PRACTICE MIDTERM AND ANSWERS

Question 1

a)If $f(x)$is defined at $x = a$and $\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = 4$, then $f(x)$is continous at a.

TRUE

By definition, $f(x)$ is differentiable at a if $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists. We are told that this limit exists and equals 4, so $f(x)$ is differentiable at a. Thus, $f(x)$ is continuous at a.

b) If f(x) is defined and continuous on the interval $[1, 2]$, and $f(1) = 0$ and $f(2) = 1$, then $0 \leq f(\frac{1}{2}) \leq 1$ by the Intermediate Value Theorem.

FALSE

As stated, nothing in the problem says anything about $f(1/2)$, so $f(1/2)$ could be anything. I meant to ask about $f(3/2)$, not $f(1/2)$. That's also false; see if you can figure out why.

c) If $\lim_{x\to 2} f(x)g(x) = 24$, then $\lim_{x\to 2} f(x) = 6$ and $\lim_{x\to 2} g(x) = 6$.

FALSE

No way. The limits of $f(x)$ and $g(x)$, if they exist, could be any two numbers that multiply to 24.

d) $\lim_{x\to\infty} ln(x) = \infty$

TRUE

Look at the graph of $ln(x)$.

e) If $g(x)$ is a continuous function, $g(2) = 4$ and $f(4) = 8$, then $\lim_{x\to 2} f(g(x)) = 8$.

FALSE

The statement would be true if $f(x)$ was continuous as well. As stated, $f(x)$ could have a jump at $x = 4$, so the limit need not even exist.

Question 2

a) $\lim_{x\to\frac{\pi}{2}} e^{sin(x)}$

e^x and $sin(x)$ are continuous functions, so we can plug in:

$$\lim_{x\to\frac{\pi}{2}} e^{sin(x)} = e^{sin(\frac{\pi}{2})} = e^1 = \mathbf{e}$$

b) $\lim_{x\to-\infty} \frac{x^3-7x+1}{x^2-12x}$

Limits of rational functions and $\pm\infty$ are determinted by the limits of the term withe highest exponent, so:

$$\begin{aligned}\lim_{x\to-\infty} \frac{x^3-7x+1}{x^2-12x} &= \lim_{x\to-\infty} x^3 \\ &= -\infty\end{aligned}$$

c) $\lim_{x\to-\infty} \frac{4x^4+121x^2-2006}{16x^4 1998x^3+2}$

Once again, only the leading term matters for limits of rational functions at ∞, so:

$$\begin{aligned}\lim_{x\to-\infty} \frac{4x^4+121x^2-2006}{16x^4 1998x^3+2} &= \lim_{x\to-\infty} \frac{4x^4}{16x^4} \\ &= \lim_{x\to-\infty} \frac{4}{16} \\ &= \mathbf{\frac{1}{4}}\end{aligned}$$

d) $\lim_{x\to 7} \frac{x+2}{x^2-5x-14}$

Here the top is going to 9 and the bottom is going to 0, so the limit **does not exist**.

e) $\lim_{x\to\infty} \sqrt{x^2-2} - x$

$$\begin{aligned}\lim_{x\to\infty} \sqrt{x^2-2} - x &= \lim_{x\to\infty} \sqrt{x^2-2} - x\frac{\sqrt{x^2-2}+x}{\sqrt{x^2-2}+x} \\ &= \lim_{x\to\infty} \frac{x^2-2+x^2}{\sqrt{x^2-2}+x} \\ &= \lim_{x\to\infty} \frac{-2}{\sqrt{x^2-2}+x} \\ &= \mathbf{0}\end{aligned}$$

f) $\lim_{h\to 0} \frac{\sqrt{4+h}-2}{h}$

$$\begin{aligned}
\lim_{h\to 0}\frac{\sqrt{4+h}-2}{h} &= \lim_{h\to 0}\frac{\sqrt{4+h}-2}{h}\frac{\sqrt{4+h}+2}{\sqrt{4+h}+2}\\
&= \lim_{h\to 0}\frac{4+h-4}{h\sqrt{4+h}+2}\\
&= \lim_{h\to 0}\frac{1}{\sqrt{4+h}+2}\\
&= \mathbf{\frac{1}{4}}
\end{aligned}$$

g) $\lim_{x\to 3} x^4 - x^2 + 7$

This a continous function, so we can plug in and get **79**.

h) $\lim_{x\to -1} \sqrt{x^2}$

Note that $\sqrt{x^2} = |x|$, which is a continuous function. Thus we can plug in, and the limit is 1.

i) $\lim_{x\to 3} \frac{\frac{3}{x}-1}{x-3}$

$$\begin{aligned}
\lim_{x\to 3}\frac{\frac{3}{x}-1}{x-3} &= \lim_{x\to 3}\frac{\frac{3-x}{x}}{x-3}\\
&= \lim_{x\to 3}\frac{3-x}{x(x-3)}\\
&= \lim_{x\to 3}\frac{-1}{x}\\
&= -\mathbf{\frac{1}{3}}
\end{aligned}$$

j) $\lim_{x\to \frac{\pi}{2}^+} e^{tan(x)}$

$\lim_{y\to \frac{\pi}{2}^+} tan(y) = -\infty$ and $\lim_{y\to -\infty} e^y = 0$, so $\lim_{x\to \frac{\pi}{2}^+} e^{tan(x)} = \mathbf{0}$.

Question 3

a) Where is $f(x)$ increasing?

$f'(x) > 0$ on $(-2, 0)$ and $(2, \infty)$, so $f(x)$ is increasing on $(-2, 0)$ and $(2, \infty)$.

b) Where is $f(x)$ decreasing?

$f'(x) < 0$ on $(-\infty, -2)$ and $(0, 2)$, so $f(x)$ is decreasing on $(-\infty, -2)$ and $(0, 2)$.

c) At was values of x does $f(x)$ have local maxima?

At $x = 0$ $f'(x)$ goes from positive to negative, so $f(x)$ has a local max at 0.

d) At was values of x does $f(x)$ have local minima?

At $x = -2$ and $x = 2$ $f'(x)$ goes from negative to positive so $f(x)$ has a local min at -2 and 2.

e) Where is $f(x)$ concave up?

$f'(x)$ is increasing on $(-\infty, -1)$ and $(1, 3)$, so $f''(x)$ is positive on $(-\infty, -1)$ and $(1, 3)$, so $f(x)$ is concave up on $(-\infty, -1)$ and $(1, 3)$.

f) Where is $f(x)$ concave down?

$f'(x)$ is decreasing on $(-1, 1)$ and $(3, \infty)$, so $f''(x)$ is negative on $(-1, 1)$ and $(3, \infty)$, so so $f(x)$ is concave down on $(-1, 1)$ and $(3, \infty)$.

g) At what values of x does $f(x)$ have inflection points.

$f'(x)$ has local maxima or minima at $x = -2, 1$ and 3, so $f''(x)$ changes sign at $-2, 1$, and 3, so $f(x)$ has inflection points at $-2, 1$ and 3.

h) Sketch a graph of $f''(x)$

I don't know how to draw it on the computer, but $f''(x)$ should be positive on $(-\infty, -1)$ and $(1, 3)$, negative on $(-1, 1)$ and $(3, \infty)$, have a local mins at 0 and 4 and a local max at 2.

Question 4

a) Find an inverse function of $f(x) = ln\sqrt{x-2}$.

Set $y = ln\sqrt{x-2}$ and solve for y:

$$
\begin{aligned}
y &= ln\sqrt{x-2} \\
e^y &= e^{ln\sqrt{x-2}} \\
e^y &= \sqrt{x-2} \\
(e^y)^2 &= x-2 \\
e^{2y} &= x-2 \\
x &= e^{2y}+2
\end{aligned}
$$

Thus $f^{-1}(x) = e^{2x} + 2$.

b) What is the domain and range of $f^{-1}(x)$?

Domain is all real numbers and range is $(2, \infty)$. Note that the domain of $f(x)$ is $(2, \infty)$ and the range is all real numbers.

Question 5

a) Use the definition the derivative to compute $f'(x)$

There are two equivalent definitions of the derivative; let's use each one:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{2(x+h)^2-\sqrt{x+h}-(2x^2-\sqrt{x})}{h} \\
&= \lim_{h\to 0} \frac{2(x+h)^2-2x^2}{h} + \lim_{h\to 0} \frac{-\sqrt{x+h}+\sqrt{x}}{h} \\
&= \lim_{h\to 0} \frac{2(x^2+2xh+h^2)-2x^2}{h} + \lim_{h\to 0} \frac{-\sqrt{x+h}+\sqrt{x}}{h}\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} \\
&= \lim_{h\to 0} \frac{4xh+2h^2}{h} + \lim_{h\to 0} \frac{-(x+h)+x}{h(\sqrt{x+h}+\sqrt{x})} \\
&= \lim_{h\to 0} 4x+2h + \lim_{h\to 0} \frac{-1}{\sqrt{x+h}+\sqrt{x}} \\
&= 4x-\frac{1}{2\sqrt{x}}
\end{aligned}$$

Note that this limit is a bit tricky to compute because we need to split it into two different limits. I won't put a limit which needs to be split up as two different limits on the exam.

Now, let's compute the derivative using the other definition of the derviative:

$$\begin{aligned}
f'(x) &= \lim_{a\to x} \frac{f(x)-f(a)}{x-a} \\
&= \lim_{a\to x} \frac{2x^2-\sqrt{x}-(2a^2-\sqrt{a})}{x-a} \\
&= \lim_{a\to x} \frac{2x^2-2a^2}{x-a} - \lim_{a\to x} \frac{\sqrt{x}-\sqrt{a}}{x-a} \\
&= \lim_{a\to x} \frac{2(x+a)(x-a)}{x-a} - \lim_{a\to x} \frac{\sqrt{x}-\sqrt{a}}{(\sqrt{x}+\sqrt{a})(\sqrt{x}-\sqrt{a})} \\
&= \lim_{a\to x} 2(x+a) - \lim_{a\to x} \frac{1}{\sqrt{x}+\sqrt{a}} \\
&= 4x-\frac{1}{2\sqrt{x}}
\end{aligned}$$

So we get the same answer.

b) Find the equation of the tangent line to $f(x)$ at $x=4$.

Any line has the form $y - y_1 = m(x - x_1)$, and the equation of the tangent line to $f(x)$ at 4 has the form:

$$\begin{aligned} y - f(4) &= f'(4)(x-4) \\ y - 2(4^2) - \sqrt{4} &= (4(4) - \frac{1}{2\sqrt{4}})(x-4) \\ y - 30 &= \frac{63}{4}(x-4) \end{aligned}$$