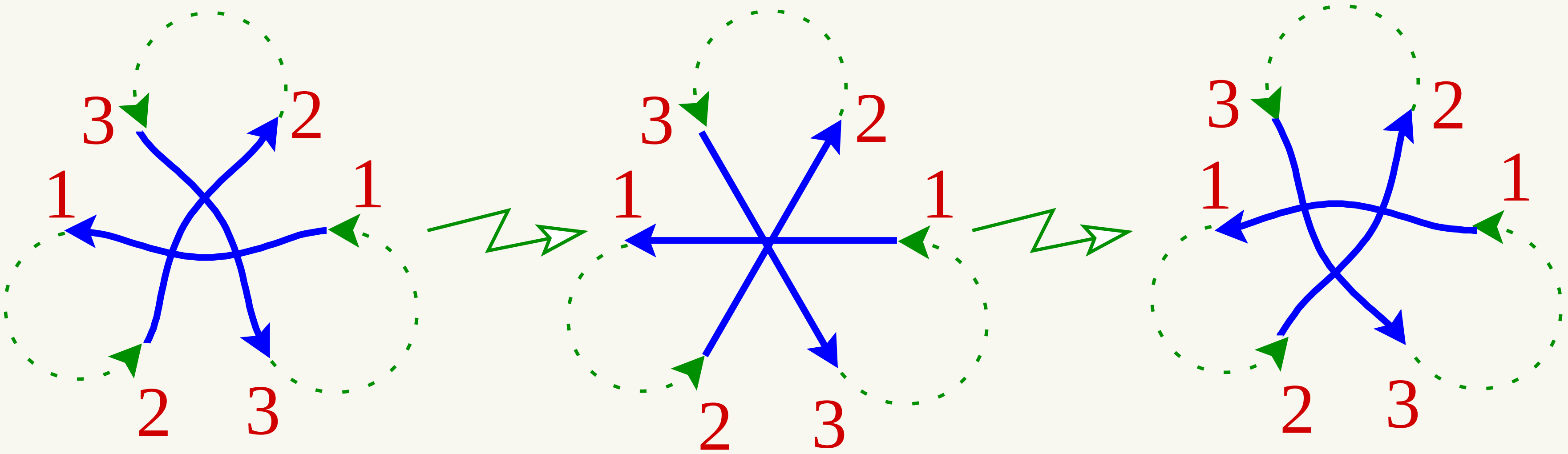

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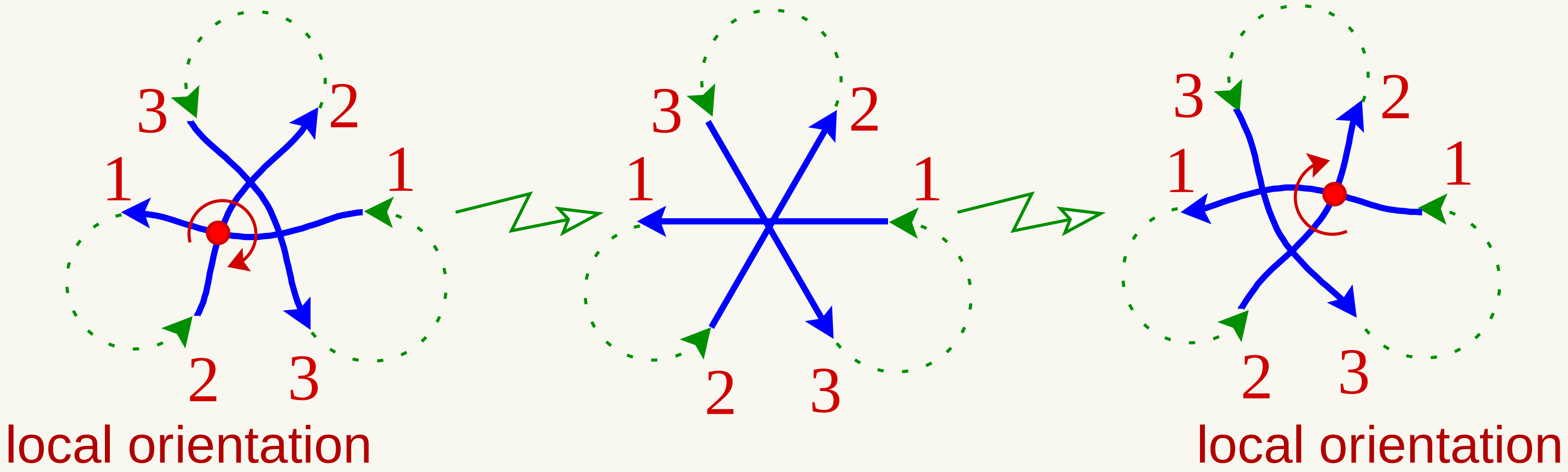

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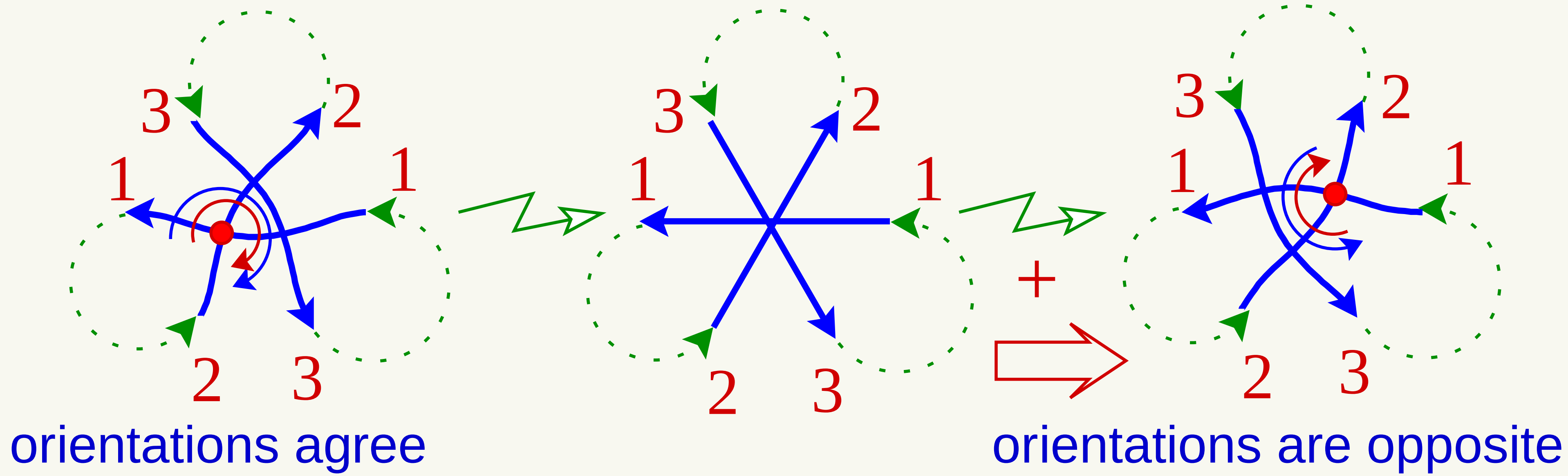

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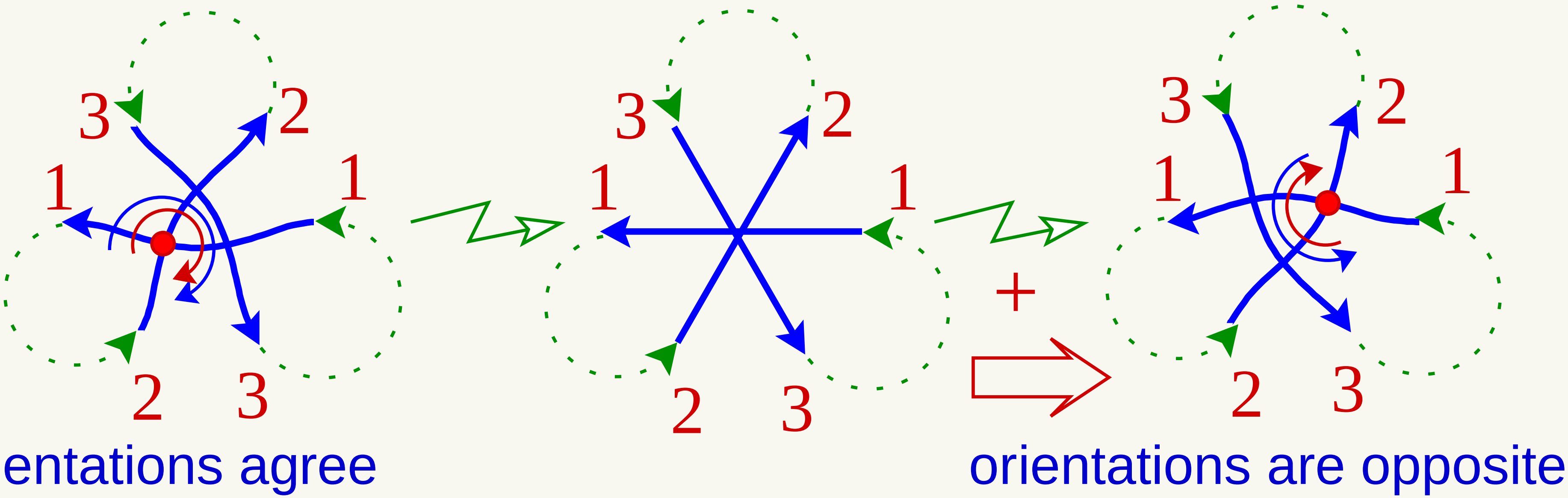

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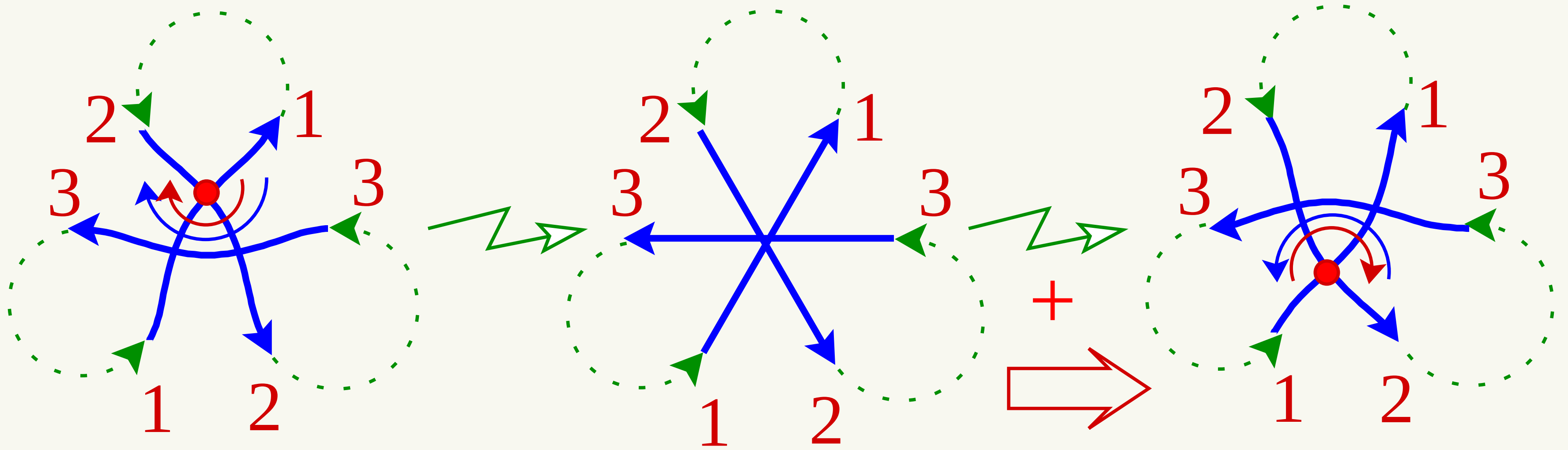

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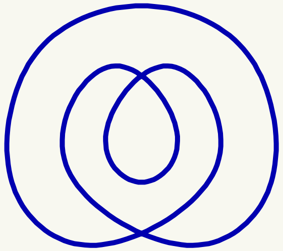
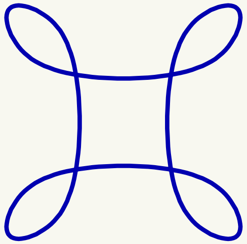
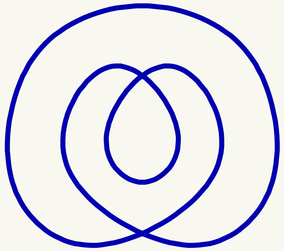
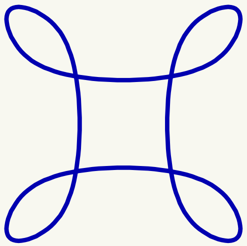
Curves  and  have Whitney number 3.

Curves  and  have Whitney number 3.

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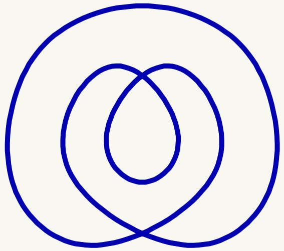
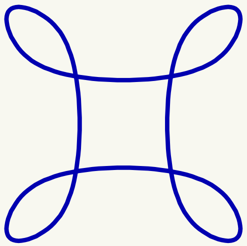
Curves  and  have Whitney number 3.

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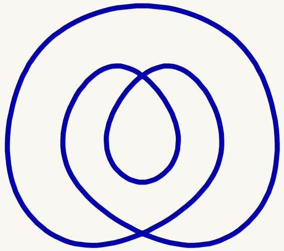
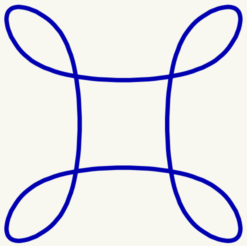
Curves  and  have Whitney number 3.

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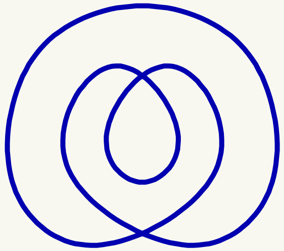
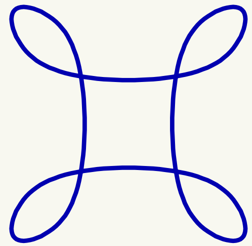
Curves  and  have Whitney number 3.

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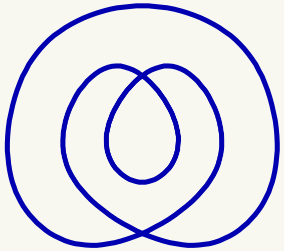
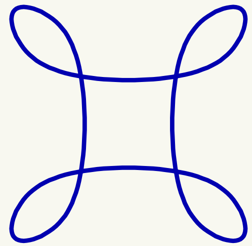
Curves  and  have Whitney number 3.

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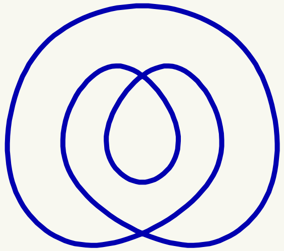
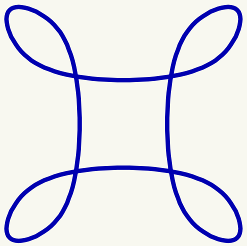
Curves  and  have Whitney number 3.

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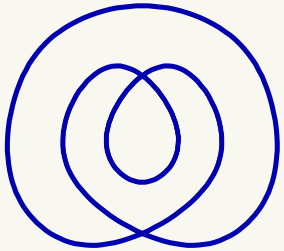
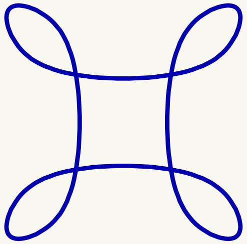
Curves  and  have Whitney number 3.

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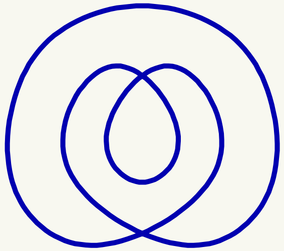
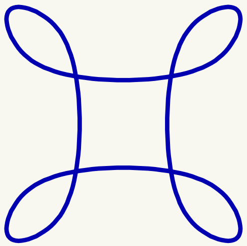
Curves  and  have Whitney number 3.

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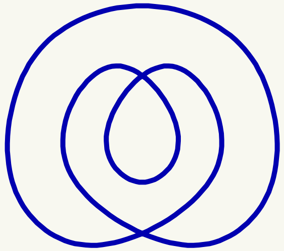
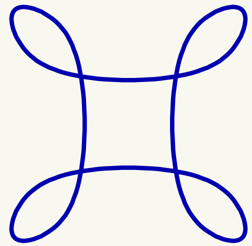
Curves  and  have Whitney number 3.

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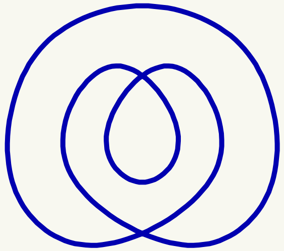
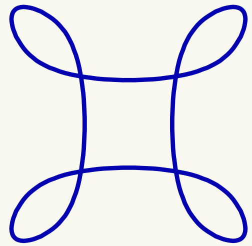
Curves  and  have Whitney number 3.

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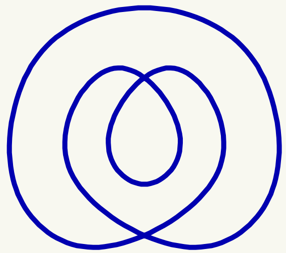
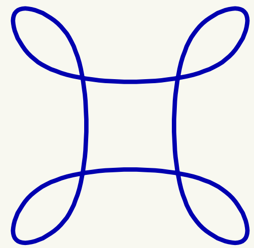
Curves  and  have Whitney number 3.

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Formulas for J^+, J^- and St wanted!

Formulas for Arnold’s invariants

Prepare to formulas

Be patient: ingredients to cook up the formulas are introduced!

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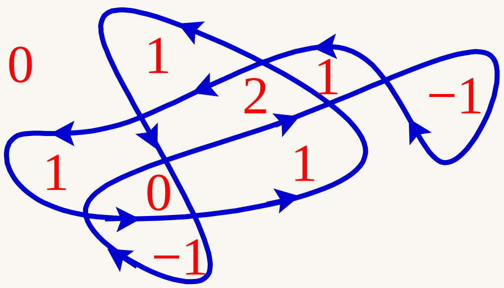
Example:



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A locally constant function f on the complement of a curve extends to arcs and double points
by taking average of values on the adjacent domains.

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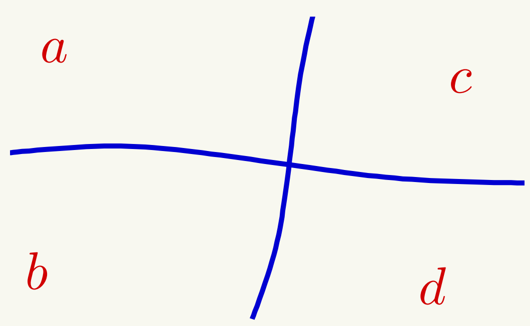

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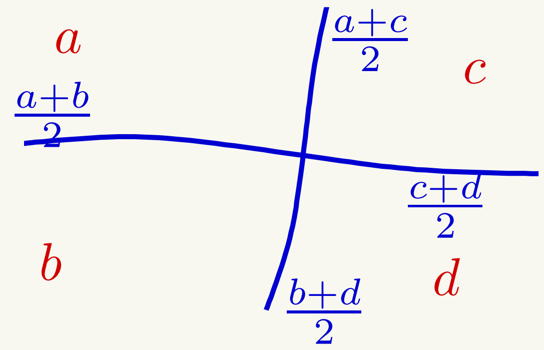

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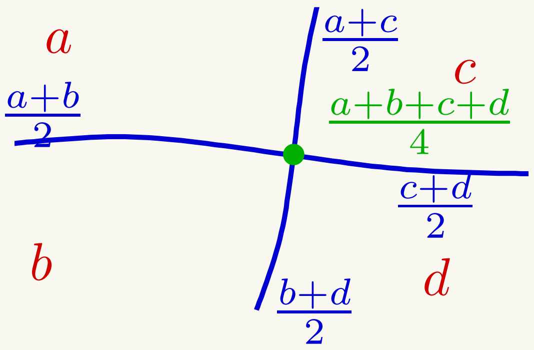

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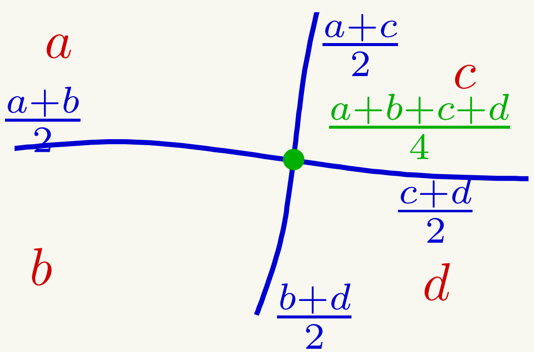

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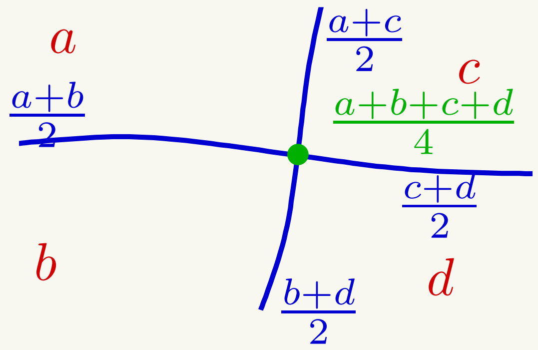

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Formulas for J^- and J^+

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Not surprising: they came from the complex orientation formula.

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Hence $\mathrm{ind}^2_C(x)$ is a well-defined number independent on the local orientation.

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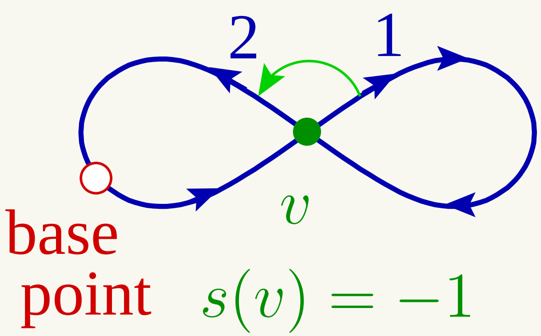

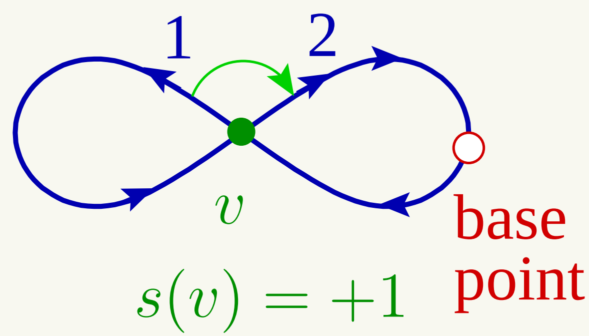

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Another way to understand the formula.
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and sum up over all double points.

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The number given by the formula has all the properties expected from $St(C)$.

Back to algebraic curves

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Choice of curves

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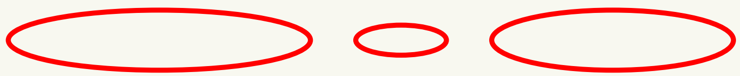

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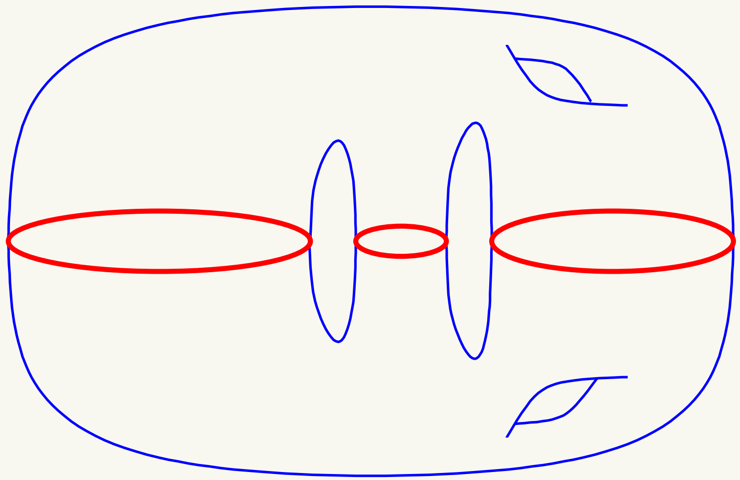

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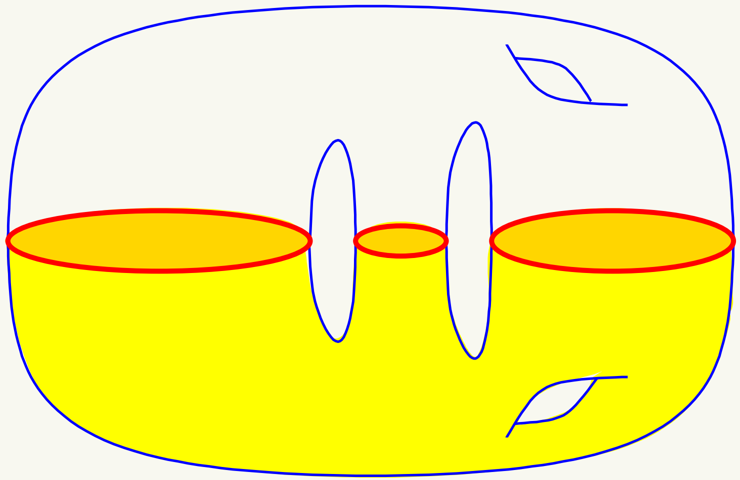

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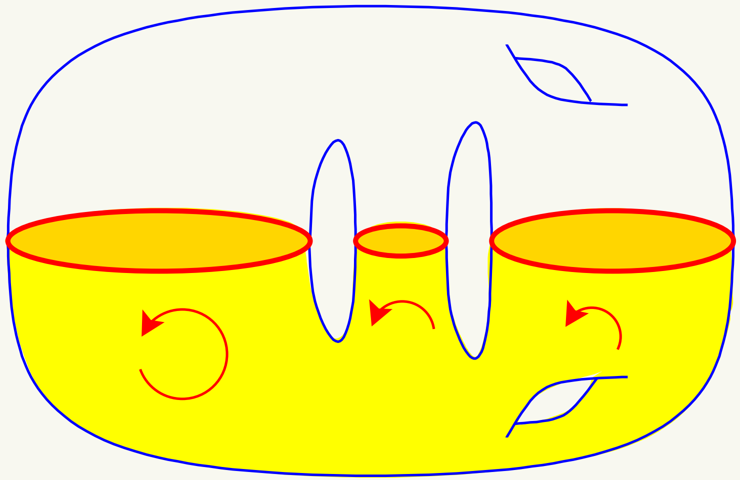

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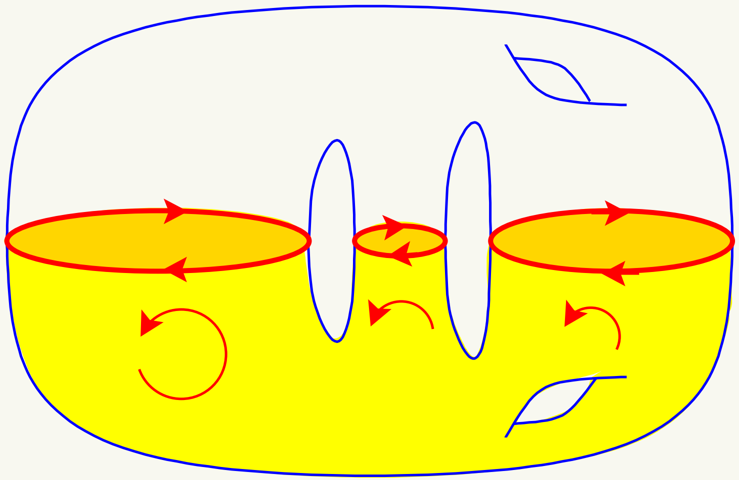

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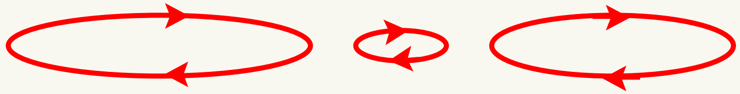

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with $\mathbb{R}A$ equipped with a complex orientation.

Choice of curves

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The latter is equivalent to a choice of a half $\mathbb{C}A_+$ of $\mathbb{C}A$.

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Consider irreducible real algebraic plane projective curves A of degree d, genus g and type I, with $\mathbb{R}A$ equipped with a complex orientation.

A generic curve A of this kind has only non-degenerate double singular points

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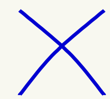
- real double points with two real branches ,

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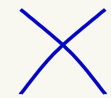
- real double points with two real branches ,
- solitary real double point with two imaginary conjugate branches, isolated point in $\mathbb{R}A$, local normal form $x^2 + y^2 = 0$.

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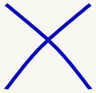
- real double points with two real branches ,
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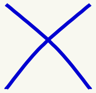
- real double points with two real branches ,
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At a solitary ordinary double point, the choice of $\mathbb{C}A_+$ determines a local orientation of $\mathbb{R}P^2$ such that $\mathbb{R}P^2$ equipped with this local orientation intersects $\mathbb{C}A_+$ at this point with intersection number $+1$.

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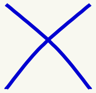
- real double points with two real branches ,
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Another way to get this local orientation: perturb the curve keeping type I and converting the solitary point into an oval.

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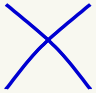
• real double points with two real branches ,

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At a solitary ordinary double point, the choice of $\mathbb{C}A_+$ determines a local orientation of $\mathbb{R}P^2$.

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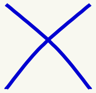
- real double points with two real branches ,
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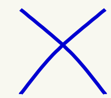
• real double points with two real branches ,
• solitary real double point with two imaginary conjugate branches,
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Denote the number of the latter points by τ .

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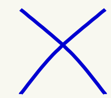
- real double points with two real branches ,
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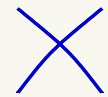
- real double points with two real branches ,
- solitary real double point with two imaginary conjugate branches,
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Denote the number of the latter points by τ.

- imaginary intersection point of $\mathbb{C}A_+$ and $\mathbb{C}A_-$.

Denote the number of the latter points by σ.

New perestrojkas

Generic $\mathbb{R}A$ experiences perestrojkas considered above plus the following three new ones.

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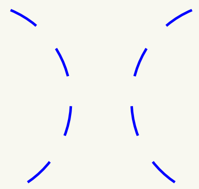

Solitary self-tangency perestrojka.

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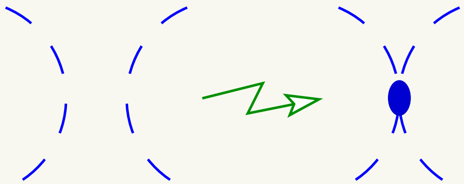

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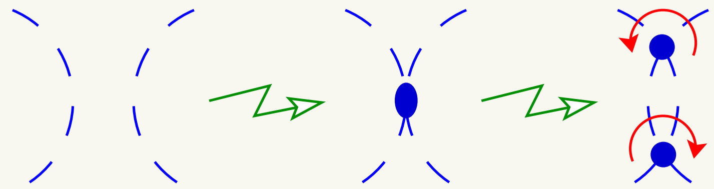

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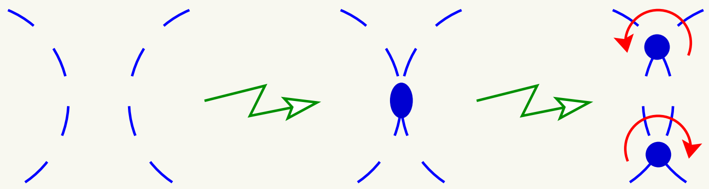

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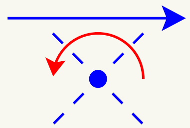

Triple point perestrojka with two imaginary branches.

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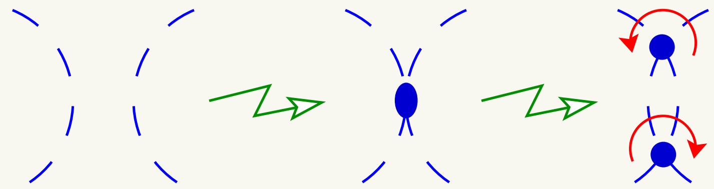

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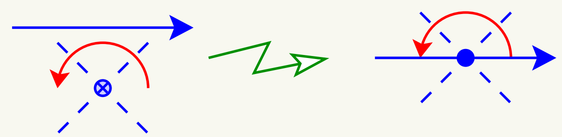

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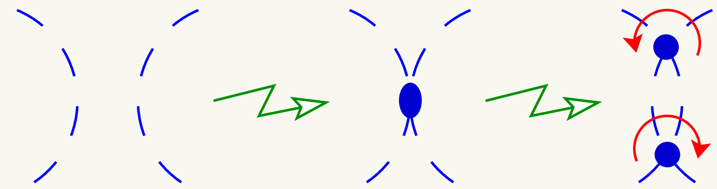

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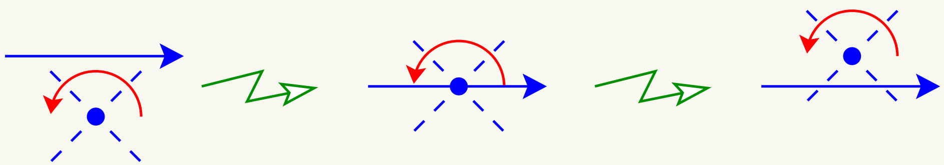

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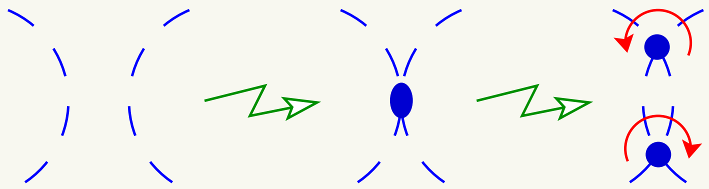

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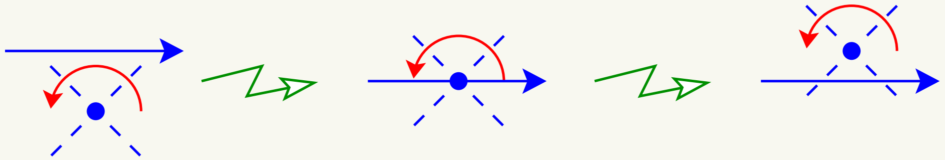

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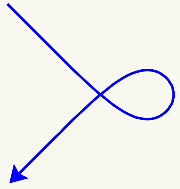

Cusp perestrojka.

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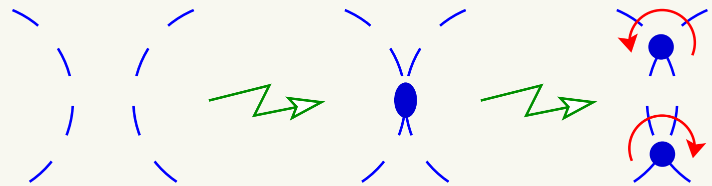
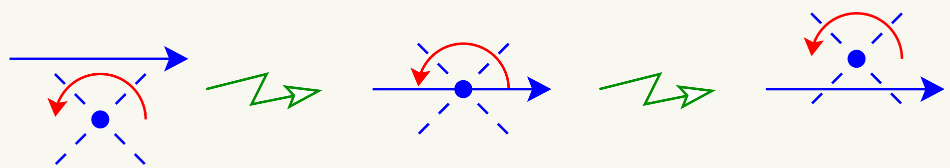


Solitary self-tangency perestrojka.



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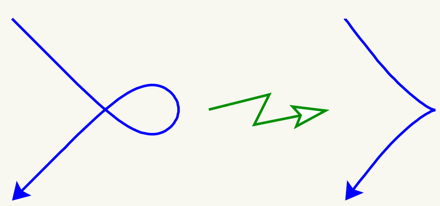

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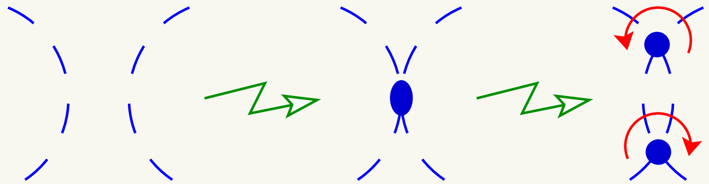

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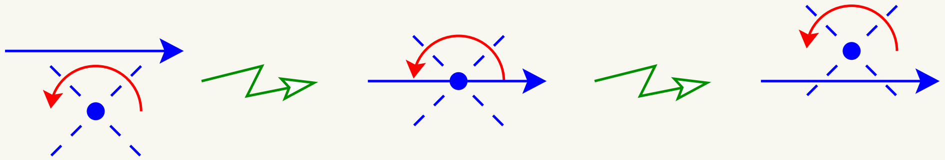

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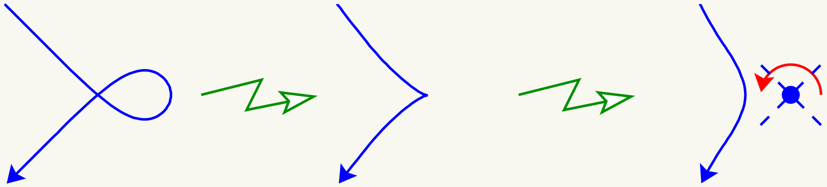

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The behavior of σ and τ under perestrojkas is shown in the following table.

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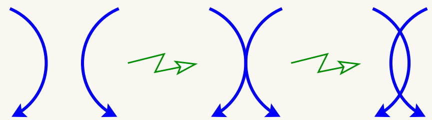
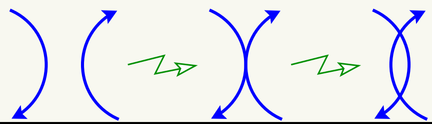
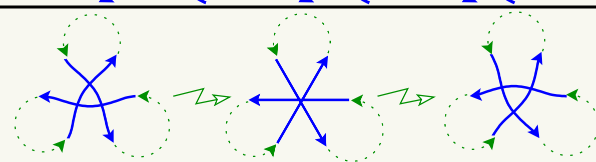
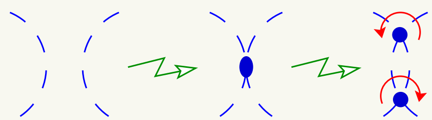
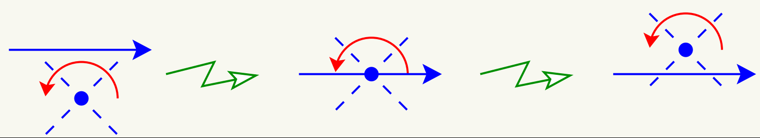
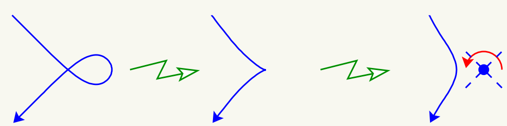
	0	-2
	-2	0
	0	0
	-2	0
	2	-2
	0	0

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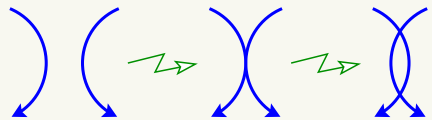
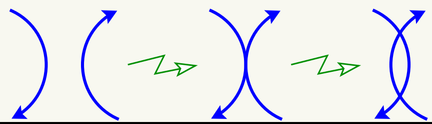
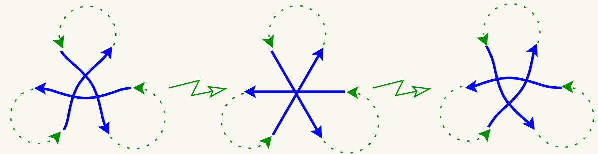
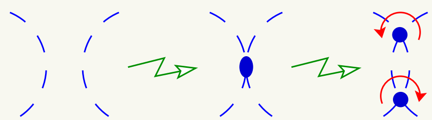
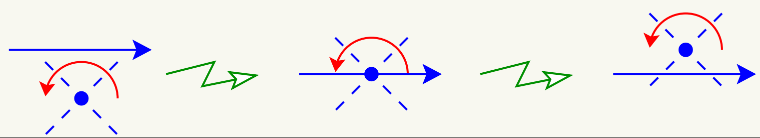
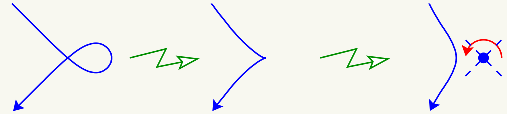
	0	-2
	-2	0
	0	0
	-2	0
	2	-2
	0	0

Obviously, σ and τ are invariants of degree 1.

Imaginary self-intersections

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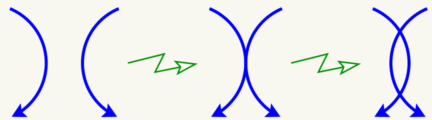
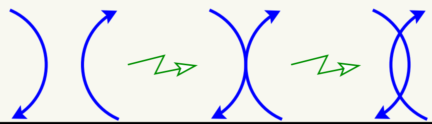
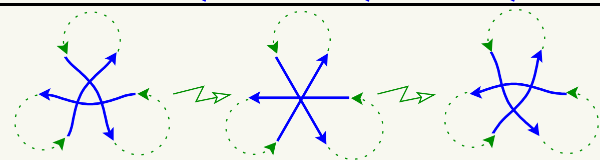
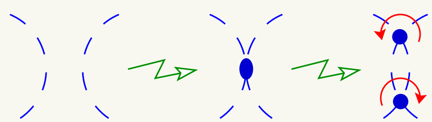
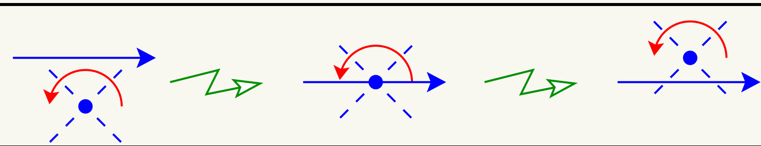
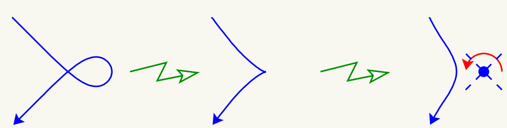
	0	-2
	-2	0
	0	0
	-2	0
	2	-2
	0	0

Obviously, σ and τ are invariants of degree 1.
Under the first 3 perestrojkas σ behaves like J^- while τ behaves like $-J^+$.

Strangeness of $\mathbb{R}A$

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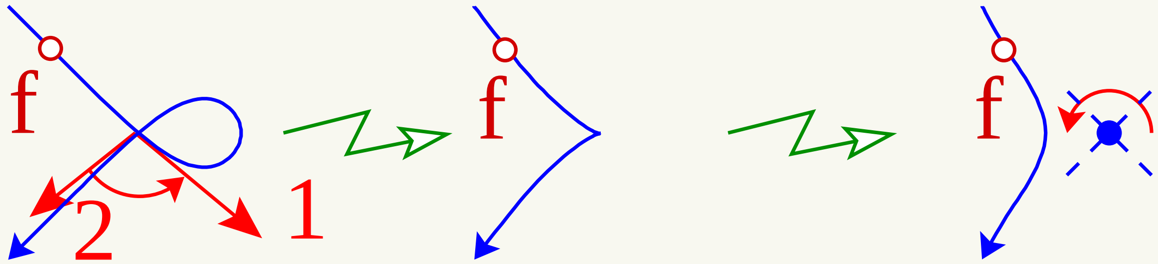
Under all the perestrojkas, except the cusp perestrojka, it behaves as an invariant of order one, but under the cusp perestrojka it changes by the index of the vanishing double point.



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A true Strangeness, which is an invariant of degree one, can be obtained by constructing a co-orientation of the union of the triple point strata both with all three branches real and with one real and two imaginary branches.

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Strangeness of $\mathbb{R}A$

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but depends on ordering of the components.

Under all the perestrojkas, except the cusp perestrojka,
it behaves as an invariant of order one,
but under the cusp perestrojka it changes by the index of the vanishing double point.
A true Strangeness, which is an invariant of degree one, can be obtained by constructing a co-orientation of the union of the triple point strata both with all three branches real and with one real and two imaginary branches. I did this in 1994.
We will avoid that consideration of codimension two strata of the discriminant and the fundamental group of the space of all curves.

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Let

$$St(A) = \sum_{\text{real double points } v \text{ of } A} \mathrm{rot}_v(A) + \sum_{\text{components } K \text{ of } \mathbb{R}A} \left(\mathrm{ind}^2_{\mathbb{R}A}(f_K) - \frac{1}{2}\right)$$

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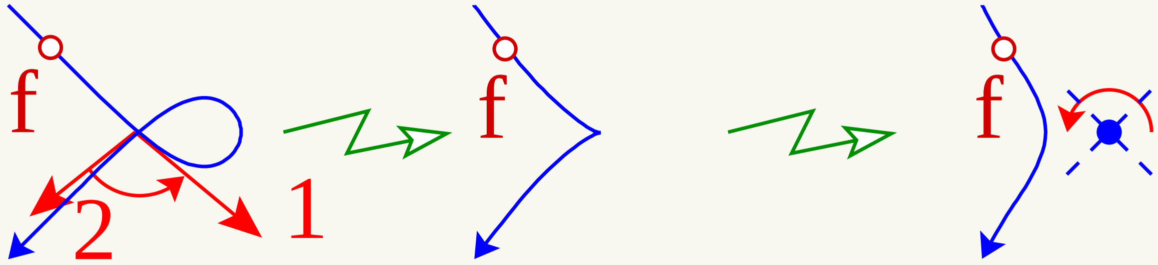
If the branches of A at v are real, this is the orientation defined by the orientations of the second branch followed by the first branch.
If the branches are imaginary, this is the orientation defined by the complex orientation of the curve.

Cusp perestrojka

This is why $St(A)$ does not change under the cusp perestrojka:



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