MAT 125 Summer II 2006

Practice Final Answers

1. Circle "true" or "false":

(a) If $f(x)$ is continuous on the open interval (a, b), then $f(x)$ has an abolute maximum in the interval (a, b)

TRUE FALSE

FALSE! A continous function on a **closed** interval must have a max and a min by the Extreme Value Theorem. But a continuous function on an open interval need not; consider $1/x$ on the interval $(0, 1)$.

(b) If $g(x)$ has a local maximum at $x = 3$ then $g'(3) = 0$

TRUE FALSE

FALSE! $g'(3)$ could be undefined at $x = 3$ as well.

(c) If $f(x)$ is a differenentiable function, then $\frac{d}{dx} ln(f(x)) = \frac{f'(x)}{f(x)}$

TRUE FALSE

TRUE! This follows from the chain rule, where the outer function is $ln(x)$ and the inner function is $f(x)$.

(d) If $h(x)$ is differentiable on the interval $[1, 2]$, $h(1) = 3$ and $h(2) = 5$, then for some number c between 3 and 4, $h'(c) = 2$

TRUE FALSE

TRUE! This is the Mean Value Theorem applied to the function h on the interval $[1, 2]$.

(d) If $\lim_{h \to 0} \frac{f(3+h) - f(3)}{h} = 17$, then $f(x)$ is continuous at $x = 3$

TRUE FALSE

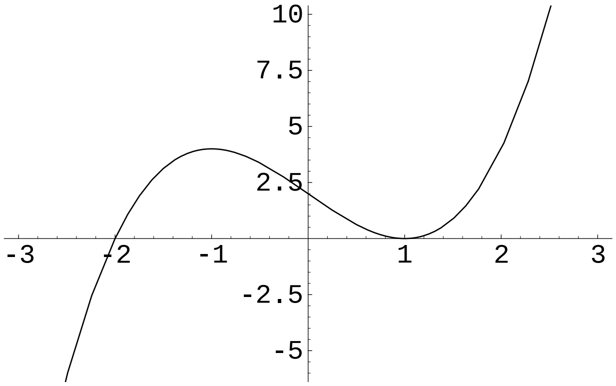
TRUE! This is the definition of $f(x)$ being differentiable at $x = 3$, and if $f(x)$ is differentiable at 3, then $f(x)$ is continuous at 3.

(e) If $f'(x)$ is increasing, then $f(x)$ is positive.

TRUE FALSE

FALSE! It's true that if **f'** is positive, then **f** is increasing, but the above statement is not true.

2. Here is a graph of $f'(x)$, the **derivative** of $f(x)$. Decide whether the following statements about $f(x)$ are true or false:



(a) $f(x)$ has a local maximum at $x = -1$

TRUE FALSE

FALSE! The graph above of f' has a local max at $x = -1$ but f doesn't have a local max there, as $f'(-1) > 0$.

(b) $f(x)$ has a local minimum at $x = -2$

TRUE FALSE

TRUE! $f'(x)$ goes from negative to positive at -2, so $f(x)$ has a local minimum at -2.

(c) $f(x)$ is increasing on the interval $(-2, \infty)$

TRUE FALSE

TRUE! $f'(x) \geq 0$ on this interval, so $f(x)$ is increasing on this interval.

(d) $f(x)$ is concave up on the interval $(-1, 1)$

TRUE FALSE

FALSE! $f'(x)$ is **decreasing** on this interval, so $f(x)$ is concave down on $(-1, 1)$.

(e) $f(x)$ has an inflection point at $x = 1$

TRUE FALSE

TRUE! $f'(x)$ has a local minimum at 1, so $f(x)$ has an inflection point at 1. Note that -1 is also an inflection point of $f(x)$.

3. Compute the following limits. If a limit does not exist, write “$+\infty$”, “$-\infty$”, or “DNE” as appropriate. Do not write “undefined”, because I won’t know what you mean.

(a) $\lim_{x\to\frac{\pi}{2}} e^{\sin x}$
This is a continuous function, so we can plug in:

$$\begin{aligned}\lim_{x\to\frac{\pi}{2}} e^{\sin x} &= e^{\sin(\frac{\pi}{2})}\\ &= e^1\\ &= e\end{aligned}$$

(b) $\lim_{x\to 3} \frac{x-3}{x^2-9}$

$$\begin{aligned}\lim_{x\to 3} \frac{x-3}{x^2-9} &= \lim_{x\to 3} \frac{x-3}{(x-3)(x+3)}\\ &= \lim_{x\to 3} \frac{1}{x+3}\\ &= \frac{1}{6}\end{aligned}$$

(c) $\lim_{h\to 0} \frac{\sqrt{h^2+1}-1}{h}$

$$\begin{aligned}\lim_{h\to 0} \frac{\sqrt{h^2+1}-1}{h} &= \lim_{h\to 0} \frac{\sqrt{h^2+1}-1}{h}\frac{\sqrt{h^2+1}+1}{\sqrt{h^2+1}+1}\\ &= \lim_{h\to 0} \frac{h^2+1-1}{h(\sqrt{h^2+1}+1)}\\ &= \lim h \to 0 \frac{h}{\sqrt{h^2+1}+1}\\ &= \frac{0}{2}\\ &= 0\end{aligned}$$

(d) $\lim_{x\to\infty} \sin(x)$
$\sin x$ has no horozontal assymtopes; the graphs oscilates forever between 1 and -1, so there is no limit at ∞
DNE

(e) $\lim_{x\to\infty} \frac{x-x^3}{3x^2+2x-1}$ Only the leading term matters for the limit of rational functions at ∞, so:

$$\begin{aligned}\lim_{x\to\infty} \frac{x-x^3}{3x^2+2x-1} &= \lim_{x\to\infty} \frac{-x^3}{1}\\ &= -\infty\end{aligned}$$

4. Differentiate the following funtions with respect to x.

(a) $7x^7 - 6x^6$

$$49x^6 - 36x^5$$

(b) $\cos(2x)$

Chain rule:

$$-2\sin(2x)$$

(c) xe^{3x}

Product rule:

$$3xe^{3x} + e^{3x}$$

(d) $\frac{\ln x}{x}$

Quotient rule:

$$\frac{x\frac{1}{x} - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$$

(e) $\sqrt{1 + x^2}$

Chain rule:

$$\frac{1}{2\sqrt{1 + x^2}}(2x) = \frac{x}{\sqrt{1 + x^2}}$$

(f) $\sec(x^2)$

Chain rule:

$$2x\sec(x^2)\tan(x^2)$$

5. Find $\frac{dy}{dx}$ in each of the following problems:

(a) $y = e^{2x} \cos \sqrt{7x^2 - 3x}$

Product rule and then chain rule twice:

$$e^{2x}(-\sin \sqrt{7x^2 - 3x})(\frac{1}{2\sqrt{7x^2 - 3x}})(14x - 3) + 2e^{2x} \cos \sqrt{7x^2 - 3x}$$

Don't even **think** about cancelling the two $\sqrt{7x^2 - 3x}$ terms in the fist half of the sum!

(b) $xy^3 - yx^3 = \sin(xy)$ (Hint: Use implicit differentiation)

$$\begin{aligned}
xy^3 - yx^3 &= \sin(xy) \\
\frac{d}{dx}(xy^3 - yx^3) &= \frac{d}{dx}\sin(xy) \\
3xy^2\frac{dy}{dx} + y^3 - 3x^2y - x^3\frac{dy}{dx} &= \cos(xy)(x\frac{dy}{dx} + y) \\
3xy^2\frac{dy}{dx} - x^3\frac{dy}{dx} - x\cos(xy)\frac{dy}{dx} &= -y^3 + 3x^2y + y\cos(xy) \\
\frac{dy}{dx} &= \frac{-y^3 + 3x^2y + y\cos(xy)}{3xy^2 - x^3 - x\cos(xy)}
\end{aligned}$$

(c) $y = x^{\sin x}$ (Hint: Use logarithmic differentiation)

$$\begin{aligned}
y &= x^{\sin x} \\
\ln y &= \ln(x^{\sin x}) \\
\ln y &= \sin x \ln x \\
\frac{d}{dx}\ln y &= \frac{d}{dx}\sin x \ln x \\
\frac{1}{y}\frac{dy}{dx} &= (\sin x)\frac{1}{x} + \cos x \ln x \\
\frac{dy}{dx} &= y(\frac{\sin x}{x} + \cos x \ln x) \\
&= x^{\sin x}(\frac{\sin x}{x} + \cos x \ln x)
\end{aligned}$$

6. Let $f(x) = \sqrt{x}$

(a) Find the equation of the tangent line to $f(x)$ at $x = 4$

In general, the equation of the tanget line to $f(x)$ at $x = a$ is:

$$y - f(a) = f'(a)(x - a)$$

Thus in this case the equation is:

$$\begin{aligned} y - \sqrt{4} &= \frac{1}{2\sqrt{4}}(x-4) \\ y &= \frac{1}{4}(x-4) + 2 \end{aligned}$$

Here I have used the fact that $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$.

(b) Use your from part (a) to approximate $\sqrt{3.9}$

We use the tangent line to approximate:

$$\begin{aligned} \sqrt{3.9} &\approx \frac{1}{4}(3.9 - 4) + 2 \\ &= \frac{1}{4}(-.1) + 2 \\ &= -.025 + 2 \\ &= 1.975 \end{aligned}$$

Note that $\sqrt{3.9} \approx 3.9748418$, so the tangent line approximation is good to 3 decimal places.

7. Consider $g(x) = \frac{x^4}{4} + x^3 - 2x^2$

(a) List the local maxima of $g(x)$

(b) List the local minima of $g(x)$

$$\begin{aligned} g'(x) &= x^3 + 3x^2 - 4x \\ &= x(x+4)(x-1) \end{aligned}$$

Thus the cricital points of $g(x)$ are -4,0, and 1. Looking at the sign of $g(x)$, we see:

$$\begin{array}{cccc} - & + & - & + \\ \hline & -4 & 0 & 1 \end{array}$$

Thus $x = -4, 1$ are local minima, and $x = 0$ is a local maximum by the first derivative test.

(c) Find the absolute maximum and minimum values of $g(x)$ on the interval $[0, 2]$

We must check the values of $g(x)$ at the endpoints and at all local exterma in the interval:

$$\begin{aligned} g(0) &= 0 \\ g(1) &= -\frac{3}{4} \\ g(2) &= 4 \end{aligned}$$

Thus the maximum value is 4 and the minimum value is $-\frac{3}{4}$

(d) List the inflection points of $g(x)$

Inflection points are where $g''(x)$ changes sign:

$$\begin{aligned} g''(x) &= 3x^2 + 6x - 4 \\ &= (x + 1 + \frac{2}{3}\sqrt{21})(x + 1 - \frac{2}{3}\sqrt{21}) \end{aligned}$$

Here I have used the quadradic formual to find the roots of $g''(x)$. The inflection points are thus $x = -1 \pm \frac{2}{3}\sqrt{21}$.

(e) Where is $g(x)$ concave up?

Looking at the sign of $g''(x)$, we see that $g(x)$ is concave up on the intervals $(-\infty, -1 - \frac{2}{3}\sqrt{21})$ and $(-1 + \frac{2}{3}\sqrt{21}, \infty)$

8. As Santa Claus attempts to descend a chimney, Rudolph heatbutts him and knocks him off the roof. Santa's position as he falls is described by:

$$s(x) = 32 - 16t - 16t^2$$

where $s(x)$ is Santa's height in feet and t is time in seconds.

(a) Give a formula for Santa's velocity as a function of time.

$$v(x) = s'(x) = -16 - 32t$$

(b) How fast is Santa falling when he hits the ground? Santa hits the ground when $s(x) = 0$:

$$\begin{aligned} 0 &= 32 - 16t - 16t^2 \\ &= -16(t^2 + t - 2) \\ &= -2(t-1)(t+2) \end{aligned}$$

We have two roots, 1 and -2, but only the positive root matters. Santa hits the ground after a second, so:

$$v(1) = -16 - 32(1) = -48$$

Thus Santa is traveling $-48\frac{ft}{s}$ when he hits the ground.