# Introduction to graphs

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What: Vertices and edges connected to each other    Why

When: 1700

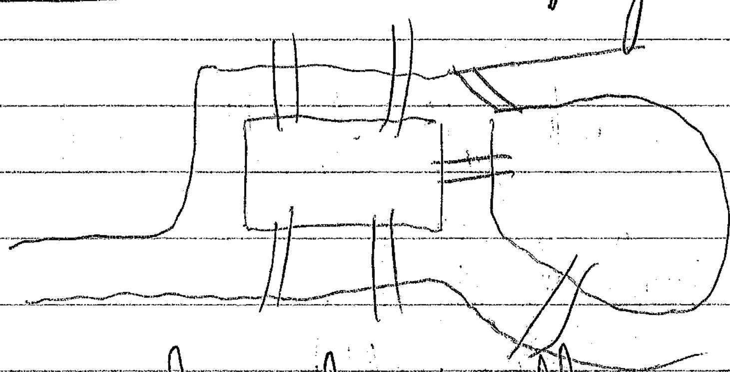
Where: Prussia

Who: The hero of today's story Euler (1707-1783).

Why? is Euler famous and why do we care about graphs?

Answer: For modern mathematicians, it is like answering the same question!!

Euler's motivation: Konigsberg bridge problem



Is there a way to cross all bridges only once.

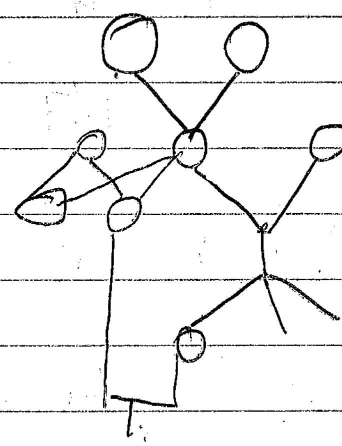
AKA Traveling salesman's problem ⇒ famous problem in Complexity theory

TR Euler solved this problem, but doing so he developped the whole background for working on graphs.

You should say, well it was a long time ago and we are in the 21 st century, isn't it outdated?

## Modern day graphs and problems

You live inside a graph!!



The Uber type of problems TR Nowadays they are much more relevant to us than for Euler because of the emergence of computers.

Starting from 1950's = shortest path algorithm developped

Applications: Page rank (Google algorithm)
Optimization (Google maps)
Decision Tree in computer science.

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II Now to work with them we need **words**, otherwise we will not be able to formulate any statements

**Graph** **Vertices** o

**Edges.** o — o

**Oriented**: o → o (one way street).

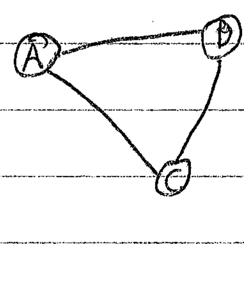
**Degree of a vertex**: (V) degree of V = 3.

\# edges connecting with V.

Disconnected: (A) — (B) (C) — (D)

(2 components "pieces").

**Path**, a route inside the graph.




A B C B A B A B C A B

circuit: a path returning to the same pt.

Euler path: path that covers all edges!, and each exactly ONCE.

bridge: an edge is a bridge if without it the graph would be disconnected!

Theorem (Euler's circuit theorem).
Take a connected graph,
it has an Euler circuit $\boxed{\text{if and only if}}$
every vertex has an even degree.

Idea is that an Euler circuit will enter ~~an edge~~ a vertex then leave that vertex through different edges!!

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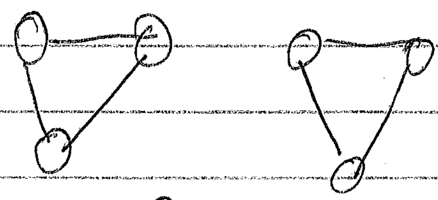
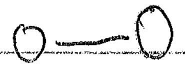
We tell the theorem

Connected?



Fail

Odd degree vertex? 

Strong statement is only even degree ⇒ has Euler circuit.

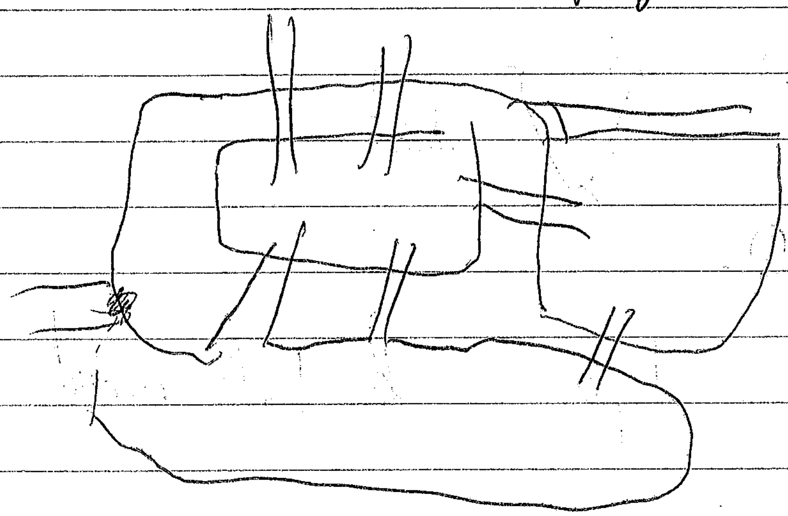
Euler's path theorem

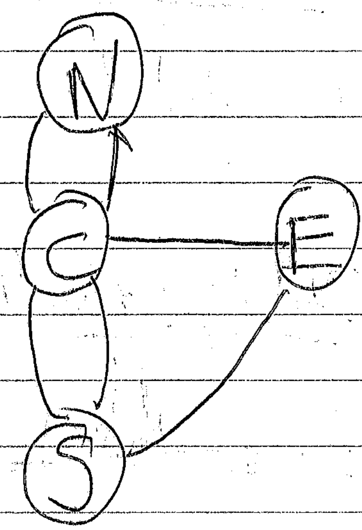
- If a graph is connected and has exactly two odd degree vertices, then it has a Euler path.

Moreover every path must start and end via those edges.

- If a graph has more than 2 odd degree vertices, it has no Euler path.

## Application Euler Konigsberg problem






deg S = 3. No Euler path

deg N, deg ~~S~~C deg E even.

# Graph II

Last time, we saw the following result

- A graph is connected and every vertex has even degree iff it has an Euler circuit
- A graph is connected and has exactly two odd vertices if and only if it has an Euler path.

TR Today I tell more on graphs.

General fact The sum of the degrees of all the vertices equals twice the number of edges.

Explanation for each edge this edge will be counted twice in the sum of the degrees because it is counted once for each of its vertices.

So far we have seen applications to Konigsberg problem and the existence or not of Euler path.

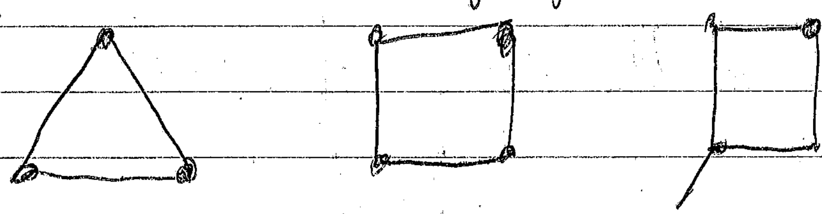
In fact, there is an even more fundamental result than that!

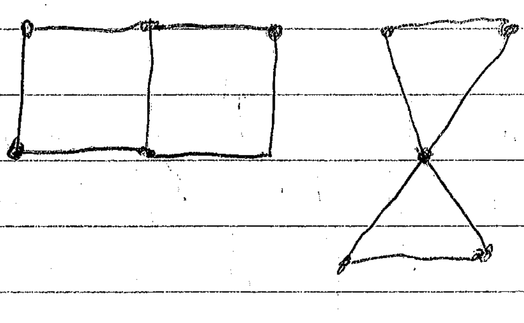
~~Theorem (Euler characteristic)~~

Pb is there a way to distinguish two graphs,

Imagine that I form my graphs using modeling clay, when ~~are two~~ am I building the same graph?

(the vertices are the gluing points).



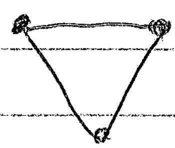


Thm (Euler characteristic).
Fix a connected graph.

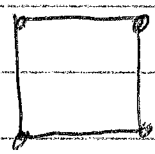
Take $\chi$ = # vertices − # edges.

Then $\chi = 1$ − number of ~~loops~~ holes in the graph.

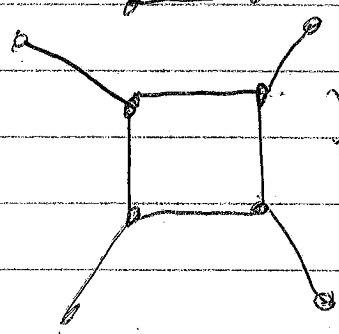
Example:



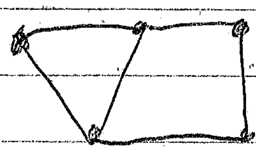
$\chi = 3 - 3 = 0$
# holes 1



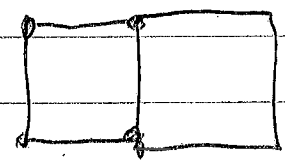
$\chi = 4 - 4 = 0$
# holes = 1.



$\chi = 8 - 8 = 0$
# holes = 1.



$\chi = 5 - 6 = -1$
# holes = 2

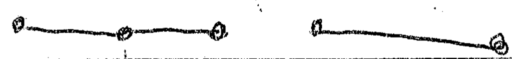


$\chi = 6 - 7 = -1$

$\#\text{holes} = 2$

$1 - \#\text{holes} = -1$.

$\chi$ is called the Euler caracteristic of the graph.

<u>Idea</u> <u>Removal</u> 

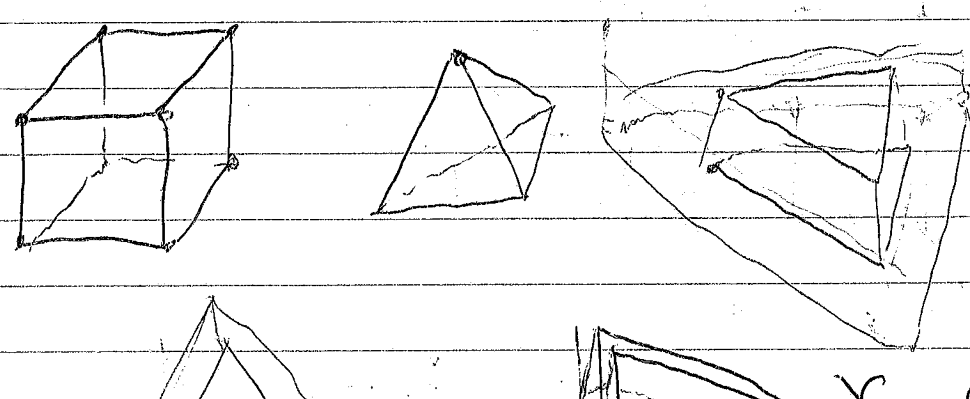
<u>remove 1 vertex 1 edge</u>

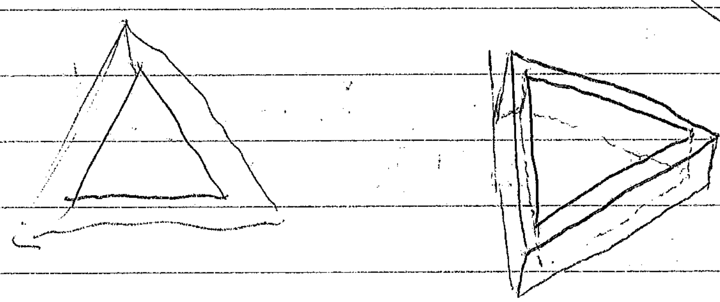
<u>Digression</u> In fact it works not only for graphs

For surfaces:

$$\chi = \#\text{vertices} - \#\text{edges} + \#\text{faces}$$

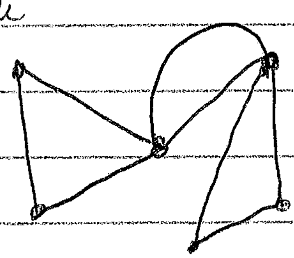
$$\chi = 2 - 2\times(\#\text{ handles})$$





$\chi = 0$

Example



Now I go back to graphs.

So far Euler's theorem allow us to know if there is an Euler path or not.

Is it sufficient? The main issue, it does not provide a way to construct it!

Fleury's algorithm (allows to construct an explicit Euler path).

Step 1 Check the conditions of the theorem.

Step 2 Choose a starting vertex (If possible start at an odd degree vertex).

Step 3 Don't choose a bridge unless it is your only choice.

Bridge = edge ~~th~~ which makes the graph disconnected if removed.

Week 4 III 1

# Graph III

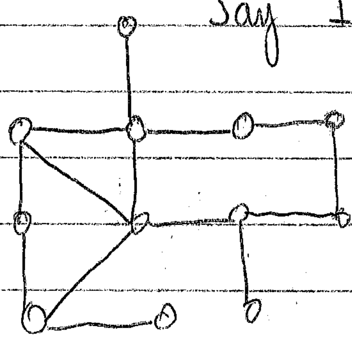
Last time, we saw how to find an Euler path using an algorithm, when there is.

What if there is no Euler path or circuit?

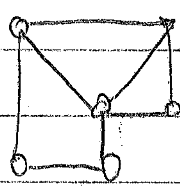
Can we still find an optimal path in an graph?

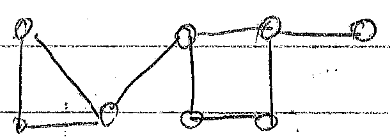
which would cover our graph in the most convenient way?

__Example__ Say I have to deliver something at each vertices.

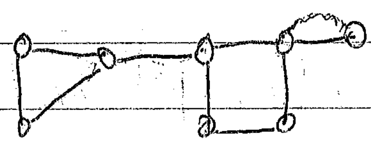


Or





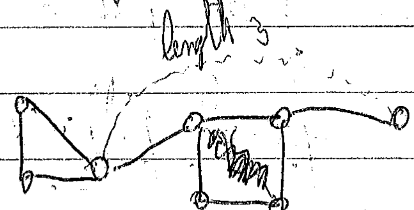
Idea we should add some virtual edges.



10 step

That is called eulerizing.

Don't:




12 steps

What was wrong I added an edge which did not exist but the two pts were too far away.

General method: . Find the odd degree vertices.
. Add some edges and try to minimize the length of those edges.
Apply Fleury's algorithm.

Eulerization all vertices become even degree vertices.
Semi " " only two odd degree vertices.

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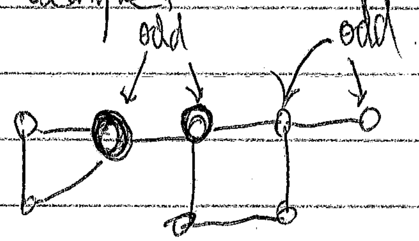
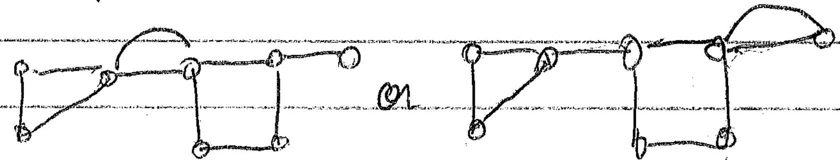
Baby example:



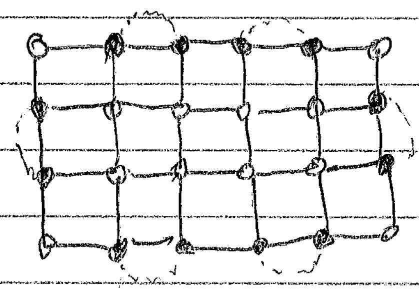

need 2 odd vertices.

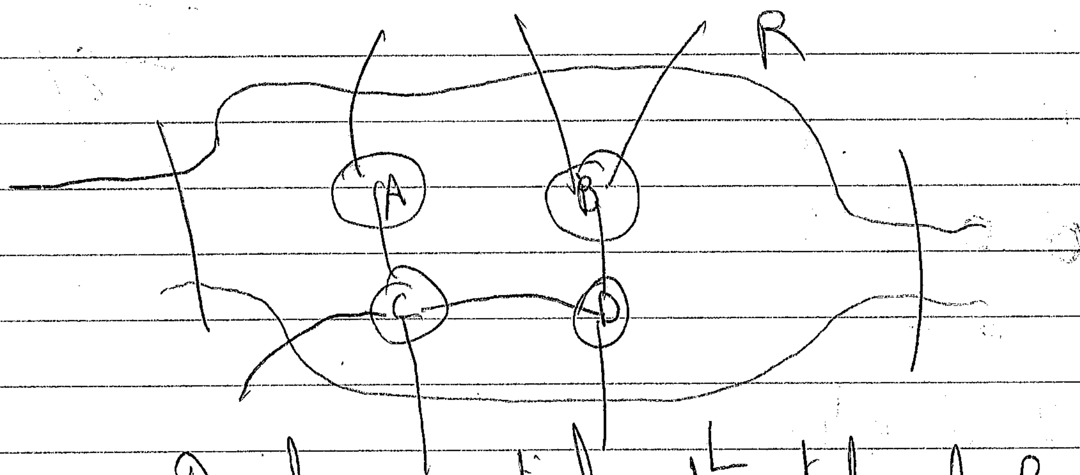
two possibilities



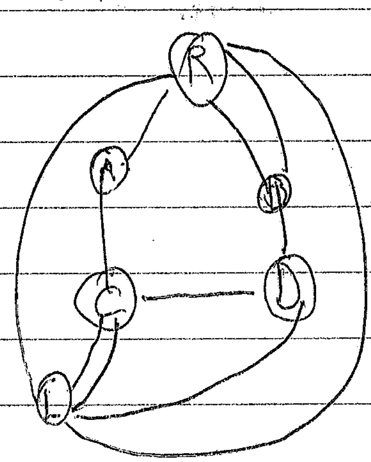

length 10 path length 10.

Example 5.4 for 4.3






Describe an optimal route starting from B
ending at L which crosses all bridges?



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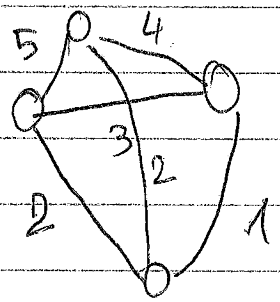
# What now?

One problem with our theory so far, we have simplified the reality, some bridges in Konigsberg can take longer time to cross so our path might not be optimal in terms of the time I take.

We get the traveling salesman problem.




Visit all points of the graph and spend least time as possible

We shall distinguish our study in 2 parts.

First can we pass by all vertices exactly once?

Second how can we optimize our travel.

Euler path = path passing through all edges and each exactly once

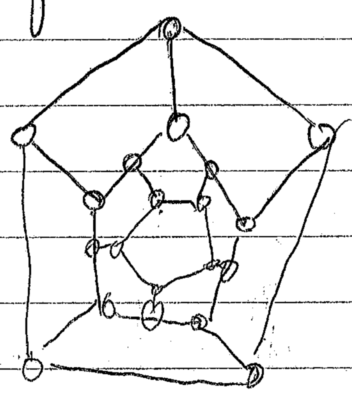
Hamilton path: path passing through all vertices and each exactly once.

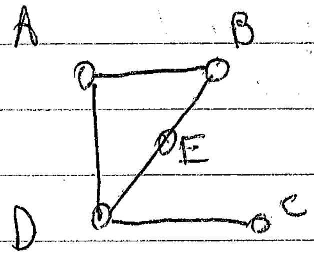
Hamilton (1805 - 1865).

Contributor on calculus of variation
worked on mechanics problem

Invented the least varying action principle.

Hamilton's game:






No Hamiltonian circuit.
C D A B E Hamiltonian path.

Week 4

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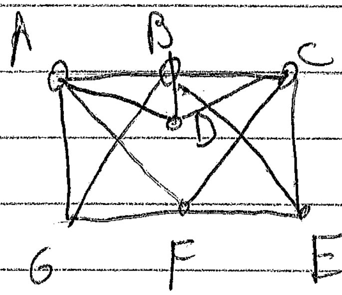


Hamiltonian circuit

A G F E C D B