Week 2 I 1/

Weighted voting and Banzhaf's power (2.1, 2.2)

Now we study a different point of view in voting problems, we will be interested in the outcome of an election if not all voters are considered equal.
Surprisingly, there are many examples:

- In companies, the ~~number of~~ voices of a shareholder depends on his share of the companies capital.
- In a party different people have the same weight but movements don't have the same number of adherent.

Let me explain how a weighted voting systems.

- The voters are called <u>players</u>. $P_1, P_2, \dots P_N$
- Each player has a weight w_1 for P_1 (number of votes for P_1)
  w_2 for P_2
  $\vdots$
  w_n for P_n.
- <u>Quota</u>. To pass a motion need to have more than a minimum required q.

To simplify our discussion, we introduce a notation.

If we have 5 players whose weight are 1, 2, 3, 4, 5
and the quota is $q = 14$, then the situation is denoted:

$[14, 1, 2, 3, 4, 5]$

quota weight for player 1

N players, $w_1, \dots, w_n$ quota q
$[q, w_1, \dots, w_n]$.

Rule

Usually we want $V > q > \frac{V}{2}$ where $V = w_1 + \dots + w_n$ is the total number of voters.

Compared to the usual situation, what can happen?
There are special situations:

Dictator: When a player has a weight $\geq q$.
(he has a majority of votes)
$\Rightarrow$ He decides if a motion passes or not.

• Example $[3, 3, 1, 1]$.

♡ <u>Veto power</u>. A player has a veto power if a motion cannot pass unless this player votes in favor of it.

Player with weight w has veto power

$$\Updownarrow$$

$$w < q \text{ and } V - w < q.$$

<u>Remark</u> when $w > \frac{V}{2}$ (majority).

then ~~V~~ the player has veto power.

LR In fact, the veto player is an extreme situation where his vote is <u>critical</u> in all situation

♡ <u>Winning coalition</u>. a situation where the motion passes.

<u>Ex</u>: $[3, 2, 1, 1]$. $\{P_1, P_2\}$ $\{P_1, P_3\}$ $\{P_1, P_2, P_3\}$ winning coalition

$\{P_1\}, \{P_2\}, \{P_3\}, \{P_2, P_3\}$ loosing.

♡ A player in a coalition is <u>critical</u> if without him the motion does not pass.

In $[3, 2, 1, 1]$ P_1 is critical in $\{P_1, P_2\}$ $\{P_1, P_3\}$ and $\{P_1, P_2, P_3\}$

$\Rightarrow$ P_1 has veto.

P_2 is critical in $\{P_1, P_2\}$, P_3 crit in $\{P_1, P_3\}$.

③ critical count: the number of times a player is critical.

In $[3, 2, 1, 1]$.

Critical count of P_1 : $B_1 = 3$.
" " P_2 : $B_2 = 1$
$B_3 = 1$.

$$\boxed{\text{Total critical count: } T = B_1 + B_2 + B_3 + \dots + B_N.}$$

In $[3, 2, 1, 1]$
$T = B_1 + B_2 + B_3 = 5$.

Banzhaf power index (measures the power of a player).

BPI of P_1 : $\beta_1 = \frac{B_1}{T}$

" " P_2 : $\beta_2 = \frac{B_2}{T}$

$\vdots$

P_N : $\beta_N = \frac{B_N}{T}$.

Remark : The sum $\beta_1 + \dots + \beta_N = 1$. (Always).

In $[3, 2, 1, 1]$ $\beta_1 = \frac{3}{5}$ $\beta_2 = \frac{1}{5}$ $\beta_3 = \frac{1}{5}$.

How to compute the BPI.

<u>Step 1</u> List all winning coalitions.

<u>Step 2</u> In each of them, find the critical players (underline them).

<u>Step 3</u> Compute each player's critical count. (B_i)

<u>Step 4</u> Compute the total critical count ($T = B_1 + \dots + B_N$)

<u>Step 5</u> Compute each BPI $\beta_i = \frac{B_i}{T}$.

<u>Example</u>: 2.1 Ex 5 (a)

[7, 4, 3, 3, 2]

<u>Step 1</u> $\{\underline{P_1}, \underline{P_2}\}$, $\{\underline{P_1}, \underline{P_3}\}$, $\cancel{\{P_1, P_4\}}$

$\{\underline{P_1}, P_2, P_3\}$ $\{\underline{P_1}, \underline{P_2}, P_4\}$ $\{\underline{P_1}, \underline{P_3}, P_4\}$

$\{\underline{P_1}, P_2, P_3, P_4\}$

<u>Step 2</u> Underline critical players.

Step 3 $B_1 = 5$ $B_2 = 2$ $B_3 = 2$ $B_4 = 0$

Step 4 $T = 9$ Step 5 $\beta_1 = \frac{5}{9}$ $\beta_2 = \frac{2}{9}$ $\beta_3 = \frac{2}{9}$ $\beta_4 = 0$

(b) $[9, 4, 3, 3, 2]$

Step 1 $\{\underline{P_1}, \underline{P_2}, \underline{P_3}\}$, $\{\underline{P_1}, \underline{P_2}, \underline{P_4}\}$, $\{\underline{P_1}, \underline{P_3}, \underline{P_4}\}$

$\{\underline{P_1}, P_2, P_3, P_4\}$

Step 3 $B_1 = 4$ $B_3 = 2$

$B_2 = 2$ $B_4 = 2$

Step 4: ~~β_1~~ $T = 10$

Step 5: $\beta_1 = \frac{4}{10}$ $\beta_2 = \frac{2}{10}$, $\beta_3 = \frac{2}{10}$, $\beta_4 = \frac{2}{10}$

week 2

II

1/

Shapley-Shubik power.

Last time, we talked about a way to quantify power of certain group of voters.
We will complicate things further.
Suppose now that each player joins a coalition one after another.

Say P_1 then P_2, then P_3 (called sequential coalition).

We denote by $\langle P_1, P_2, P_3 \rangle$.

Example $[16: 9, 8, 7]$.

Q: How can we determine the importance of each player in this setting?

A possible ~~Then~~ answer was given by Shapley-Shubik:

Take $\langle P_1, P_2, P_3 \rangle$ in $[16, 9, 8, 7]$.

First comes P_1 with 9 votes, not enough to win

then P_2 $\Rightarrow$ so P_2 is a pivotal for the sequential coalition.

Idea : Shapley Shubik's power : count ~~the number of times~~ how ~~much~~ often a player is pivotal.

Description

Step 1 list all the sequential coalitions for N players there will be

$$N! = N \times (N-1) \times (N-2) \times \cdots \times 1 \text{ coalitions.}$$

Step 2 : For each coalition, find the pivotal player.

Step 3 : For each player P_i count the number SS_i of times he is pivotal.

SS_1 = number of times P_1 is pivotal.

Step 4 : Compute

$$\sigma_1 = \frac{SS_1}{N!}$$

$$\sigma_2 = \frac{SS_2}{N!}$$

$$\vdots$$

$$\sigma_N = \frac{SS_N}{N!}$$

Week 2
II 2/

On example $[16: 9, 8, 7]$ 3 players $3!$ coalitions
$3! = 3 \times 2 \times 1 = 6$.

Step 1 $\langle \underbrace{P_1, \underline{P_2}}_{17}, P_3 \rangle$ $\langle \underbrace{P_2, P_3}_{15}, \underline{P_1} \rangle$

$\langle \underbrace{P_1, \underline{P_3}}_{16}, P_2 \rangle$ $\langle \underbrace{P_2, \underline{P_1}}_{17}, P_3 \rangle$

$\langle \underbrace{P_3, P_2}_{15}, \underline{P_1} \rangle$

$\langle \underbrace{P_3, \underline{P_1}}_{16}, P_2 \rangle$.

Step 2

Step 3 $SS_1 = 4$ $SS_2 = 1$

$SS_3 = 1$.

Step 4 $\sigma_1 = \frac{4}{6}$ $\sigma_2 = \frac{1}{6}$ $\sigma_3 = \frac{1}{6}$

Rmk: Always check that $\sigma_1 + \sigma_2 + \sigma_3 = 1$.

Week 2
II
3/

Example 2:

$[40 : 40, 10, 10, 10]$

4 players $\Rightarrow$ 4! coalition $4 \times 3 \times 2 = 24$

Step 1 $\langle P_1$

Coalitions starting with P_1 (6)

$2 = 2!$ $\langle P_1, \underline{P_2}, P_3, P_4 \rangle$ $\langle P_1, \underline{P_3}, P_4, P_2 \rangle$

$\langle P_1, \underline{P_2}, P_4, P_3 \rangle$ $\langle P_1, \underline{P_4}, P_2, P_3 \rangle$

$\langle P_1, \underline{P_3}, P_2, P_4 \rangle$ $\langle P_1, \underline{P_4}, P_3, P_2 \rangle$

Coalitions starting with P_2

$\langle P_2, \underline{P_1}, P_3, P_4 \rangle$ $\langle P_2, P_3, P_4, \underline{P_1} \rangle$

$\langle P_2, \underline{P_1}, P_4, P_3 \rangle$ $\langle P_2, P_4, P_1, P_3 \rangle$

$\langle P_2, P_3, \underline{P_1}, P_4 \rangle$ $\langle P_2, P_4, P_3, \underline{P_1} \rangle$

Coalition " " P_3

$\langle P_3, \underline{P_1}, P_2, P_4 \rangle$ $\langle P_3, P_2, P_4, P_1 \rangle$

$\langle P_3, \underline{P_1}, P_4, P_2 \rangle$ $\langle P_3, P_4, \underline{P_1}, P_2 \rangle$

$\langle P_3, P_2, \underline{P_1}, P_4 \rangle$ $\langle P_3, P_4, P_2, \underline{P_1} \rangle$

" " P_4

$\langle P_4, \underline{P_1}, P_2, P_3 \rangle$ $\langle P_4, P_2, P_3, \underline{P_1} \rangle$

$\langle P_4, \underline{P_1}, P_3, P_2 \rangle$ $\langle P_4, P_3, \underline{P_1}, P_2 \rangle$

$\langle P_4, P_2, \underline{P_1}, P_3 \rangle$ $\langle P_4, P_3, P_2, \underline{P_1} \rangle$

Step 2

Step 3:

$S_1 = 18 \quad S_2 = 2 \quad S_3 = 2 \quad S_4 = 2.$

Step 4 $\quad \sigma_1 = \frac{18}{24} \quad \sigma_2 = \frac{2}{24} \quad \sigma_3 = \frac{2}{24} \quad \sigma_4 = \frac{2}{24}$

Second method (Dangerous but quick).

If a coalition starts with P_1 automatically the ~~second~~ next player is pivotal, each player gets to be after P_1 only twice (order the last two).
In any other situation P_1 is pivotal.
There are 24 coalitions not starting with P_1

$24 = (3 \times 3!)$

3: choose the first player (not P_1).
3!: order the other 3.

$\Rightarrow \quad S_1 = 24 \quad S_2 = 2 \quad S_3 = 2 \quad S_4 = 2.$

(Week 2) II

4/

- [50, 40, 10, 10, 10].

[7; 4, 3, 3, 2]

4 player 4! sequence coalitions = 24.

$\langle P_1, \underline{P_2}, P_3, P_4 \rangle$ $\langle P_1, P_3, P_4, P_2 \rangle$
$\langle P_1, \underline{P_2}, P_4, P_3 \rangle$ $\langle P_1, P_3, \underline{P_2}, P_3 \rangle$
$\langle P_1, \underline{P_3}, P_2, P_4 \rangle$ $\langle P_1, P_4, \underline{P_3}, P_2 \rangle$

$\langle P_2, \underline{P_1}, P_3, P_4 \rangle$ $\langle P_2, P_3, P_1, P_4 \rangle$.
$\langle P_2, \underline{P_1}, P_4, P_3 \rangle$ $\langle P_2, P_4, \underline{P_1}, P_3 \rangle$
$\langle P_2, P_3, \underline{P_1}, P_4 \rangle$ $\langle P_2, P_4, \underline{P_3}, P_1 \rangle$

$\langle P_3, \underline{P_1}, P_2, P_4 \rangle$ $\langle P_3, P_2, P_4, P_1 \rangle$
$\langle P_3, \underline{P_1}, P_4, P_2 \rangle$ $\langle P_3, P_4, \underline{P_1}, P_2 \rangle$
$\langle P_3, P_2, \underline{P_1}, P_4 \rangle$ $\langle P_3, P_4, \underline{P_2}, P_1 \rangle$.

$\langle P_4, P_1, \underline{P_2}, P_3 \rangle$ $\langle P_4, P_2, \underline{P_3}, P_1$
$\langle P_4, P_1, \underline{P_3}, P_2 \rangle$ $\langle P_4, P_3, \underline{P_1}, P_2$
$\langle P_4, P_2, \underline{P_1}, P_3 \rangle$ $\langle P_4, P_3, \underline{P_2}, P_1$

$S_1 = 11$ $S_2 = 6$ $S_3 = 6$ $S_4 = 2$

$\sigma_1 = \frac{10}{24}$ $\sigma_2 = \frac{6}{24}$ $\sigma_3 = \frac{6}{24}$ $\sigma_4 = \frac{2}{24}$.

Lazy method

P_1 is pivotal when he is second after P_2 or P_3,

$$\langle \underbrace{p}_{\substack{P_2 \text{ or } P_3 \\ 2}}, \underline{P_1}, \underbrace{x, x}_{2!} \rangle$$

$2 \times 2 = 4$.

P_1 is pivotal when he is third after anyone

$$\langle x, x, \underline{P_1}, x \rangle$$

3 choices 2 choices. 6.

$3 \times 2 = 6$

$\cdot\, S_1 = 6 + 4$

P_2 is pivotal:

when he is second ~~of~~ after P_1

$\langle P_1, P_2, x, x \rangle$

$2!$

when he is third and the first two are not P_1

$\langle x, x, P_2, P_1 \rangle$

2

" " are $\{P_1, P_4\}$.

$\langle \underbrace{\ ,\ }_{\{P_1, P_3\}} P_2, x \rangle \to 2$ choices.

$S_2 = 2 + 2 + 2 = 6 = S_3$ (symmetry). $S_4 = 24 - S_1 - S_2 - S_3$

Week ~~2~~ 5

III

1

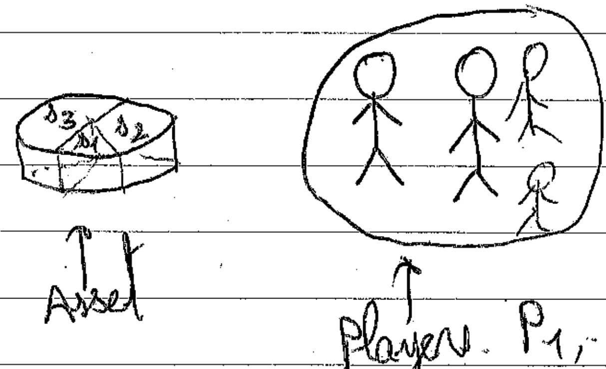
Fair division games.

Previously we were interested in voting methods and quantifying one's power in a vote.

Now we are interested in different type of ~~games~~ problems.

Namely the mathematics behind sharing and division.

How this works:



Value system: s_1 \$\$, s_2 ~~\$~~

P_1: s_1: 10% s_2: 35% s_3 57%

Objective: Find a method so that each player gets what he thinks is a fair share $\frac{1}{N}$ th the asset.

The division method decides who gets what and how.

division → Continuous fair division S is always divisible.
→ Discrete fair division S made up of objects.
→ Mixed fair division Some are divisible, and some are not.

The first method I present:

Divider chooser method.

"You cut, I choose".

D
- The reasoning: If we have the same value
  If I cut unequally the chooser will choose the advantageous part.
  Otherwise I cut equally.
- If the chooser thinks one of the two part is more advantageous, he will take it
  The chooser gets more.