

28/01/19

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Chapter 1, 1.1, 1.2

Basic facts and the plurality method

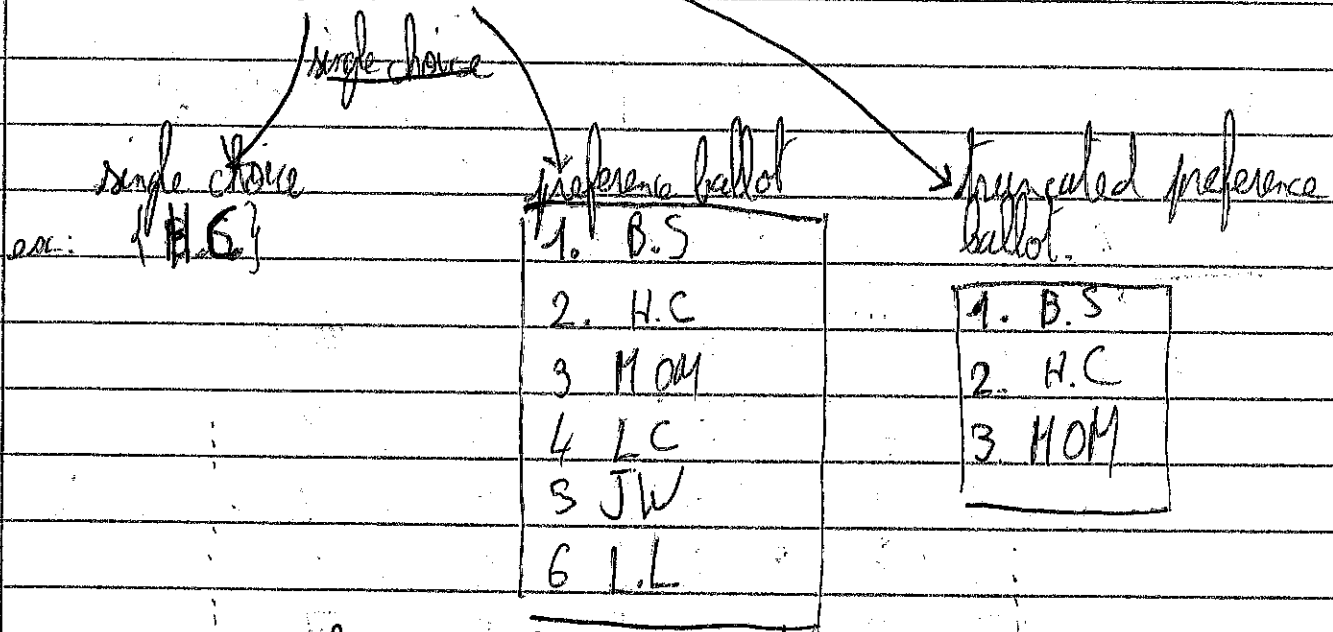
How an election works

Markus O'Halley

Phase 1: List of candidates $\mathcal{C} = \{H.C, B.S, MOM, L.C, JW, L.L\}$

↑
 P. Wells ↑ Lawrence Lemay ↑
 Lincoln Clarke

Phase 2: Voters make their choice on ballots



→ Phase 3: Apply a voting method to determine an outcome.

Unique winner

winning list

1.	A
2.	B
3.	6
4.	0

ranking

1.
2.
3.

IR: we won't be dealing with ties, so let us see the first example of voting method.

Plurality method

winner only

ranked election

Candidates who has the most number of votes

- outcome:
- 1 - Candidate who has the largest number of first votes
 - 2 - The second candidates with most 1. votes
 - 3 - The third ...

Majority: When a candidate has more than 50% of votes.

Example A

Preference ballot						# of 1st votes
# voters	14	10	8	7	4	
A	2	3	1	5	3	A → 8
B	1	1	2	3	2	B → 14 + 10 = 24
C	4	5	5	2	4	C → 0
D	5	2	4	1	5	D → 7
E	3	4	3	4	1	E → 4

Winner B
 2 - A
 3 - D
 4 - E
 5 - C

Winner B

Complete ranking

- 1 B
- 2 A
- 3 D
- 4 E

Example 2 (1.2 Ex 11)

# of voters	27	15	11	9	8	1
1st	C	A	B	D	B	B
2nd	D	B	D	A	A	A
3rd	B	D	A	B	C	D
4th	A	C	C	C	D	C

Step 1 Compute the number of voters who voted 1st place for each candidate
 Number of 1st place for A: 15

for B: $11 + 8 + 1 = 20$
 for C: 27
 for D: 9
 # of voters: $27 + 15 + 11 + 9 + 8 + 1 = 71$

Step 2
 Conduct

Winner C

The ranking of the candidates

1. C
2. B
3. A
4. D

Condorcet candidate: preferred candidate compared head to head.

voters who prefer D to C = $15 + 11 + 9 + 1 = 36$ (Majority)
 # voters who prefer B to C = $27 + 9 = 36$
 # " " D to A = $27 + 11 + 9 = 47$

D is a Condorcet candidate

More examples: see the ones treated in the book

More if students say's
 Ex 1.1 Ex 1.2

# voters	29	21	18	10	1
1	D	A	B	C	C
2	C	C	A	B	B
3	A	B	C	A	D
4	B	D	D	D	A

A: 21

B: 18

C: 11 = 10 + 1

D: 29

Winner D	Ranking	1 D
		2 A
		3 B
		4 C

Examples 4: Ex 13

# voters	6	5	4	2	2	2	2
1	C	A	B	B	C	C	C
2	D	D	D	A	B	B	D
3	A	C	C	C	A	D	B
4	B	B	A	D	D	A	A

A: 5

B: 6

C: 12

D: 0

Winner C	Ranking	1 C
		2 B
		3 A
		4 D

Borda method and plurality with elimination methods:

1/II

• Borda count

* each player gets some points for each vote depending on his ranking.

→ The winner is the candidate with most points

Example 2.1.2 Ex 11

# of voters	27	15	11	9	8	1
1	C	A	B	D	B	B
2	D	B	D	A	A	A
3	B	D	A	B	C	D
4	A	C	C	C	D	C

First 4 pts, Step 1 count points for each player.

2nd → 3 pts

3rd → 2 pts, Step 2 order

4th → 1 pt

Step 1 points for A: $15 \times 4 + (9+8+1) \times 3 + 11 \times 2 + 27$

$$\underline{60} + 54 + 22 + 27 = 163$$

$$\begin{aligned} \text{points for B: } & \left(\frac{11+8+1}{80} \right) \times 4 + 15 \times 3 + (27+9) \times 2 \\ & = 45 + 72 \\ & = 117 \end{aligned}$$

$$\begin{aligned} \text{" C: } & 27 \times 4 + 8 \times 2 + (15+11+9) \times 1 \\ & = 108 + 16 + 35 \\ & = 159 \end{aligned}$$

$$\begin{aligned} \text{" D: } & 9 \times 4 + (27+11) \times 3 + (15+1) \times 2 + 8 \\ & = 36 + 114 + 32 + 8 \\ & = 190 \end{aligned}$$

Winner	Ranking
D	1. B
	2. D
	3. A
	4. C

Example 1

# voters	14	10	8	7	4
A	2	3	1	5	3
B	1	1	2	3	2
C	4	5	5	2	4
D	5	2	4	1	5
E	3	4	3	4	1

points A: $14 \times 4 + 10 \times 3 + 8 \times 5 + 7 \times 1 + 4 \times 3$
 $= 56 + 30 + 40 + 7 + 12$
 $= 145$ pts

B: $(14+10) \times 5 + (8+7) \times 4 + 7 \times 3$
 $= 120 + 48 + 21$
 $= 189$ pts

C: $(14+4) \times 2 + (10+8) \times 1 + 7 \times 4$
 $= 36 + 18 + 28$
 $= 82$ pts

D: $(14+4) \times 1 + 10 \times 4 + 8 \times 2 + 7 \times 5$
 $= 18 + 40 + 16 + 35$
 $= 109$

E: $(14+8) \times 3 + (10+7) \times 2 + 4 \times 5$
 $= 66 + 34 + 20$
 $= 120$

Winner B Ranking: B, A, E, D, C

Example 3 Ch 1.1 Ex 12

Borda method for

# voters	29	21	18	10	1
(4pt) 1	D	A	B	C	C
(3pt) 2	C	C	A	B	B
(2pt) 3	A	B	C	A	D
1pt 4	B	D	D	D	A

Step 1

$$A: (29+10) \times 2 + 21 \times 4 + 18 \times 3 + 1$$

$$= 78 + 84 + 54 + 1$$

$$= 217 \text{ pts}$$

$$B: 18 \times 4 + (10+1) \times 3 + 21 \times 2 + 29$$

$$= 72 + 33 + 42 + 29$$

$$= 176 \text{ pts}$$

$$C: (10+1) \times 4 + (29+21) \times 3 + 18 \times 2 + 29$$

$$= 44 + 150 + 36$$

$$= 230 \text{ pts}$$

$$D: 29 \times 4 + 1 \times 2 + (21+18+10)$$

$$= 116 + 2 + 49$$

$$= 167 \text{ pts}$$

Winner C
Ranking: C, A, B, D

Ex 1.1

ch 1.1 Ex 1.3

	# notes	6	5	4	2	2	2	2
(4 pt)	1	C	A	B	B	C	C	C
(3 pt)	2	D	D	D	A	B	B	D
(2 pt)	3	A	C	C	C	A	D	B
(1 pt)	4	B	B	A	D	D	A	A

Step 1

$$A \rightarrow 5 \times 4 + 2 \times 3 + (6+2) \times 2 + (4+2) \times 1$$

$$= 20 + 6 + 16 + 6$$

$$= 48 \text{ pts}$$

$$B \rightarrow (4+2) \times 4 + (2+2) \times 3 + 2 \times 2 + (6+5)$$

$$= 24 + 12 + 4 + 11$$

$$= 51 \text{ pts}$$

$$C \rightarrow (6+2+2+2) \times 4 + (5+4+2) \times 2$$

$$= 48 + 22$$

$$= 70 \text{ pts}$$

$$D \rightarrow (6+5+4+2) \times 3 + 2 \times 2 + 2$$

$$= 51 + 4 + 2$$

$$= 57 \text{ pts}$$

Winner C
Ranking C, D, B, A.

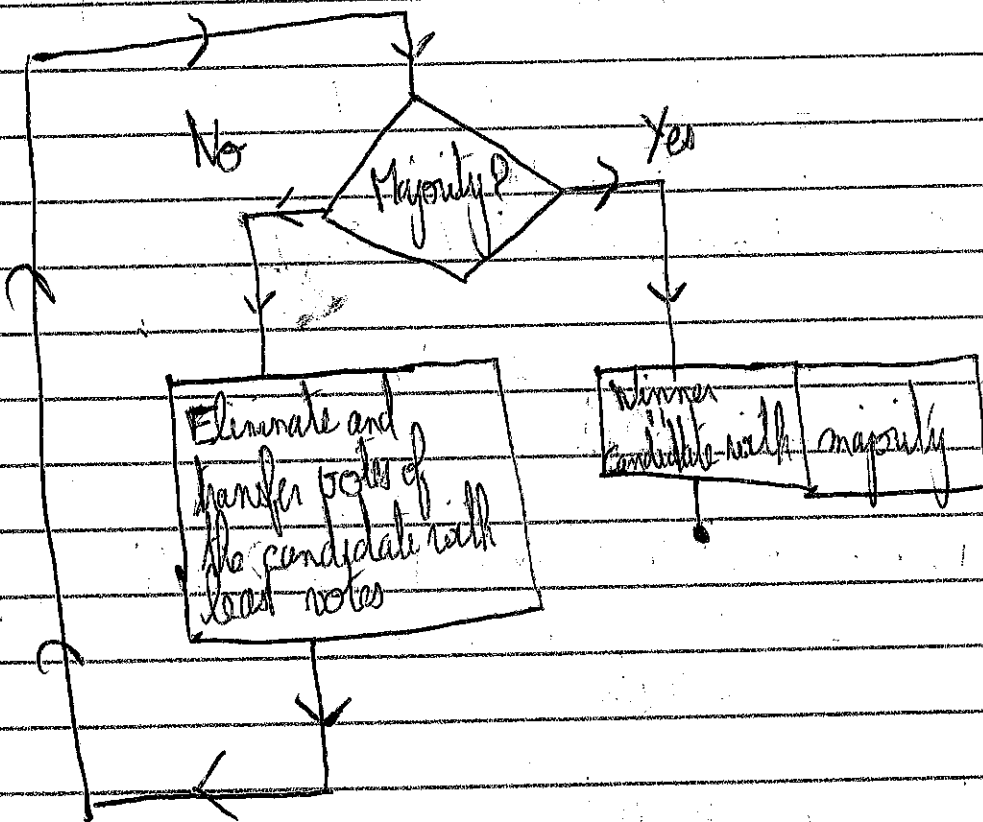
Plurality with elimination (winner only)

IR (it starts to get more complicated now)

Step 1 Check if there is a majority
If yes, the winner is the candidate with the majority of votes.

Step 2 If not, eliminate the candidate with least votes, transfer his votes to the next candidate with least votes.

Step 3 Go back to step 1.



Example 2 (1.2 Ex 11)

	27	15	11	9	8	1
1	C	A	B	D	B	B
2	D	B	D	A	A	A
3	B	D	A	B	D	D
4	A	C	C	C	D	C

C	27
A	15
B	20
D	9

L = [D]

C	27
B	20
A	24

Eliminate B
 \Rightarrow L = [B, D]

A	27
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Ranking(A, C, B, D)
 Winner A

A	44
C	27

Example 3 1.1 Ex 12

	29	21	18	10	1
1	D	A	B	C	C
2	C	C	A	B	B
3	A	B	C	A	D

A	21
B	18
C	11
D	29

L = [C]

A	21
B	29
D	29

~~1st B~~
~~2nd D~~
~~3rd A~~
 \Rightarrow L = [A, C]

B	50
D	29

Ranking B, D, A, C

Example 4
13

	6	5	4	2	2	2
1	C	A	B	B	C	C
2	D	D	D	A	B	D

A	5
B	6
C	12
D	0

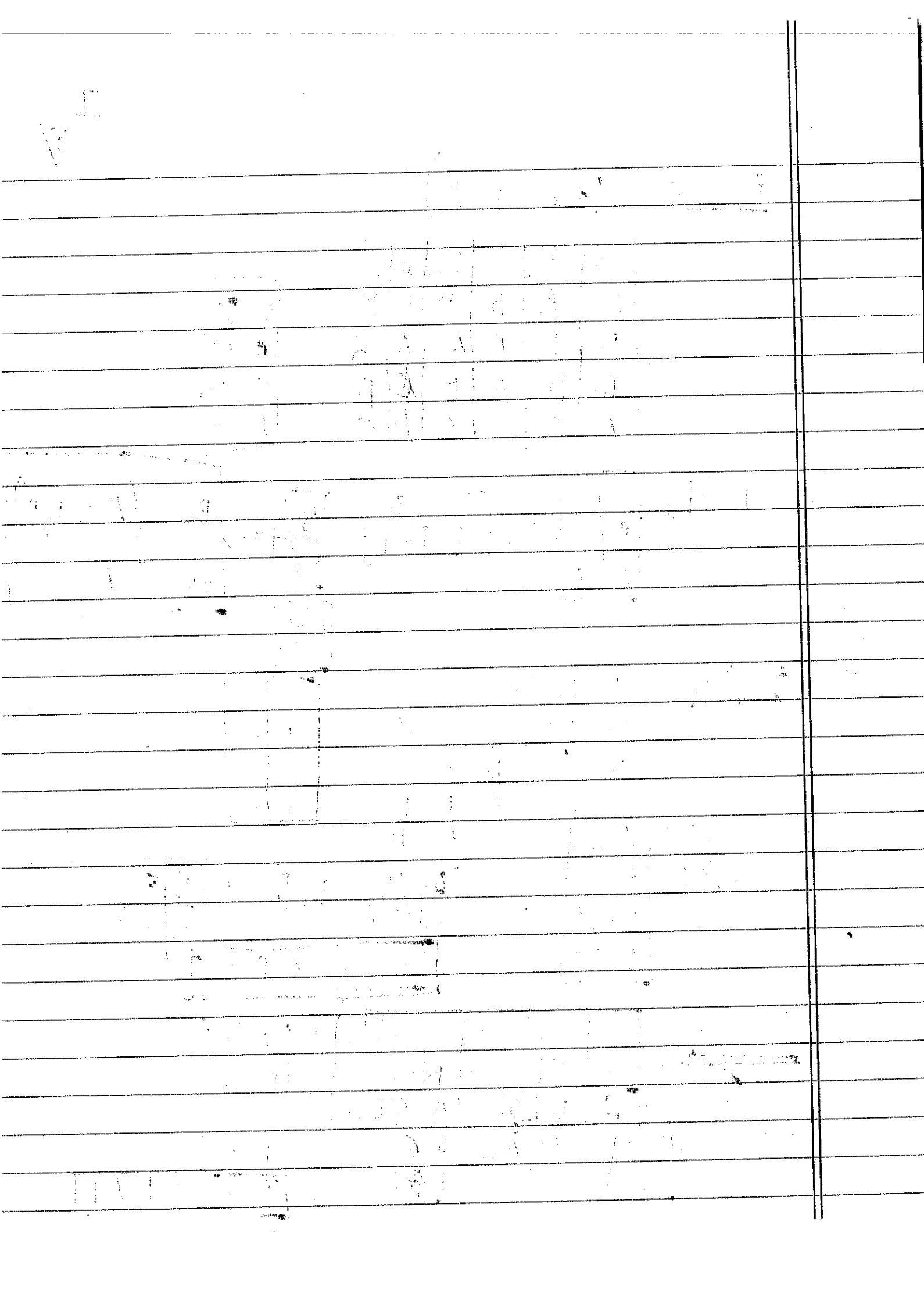
L = [D]

A	5
B	6
C	12

\Rightarrow L = [A, D]

B	6
C	12

\Rightarrow Ranking C, B, A, D



Ranked plurality with elimination

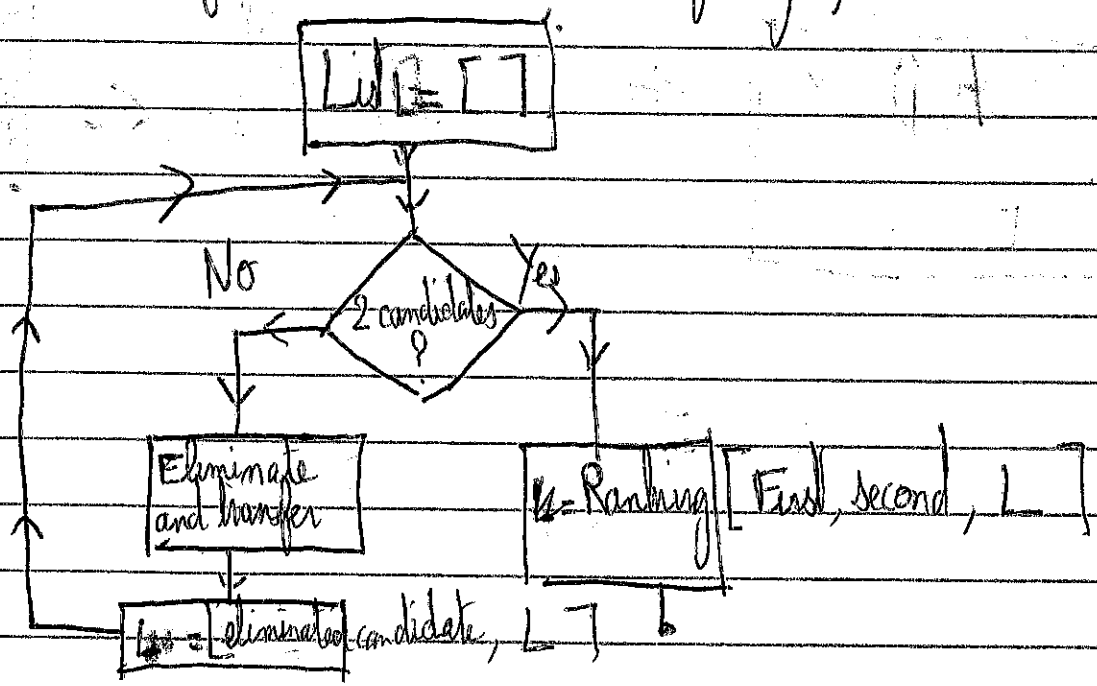
Step 1 If there are 2 candidates, order the 2 candidates at the top of the list.

Step 2 Eliminate candidates with least votes and transfer his votes to the next candidate (with least votes)

Step 3 Add the eliminated to the top of the list

Step 4 Go back to step 1.

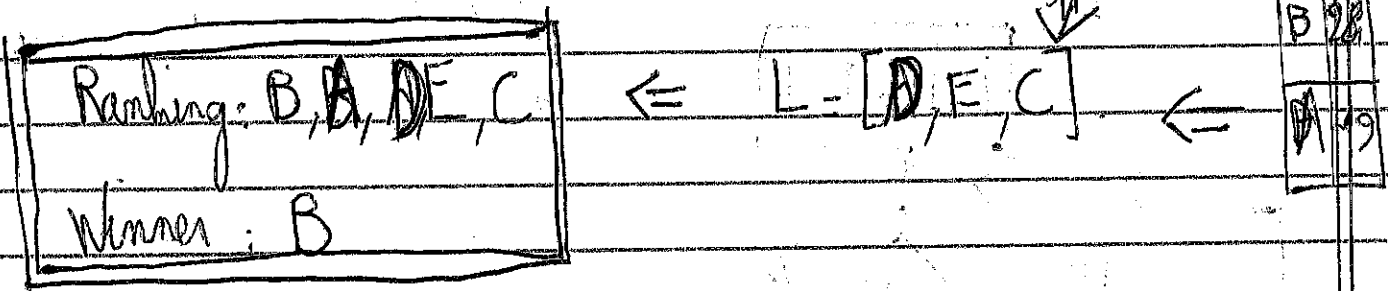
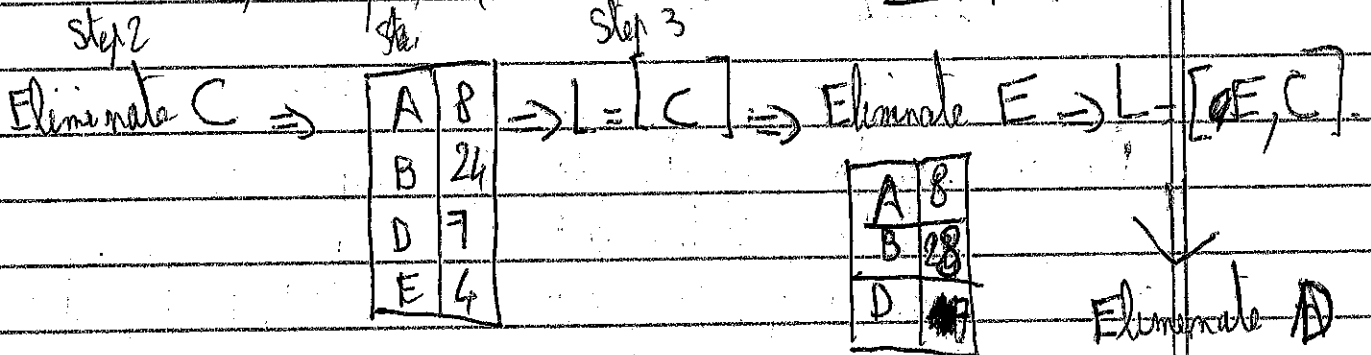
Remark. In this algorithm, the first eliminated gets to the bottom of the list, as and the last eliminated is third. If a candidate has a majority, he ends up first.



Advice If you have to do a ranking do the second one directly and you decide the winner by taking the first of the list.

Example 1 1.1 Bn/4

# voters	1	10	8	7	4											
A	2	3	1	5	3	<table border="1"> <tr><td>A</td><td>8</td></tr> <tr><td>B</td><td>24</td></tr> <tr><td>C</td><td>0</td></tr> <tr><td>D</td><td>7</td></tr> <tr><td>E</td><td>1</td></tr> </table>	A	8	B	24	C	0	D	7	E	1
A	8															
B	24															
C	0															
D	7															
E	1															
B	1	1	2	3	2	⇒										
C	4	5	5	2	4											
D	5	2	4	1	5											
E	3	4	3	4	1											



III
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Method of pairwise comparison and fairness criteria. (1.5, 1.6)

Goal In the method of pairwise comparison,
we determine the most preferred candidate.
• Like in a tournament where each player
plays against everyone.

Method of pairwise comparison: B.

- Ballots are ranked votes.
- For each pair of candidates: X, Y

~~Assign 1 pt for each.~~

Step 1: Do a pairwise comparison between these
2 candidates.

Count the # of voters who prefer X to Y .
and the # " " who prefer Y to X .
The ~~winner~~ candidate who is preferred (more voters).

gets 1 pt, the other gets 0 and both get $\frac{1}{2}$ pt in case of a tie.

Step 2 The candidates with most pts wins the election.

Oral sum the total points and decide the ranking.

Tips to do it fast compute the total numbers of voters.

Example 1 1.1 Ex 4 half votes!! to win a pairwise comparison, you need more than half votes!!

total 43.

# voters	14	10	8	7	4
A	2	3	1	5	3
B	1	1	2	3	2
C	4	5	5	2	4
D	5	2	4	1	5
E	3	4	3	4	1

A vs B \rightarrow B
 A vs C \rightarrow A
 A vs D \rightarrow A
 A vs E \rightarrow A
 B vs C \rightarrow B
 B vs D \rightarrow B
 B vs E \rightarrow B
 C vs D \rightarrow D

Detailed A vs B

$A > B \Rightarrow 8$ B wins.

A vs C: $A > C$ $14 + 10 + 8 + 4 = 36$

A vs D: $A > D$ $14 + 8 + 4 = 26$

A vs E: $A > E$ $14 + 10 + 8 = 32$

B vs C: $B > C$ $14 + 10 + 8 + 4$

A \rightarrow 3 pts C 0 E 1 pt
 B \rightarrow 4 pts D 4

Winner B

Ranking (B, A, E, D, C).

Example 2: (1.2 Pn 11)

	27	15	11	9	8	1
1	C	A	B	D	B	B
2	D	B	D	A	A	A
3	B	D	A	B	C	D
4	A	C	C	C	D	C

Step 1 total: $27 + 15 + 11 + 9 + 8 + 1 = 71$
 Half = 35.5

Step 2 pairwise comparison. (4 candidates $\rightarrow \frac{4 \times 3}{2} = 6$ comparisons)

A vs B $\rightarrow B > A$ $27 + 11 + \dots$ $\rightarrow B$

A vs C $\rightarrow A > C$ $15 + 11 + 9 + 8 + 1 \rightarrow A$

A vs D $\rightarrow D > A$ $27 + 11 + \dots \rightarrow D$

B vs C $\rightarrow B$

B vs D $\rightarrow D > B$ $27 + 9 + \dots \rightarrow D$

C vs D $\rightarrow C > D$ $27 + 8 = 35 \rightarrow D$

Step 3 Count pts
 A \rightarrow 1pt B 2pts D 3pts C 0pts
 || Winner D ||

|| Ranking D, B, A, C ||

Example 3 (1.1 Ex 12)

# voters	29	21	18	10	1
1	D	A	B	C	C
2	C	C	A	B	B
3	A	B	C	A	D
4	B	D	D	D	A

Step 1 Total: 79
 Half: 34.5

Step 2 Pairwise comparison (4 candidates $\frac{4 \times 3}{2}$ comparisons)

A vs B \rightarrow A

B vs C \rightarrow C C vs D. C

A vs C \rightarrow A

B vs D \rightarrow B

A vs D \rightarrow A

Step 3

A \rightarrow 3 pts

C \rightarrow 2 pts

B \rightarrow 1 pt

D \rightarrow 0 pts

Winner A

Ranking A, C, B, D

Example 4 (1.1 & 13)

	6	5	4	2	2	2
1	C	A	B	B	C	C
2	D	D	D	A	B	B
3	A	C	C	C	A	D
4	B	B	A	D	D	A

Step 1 Total: 23 Half: 11,5.

Step 2 Pairwise comparison 4 candidates $\frac{4 \times 3}{2} = 6$ comp.

A vs B \rightarrow B B vs C \rightarrow C C vs D \rightarrow C

A vs C \rightarrow C B vs D \rightarrow D

A vs D \rightarrow D

A 0pt C 3pt
B 1pt D 2pt

Step 3 Winner C
Ranking C, D, B, A

Question: what is the best voting method?

Answer: Depends on who you want to win.

This question was answered precisely by Arrow.

Each voting method has particular features (they satisfy particular criteria)

① [A] • Majority criterion: A majority candidate always wins.

[B] • Condorcet criterion: A condorcet candidate always wins.

[C] • Monotonicity criterion: A winner also wins if he was ranked higher in the ballots.

[D] • Independence of irrelevant alternatives (IIA) criterion.

If we remove a losing candidate, the winner remains the same.

Arrow Impossibility theorem

There is no voting method who would satisfy all these criteria!

Example

- Borda count does not satisfy the majority criterion

	4	2	1
1	A	B	C
2	B	C	B
3	C	A	A

A has a majority of 1st votes
but B wins

$$A \rightarrow 4 \times 3 + 3 = 15 \text{ pt}$$

$$B \rightarrow 8 + 6 + 2 = 16 \text{ pt}$$

- Plurality method violates the Condorcet criterion

see example of last week.

	4	3	2
1	A	B	C
2	B	A	B
3	C	C	A

Plurality with elimination violates
monotonicity criterion.

	7	8	10	2
1	A	B	C	A
2	B	C	A	C
3	C	A	B	B

Eliminate B → C wins.

	7	8	10	2
1	A	B	C	C
2	B	C	A	A
3	C	A	B	B

Eliminate A → B wins.

The key is to eliminate a different candidate.

Pairwise comparison violates IIA.

	4	3	2
1	A	B	C
2	B	A	B
3	C	C	A