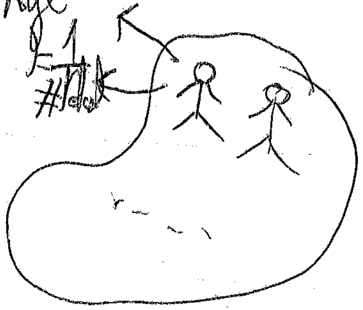
.2 18, 22, 27
6.3 33, 43
6.4 58, 62

# Expectation

**Main example**: What is the average age of the people in this class room?

1st way to compute:

$$E = \frac{\text{sum the ages of all the persons}}{\#\ \text{persons}}$$




However the sum of all the ages can be computed in another fashion:

$$20 \times (\#\ \text{persons who are 20}) + 21 \times (\#\ \text{persons who are 21}) + \dots$$

E then becomes:

$$E = 20 \times \underbrace{\left(\frac{\#\ \text{persons who are 20}}{\#\ \text{total persons}}\right)}_{\text{probability that a person is 20}} + 21 \times \left(\frac{\#\ \text{persons who are 21}}{\#\ \text{total persons}}\right) + \dots$$

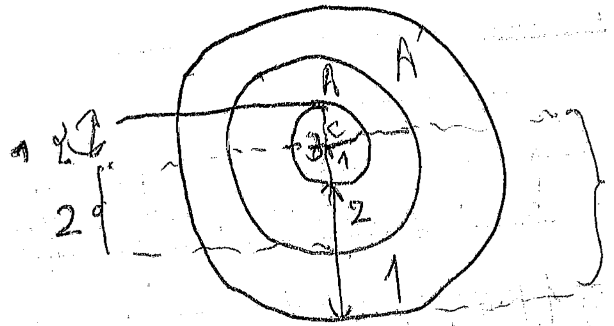
" " 21.

In other words:

$$E = \underset{\text{over all values}}{\text{sum of}} \left[ \underset{\text{ex: } 20}{\text{values}} \times \left( \text{probability of that values} \right) \right]$$

$$\text{probability} = \frac{\#\ \text{person who are 20}}{\#\ \text{total person}}$$

<u>The dart example (bis).</u>




What is the average number of pts that I get?

Recall


$\text{Area} = \pi \times R^2$

event $A' \longrightarrow$

$$\mathbb{P}(A') = \frac{\text{Area}(A')}{\text{Area}(\text{Disc})} = \frac{\pi \times 3^2 - \pi \times 2^2}{\pi \times 3^2} = \frac{5}{9}$$

$$\mathbb{P}(A) = \frac{\text{Area}(A)}{\text{Area}(\text{Disc})} = \frac{\pi\, 2^2 - \pi\, 1^2}{\pi \times 3^2} = \frac{3}{9}$$

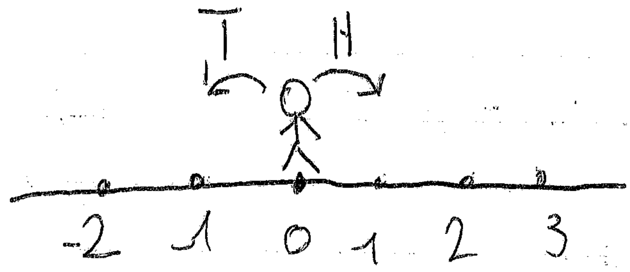
$$\mathbb{P}(C) = \frac{\text{Area}(C)}{\text{Area}(\text{Disc})} = \frac{\pi\, 1^2}{\pi\, 3^2} = \frac{1}{9}$$

| Event | $A'$ | $A$ | $C$ |
| --- | --- | --- | --- |
| Probability | $\frac{5}{9}$ | $\frac{3}{9}$ | $\frac{1}{9}$ |
| Value | 1 | 2 | 3 |

$$E = 1 \times \frac{5}{9} + 2 \times \frac{3}{9} + 3 \times \frac{1}{9}$$

$$\boxed{E = \frac{14}{9}}$$

**Drunken walker :** (perfectly balanced, indep throws)




After 4 steps, where is the walker on average ?

S = 4 tuples of elements of $\{H, T\}$.

an element is $(H, H, T, T)$

$(x_1, x_2, x_3, x_4)$

H or T

$\#S = 2 \times 2 \times 2 \times 2 = 16$.

If the walker tossed $(H, H, T, T)$ he ends at 0.

$(H, H, H, H)$ " " 4.

$A_0$ = end at 0 $\qquad$ $A_i$ = end at $i$

$A_1$ = " 1

$A_{-1}$ = " " $-1$

$\#A_0$ = # ways to put H, H in ( , , , ).

choose which positions to put the two heads.

there are 4 positions

$\#A_0 = \frac{4 \times 3}{2} = 6.$ $\quad \#A_2 = \#$ ways to put 3 H in ( , , , ) 1 T

$\#A_1 = 0$ $\quad \#A_2 = 4.$

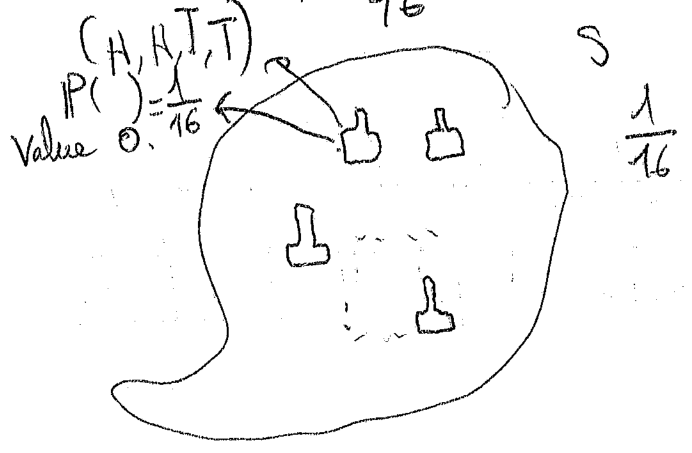
$\#A_{-1} = 0$ $\quad \#A_{-2} = 4$

$\#A_3 = 0$ $\quad \#A_{-3} = 0$

$\#A_4 = 1$ $\quad \#A_{-4} = 1.$

| Event | $A_0$ | $A_1$ | $A_{-1}$ | $A_2$ | $A_{-2}$ | $A_3$ | $A_{-3}$ | $A_4$ | $A_{-4}$ |
|---|---|---|---|---|---|---|---|---|---|
| Probability | $\frac{6}{16}$ | 0 | 0 | $\frac{4}{16}$ | $\frac{4}{16}$ | 0 | 0 | $\frac{1}{16}$ | $\frac{1}{16}$ |
| Position (Value). | 0 | 1 | -1 | 2 | -2 | 3 | -3 | 4 | -4. |

$$E = 0 \times \frac{6}{16} + 1 \times 0 + (-1) \times 0 + 2 \times \frac{4}{16} + (-2) \times \frac{4}{16} + 4 \times \frac{1}{16} - 4 \times \frac{1}{16} = 0.$$



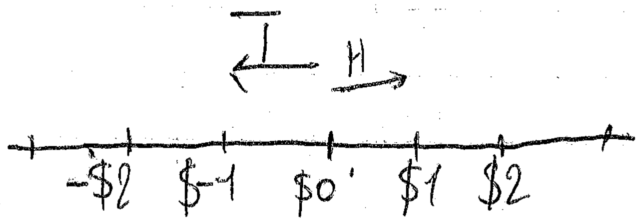
Proba IV
3/

<u>Gambler's ruins:</u> H → ~~loose~~ win 1 \$
T → loose 1\$.

coin: $P(H) = \frac{1}{3}$

$P(T) = \frac{2}{3}$.

Toss the coin 4 times




What is the average win?

$$P(A_0) = (2 \text{ combinations of } 4) \times \frac{1}{3} \times \frac{1}{3} \times \frac{2}{3} \times \frac{2}{3}.$$

$$= \underbrace{\frac{4 \times 3}{2}}_{\#A_0} \times \underbrace{\frac{1}{3} \times \frac{1}{3} \times \frac{2}{3} \times \frac{2}{3}}_{\text{probability of a single elt of } A_0.}$$

$$P(A_0) = \frac{8}{27}$$

$$P(A_{-1}) = 0$$

$$P(A_1) = 0$$

$$P(A_2) = \#A_2 \times \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} \times \frac{2}{3}$$

$$= 4 \times \frac{1}{3^4} \times 2$$

$$P(A_2) = \frac{16}{81}$$

$$\mathbb{P}(A_{-2}) = (\# A_{-2}) \times \overset{T}{\frac{2}{3}} \times \overset{T}{\frac{2}{3}} \times \overset{T}{\frac{2}{3}} \times \overset{H}{\frac{1}{3}}$$

$$= 4 \times \frac{8}{81}$$

$$= \frac{32}{81}$$

$$\mathbb{P}(A_3) = \mathbb{P}(A_{-3}) = 0$$

$$\mathbb{P}(A_4) = \# A_4 \times \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} = \frac{1}{81}$$

$$\mathbb{P}(A_{-4}) = \# A_{-4} \times \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{16}{81}$$

| Event | $A_0$ | $A_{-1}$ | $A_1$ | $A_2$ | $A_{-2}$ | $A_3$ | $A_{-3}$ | $A_4$ | $A_{-4}$ |
|---|---|---|---|---|---|---|---|---|---|
| Probability | $\frac{24}{81}$ | 0 | 0 | $\frac{16}{81}$ | $\frac{32}{81}$ | 0 | 0 | $\frac{1}{81}$ | $\frac{16}{81}$ |
| Value | 0 | -1 | 1 | +2 | -2 | 3 | -3 | 4 | -4 |

$$E = 2 \times \frac{16}{81} - 2 \times \frac{32}{81} + \frac{4}{81} - 4 \times \frac{16}{81}$$

$$= \frac{1}{81}\left[32 - 64 + 4 - 64\right]$$

$$E = -\frac{92}{81} \simeq -\$1.13$$