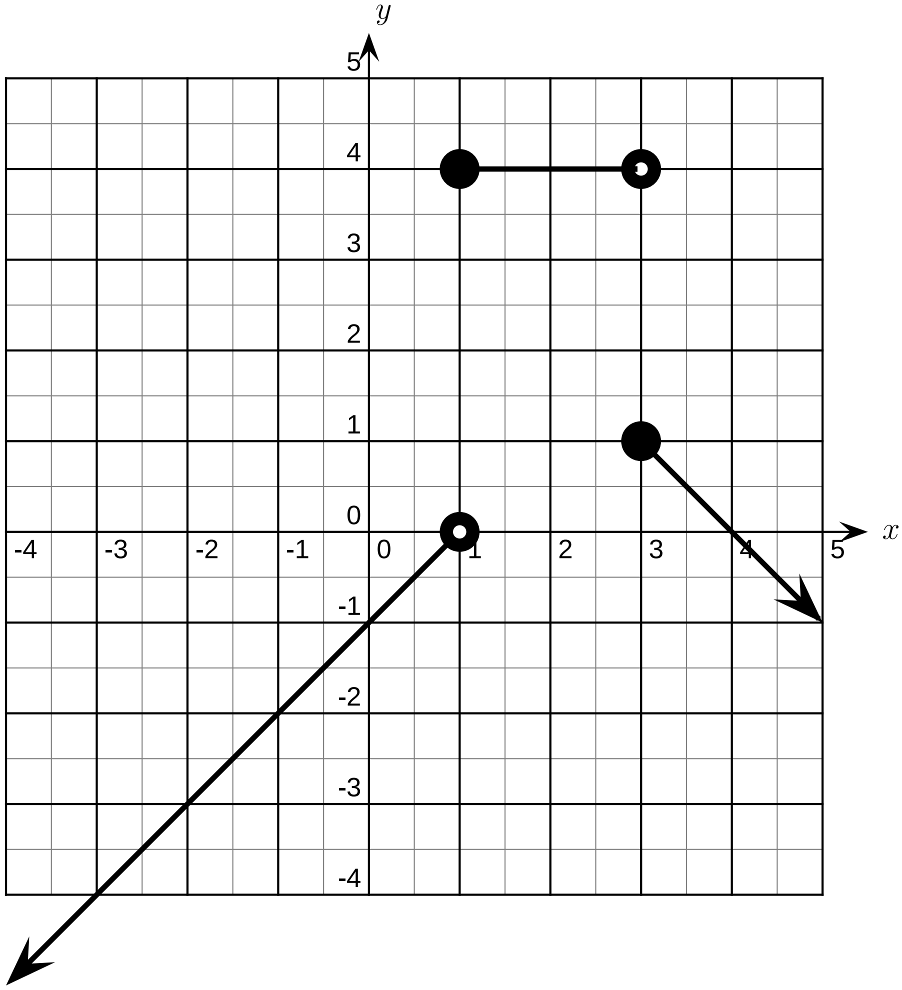
MAP 103 Spring 2008 Exam 2 Review Answers

Problems:

1. Graph the following piecewise function:

$$f(x) = \begin{cases} x - 1 & \text{if } x < 1 \\ 4 & \text{if } 1 \leq x < 3 \\ -x + 4 & \text{if } x \geq 3 \end{cases}$$



2. Rewrite the function $g(x) = |3 - x|$ as a piecewise function with two rules. Each rule should be a first-degree polynomial.

Solution.

$$g(x) = \begin{cases} 3 - x & \text{if } x \leq 3 \\ x - 3 & \text{if } x > 3 \end{cases}$$

3. Find $h(0)$, $h(5)$, $h(-3)$, and $h(k-3)$ if $k - 3 < -2$ if

$$h(x) = \begin{cases} -x^2 + 3x - 1 & \text{if } x < -1 \\ 3 - x^3 & \text{if } -1 \leq x \leq 2 \\ \sqrt{x} & \text{if } x > 2 \end{cases}$$

Solution.

(a) $h(0) = 3 - 0^3 = \boxed{3}$

(b) $h(5) = \boxed{\sqrt{5}}$

(c) $h(-3) = -(-3)^2 + 3(-3) - 1 = -9 - 9 - 1 = \boxed{-19}$

(d) Since $k - 3 < -2$, $k - 3$ is also less than -1, so we will use the top rule: $h(k-3) = -(k-3)^2+3(k-3)-1 = -(k^2-6k+9)+3k-9-1 = -k^2+6k-9+3k-10 = \boxed{-k^2 + 9k - 19}$

4. Subtract $1 - 5xy + 3xy^2 - 4y$ from $-2xy + 5xy^2 + 3y$.

Solution. $-2xy+5xy^2+3y-(1-5xy+3xy^2-4y) = -2xy+5xy^2+3y-1+5xy-3xy^2+4y = \boxed{2xy^2 + 3xy + 7y - 1}$

5. Write the area of a rectangle as a fourth-degree polynomial if its length is given by x^2+x+2 and its width is given by $2x^2 - 3x + 1$.

Solution. The area of a rectangle is its length times its width:
$(x^2+x+2)(2x^2-3x+1) = 2x^4-3x^3+x^2+2x^3-3x^2+x+4x^2-6x+2 = \boxed{2x^4 - x^3 + 2x^2 - 5x + 2}$

6. Expand completely: $(x + h)^3$.

Solution.

$$(x + h)^3 = (x + h)(x + h)(x + h)$$

Multiply the first two binomials out first and then multiply that result by the third:

$$(x + h)(x + h) = x^2 + xh + xh + h^2 = x^2 + 2xh + h^2$$

$$(x^2 + 2xh + h^2)(x + h) = x^3 + x^2h + 2x^2h + 2xh^2 + xh^2 + h^3 = \boxed{x^3 + 3x^2h + 3xh^2 + h^3}$$

7. Divide $x^3 - 4x^2 + 2x - 1$ by $x^2 + 1$. Clearly identify the quotient and remainder.

Solution. This is tough to type out, so here is the answer:

$$\boxed{x - 4 + \frac{x + 3}{x^2 + 1}}$$

8. Divide: $\dfrac{20x^3 - 8x^2 + 5x - 5}{5x - 2}$

Solution.

$$\boxed{4x^2 + 1 - \frac{3}{5x - 2}}$$

9. Factor the following polynomials completely:

(a) $16x^4 - 81y^4 = (4x^2 - 9y^2)(4x^2 + 9y^2) = \boxed{(2x - 3y)(2x + 3y)(4x^2 + 9y^2)}$

(b) $9x^2y - 9xy + 2y = y(9x^2 - 9x + 2) = \boxed{y(3y - 1)(3y - 2)}$

(c) $2a^2(y + 1) + a(y + 1) - 6(y + 1) = (y + 1)(2a^2 + a - 6) = \boxed{(y + 1)(2a - 3)(a + 2)}$

(d) $12y^6+23y^3+10 = 12y^6+8y^3+15y^3+10 = 4y^3(3y^3+2)+5(3y^3+2) = \boxed{(3y^3 + 2)(4y^3 + 5)}$

(e) $7g^2-12gh-4h^2 = 7g^2-14gh+2gh-4h^2 = 7g(g-2h)+2h(g-2h) = \boxed{(7g - 2h)(g - 2h)}$

(f) $x^2 + 16x - 80 = \boxed{(x + 20)(x - 4)}$

(g) $15x^2 - 91x + 6 = 15x^2 - 90x - 1x + 6 = 15x(x - 6) - 1(x - 6) = \boxed{(15x - 1)(x - 6)}$

(h) $2x^5 + 16x^2y^3 = 2x^2(x^3 + 8y^3)$. You can go further with this one if you know how to factor the sum of two cubes, but that is not required for the exam.

(i) $5xy^2 - 9xy + 4x = x(5y^2 - 9y + 4) = \boxed{x(5y - 4)(y - 1)}$

(j) $x^2 + 6xy + 8y^2 = x^2 + 4xy + 2xy + 8y^2 = x(x + 4y) + 2y(x + 4y) = \boxed{(x + 2y)(x + 4y)}$

(k) $(3r + 6)^2 - 12(3r + 6) - 28$. Let $u = 3r + 6$ and factor $u^2 - 12u - 28$, then replace u by $3r + 6$:$u^2 - 12u - 28 = (u - 14)(u + 2) = (3r + 6 - 14)(3r + 6 + 2) = \boxed{(3r - 8)(3r + 8)}$.

(l) $3x^2y^3 + 6x^2y^2 - 45x^2y = 3x^2y(y^2 + 2y - 15) = \boxed{3x^2y(y + 5)(y - 3)}$

(m) $3x^{2n} - 8x^n + 4$ (Assume $n \in \mathbb{N}$. This is challenging.) Since we have a trinomial, let's attempt to use the ac-method here. The factor pair of 12 that adds up to -8 is $-6, -2$ so we will split up the middle term into two terms with those coefficients:

$$3x^{2n} - 6x^n - 2x^n + 4$$

Note that $x^n \cdot x^n = x^{2n}$, so the GCF of the first two terms is $3x^n$. The GCF of the second two terms is -2, so we get

$$3x^n(x^n - 2) - 2(x^n - 2) = \boxed{(3x^n - 2)(x^n - 2)}$$

10. Find all solutions to each equation below:

(a) $-2t^3 = 108t - 30t^2$
Solution. Since this equation is not linear, we will proceed by moving all terms to one side and attempt to factor:

$$-2t^3 = 108t - 30t^2 \rightarrow 0 = 2t^3 - 30t^2 + 108t$$

We now can factor out $2t$ from the right-hand side (RHS):

$$0 = 2t(t^2 - 15t + 54)$$

Factor the trinomial:

$$0 = 2t(t-9)(t-6)$$

We now can set each factor equal to 0 and solve the resultant linear equations:

$$2t = 0 \to \boxed{t=0}, \qquad t-9=0 \to \boxed{t=9}, \qquad t-6=0 \to \boxed{t=6}$$

(b) $y^3 + 4y^2 = 9y + 36$
Solution. Since this equation is not linear, we will proceed by moving all terms to one side and attempt to factor:

$$y^3 + 4y^2 = 9y + 36 \to y^3 + 4y^2 - 9y - 36 = 0$$

We now can factor the left-hand side (LHS) by factoring by grouping:

$$y^2(y+4) - 9(y+4) = 0 \to (y^2-9)(y+4) = 0 \to (y+3)(y-3)(y+4) = 0$$

We now can set each factor equal to 0 and solve the resultant linear equations:

$$y+3=0 \to \boxed{y=-3}, \qquad y-3=0 \to \boxed{y=3}, \qquad y+4=0 \to \boxed{y=-4}$$

(c) $2z(z+6) = 2z^2 + 12z - 8$
Solution. First expand the LHS:

$$2z^2 + 12z = 2z^2 + 12z - 8$$

Notice that there is a term of $2z^2$ on both sides of the equation that could be cancelled from both sides of the equation, leaving us with

$$12z = 12z - 8$$

Subtracting $12z$ from both sides leaves us with the equation $0 = -8$, which clearly has no solutions. So the original equation has $\boxed{\text{no solutions}}$.

(d) $2x^3 + 5x^2 = 8x + 20$
Solution. Since this equation is not linear, we will proceed by moving all terms to one side and attempt to factor:

$$2x^3 + 5x^2 = 8x + 20 \to 2x^3 + 5x^2 - 8x - 20 = 0$$

We now can factor the left-hand side (LHS) by factoring by grouping:

$$x^2(2x+5) - 4(2x+5) = 0 \to (x^2-4)(2x+5) = 0 \to (x+2)(x-2)(2x+5) = 0$$

We now can set each factor equal to 0 and solve the resultant linear equations:

$$x+2=0 \to \boxed{x=-2}, \qquad x-2=0 \to \boxed{x=2}, \qquad 2x+5=0 \to \boxed{x=-5/2}$$

(e) $(x+2)(x-2) = 5(x+4)$
Solution.
First, expand both sides:

$$x^2 - 4 = 5x + 20$$

Since this equation is not linear, we will proceed by moving all terms to one side and attempt to factor:

$$x^2 - 4 = 5x + 20 \to x^2 - 5x - 24 = 0$$

We now can factor the left-hand side (LHS) by the ac-method (or guess-and-check factoring):

$$(x-8)(x+3) = 0$$

We now can set each factor equal to 0 and solve the resultant linear equations:

$$x - 8 = 0 \to \boxed{x = 8}, \qquad x + 3 = 0 \to \boxed{x = -3}$$

(f) $b - (3 + 2b) = 2b - 7(1 - b)$

Solution: Expand both sides:

$$b - 3 - 2b = 2b - 7 + 7b \to -3 - b = 9b - 7$$

This is a linear equation so we will move all b terms to one side and all constants to the other side:

$$4 = 10b \to \boxed{b = 2/5}$$

(g) $(x^2 + x - 6)(3x^2 - 14x - 5) = 0$

Solution. We have a product of two factors equal to 0. In order for this equation to be true, one of the factors must be 0. So set both factors equal to 0:

$$x^2 + x - 6 = 0 \text{ or } 3x^2 - 14x - 5 = 0$$

We will factor each trinomial to solve each equation:

$$x^2 + x - 6 = (x+3)(x-2) = 0 \to x + 3 = 0 \text{ or } x - 2 = 0 \to \boxed{x = -3 \text{ or } x = 2}$$

$$3x^2 - 14x - 5 = 3x^2 + 1x - 15x - 5 = 0 \to x(3x+1) - 5(x+1) = 0 \to (x-5)(3x+1) = 0$$

$$\to x - 5 = 0 \text{ or } 3x + 1 = 0 \to \boxed{x = 5 \text{ or } x = -1/3}$$

So the original equation has four solutions.

(h) $x^3 - 18x = 3x^2$
Solution. I will just give the solutions to this equation: $x = 0, -3, 6$

(i) $p^4 + 36 = 13p^2$
Solution. I know this wasn't on the original review sheet, but it's a good problem to try anyway. Since we don't have a linear equation, move everything to one side:

$$p^4 - 13p^2 + 36 = 0$$

We have a trinomial on the LHS that we can factor as follows:

$$p^4 - 9p^2 - 4p^2 + 36 = p^2(p^2 - 9) - 4(p^2 - 9) = (p^2 - 4)(p^2 - 9) = (p+2)(p-2)(p+3)(p-3) = 0$$

Setting each linear factor equal to 0 and solving for p in each case, you will find that $p = -2, 2, -3, 3$

11. You are so disgusted with MAP 103 that you climb to the top of the math building and you throw your textbook up into the air and off the building. Of course, since this is a math problem, we can model the height h (in feet) of the book as a function of time t (in seconds) after you threw the book into the air as $h(t) = -16t^2 + 64t + 960$. [By the way, I realize the numbers in this problem are quite unrealistic. Sorry :( ]

(a) Find the height of the object at $t = 0$ seconds, $t = 3$ seconds, $t = 6$ seconds, and $t = 9$ seconds.

Solution. $h(t)$ is a function so that, given a certain time, we can find the height of the projectile by evaluating h at the given times.
$h(0) = -16(0)^2 + 64(0) + 960 = 960$ ft (ok, the math building isn't nearly this high, so I will have to reformulate this question next time!)

$h(3) = -16(3)^2 + 64(3) + 960 = -144 + 192 + 960 = 1008$ ft

$h(6) = -16(6)^2 + 64(6) + 960 = 768$ ft

$h(9) = -16(9)^2 + 64(9) + 960 = 240$ ft

(b) You watch the book hit the ground. You then immediately realize that throwing the book off of the building was not such a good idea, and you climb down the building to retrieve the book. It takes you 30 seconds to climb down to the ground and reach the book. How much time has elapsed from the time you threw the book in the air until the time you retrieved the book?
Solution. Let's find out how long it took for the book to hit the ground. When the book has hit the ground, its height above the ground is 0, so $h(t) = 0$. We also know that $h(t) = -16t^2 + 64t + 960$, so we want to solve the equation $-16t^2 + 64t + 960 = 0$ for t. Let's factor the LHS:

$$-16t^2 + 64t + 960 = -16(t^2 - 4t - 60) = -16(t - 10)(t + 4) = 0$$

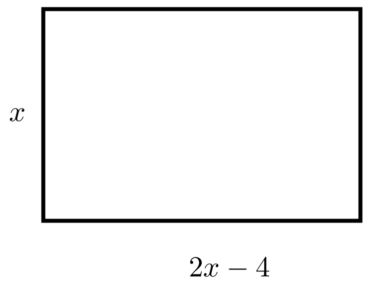
We can now set each factor to 0.

$-16 = 0$ this doesn't yield any solutions $\quad t - 10 = 0 \rightarrow \boxed{t = 10} \quad t + 4 = 0 \rightarrow t = -4$

We can't have negative time in this situation since the clock starts at 0, so it took 10 seconds for the book to fall. It took you 30 seconds after it laneded to retrieve it. So in total it took $\boxed{\text{40 seconds}}$ for you to get the book after it landed.

12. If the width of a rectangular plot of land is four less than twice the length and the area of the rectangular plot is 96 square miles, find the dimensions of the plot of land.

Solution. If we call the length x, then the width is $2x - 4$. Below is a picture labeled with this information:



The area of a rectangle is its length times its width. So if we multiply x by $2x - 4$, we need to get 96, the area of the rectangle:

$$x(2x - 4) = 96 \rightarrow 2x^2 - 4x = 96$$

We now see that we have a quadratic equation to solve, so move 96 to the LHS and factor:

$$2x^2 - 4x - 96 = 2(x^2 - 2x - 48) = 2(x - 8)(x + 6) = 0$$

Now we have the product of two linear factors equal to 0, so one of them must be 0 (2 cannot be 0 so we will just throw this factor out immediately):

$$x - 8 = 0 \rightarrow \boxed{x = 8} \text{ or } x + 6 = 0 \rightarrow x = -6$$

We will disregard the negative solution since we cannot have a negative length. So the length of the rectangle is $\boxed{\text{8 feet}}$ and the width is $2(8) - 4 = \boxed{\text{12 feet}}$.