MAP 103 Notes
Piecewise Functions
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Introduction

Suppose we were parking in a parking garage that we had to pay for by the hour. The pay structure is as follows:

- For the first 4 hours, it costs us \$3 per hour to park.
- For any number of hours parked after 4 hours, it costs \$2 per hour to park.

These rates are proportional to any amount of time spent parked within the first three hours; for example if we only stayed for ½ hour, we would pay \$1.50.

The goal of this introduction is to create a function which inputs hours parked and outputs total cost.

During the first four hours, \$3 per hour is a constant rate of change. Thus, if we were to graph time spent parked on the x-axis and cost to park that long on the y-axis, we would get a line. In the language of functions, we can say that the cost y is a *linear function* of time, x, spent parked there, at least for the first three hours. Clearly, if we spend 0 hours in the parking lot, we will pay \$0, so the y-intercept of our line is (0, 0). We will pay \$3 per hour; this is our constant rate and therefore our slope. So we could write the following function that gives the price to park (y) for any amount of time (x) between 0 and 4 hours:

$$\text{Eq. (1)} \qquad y = 3x \quad \text{if } 0 \le x \le 4$$

Notice that the x-values are restricted between 0 and 4. This is because we so far only have considered how much we have to pay for parking any time from 0 to 4 hours.

We know that after 4 hours we only have to pay \$2 per hour. So, for example, to park for 5 hours, we would pay \$3 per hour for the first 4 hours and \$2 for the number of hours parked after 4 hours, which is just 5 – 4 = 1. In total, we would pay

$$3(4)+2(5-4)=12+2=\$14$$

The table below shows similar calculations for parking a total of 6, 7 and 8 hours:

Total Hours Parked	Total Payment
6	3(4) + 2(**6 – 4**) = 12 + 4 = \$16
7	3(4) + 2(**7 – 4**) = 12 + 6 = \$18
8	3(4) + 2(**8 – 4**) = 12 + 8 = \$20

If we take a closer look at the table, we always are paying \$12 plus a quantity which depends on how many hours we parked minus four hours. Below is a table which helps us extend our numerical observations to an algebraic formula:

Total Hours Parked	Total Payment
6	3(4) + 2(**6** – 4) = 12 + 4 = \$16
7	3(4) + 2(**7** – 4) = 12 + 6 = \$18
8	3(4) + 2(**8** – 4) = 12 + 8 = \$20
x	3(4) + 2(***x*** – 4) = 12 + 2(x – 4)

So we now have a new rule:

$$\text{Eq. (2)} \qquad y=12+2(x-4)$$

This new rule only “works” when $x > 4$. Otherwise, we get the wrong total price to park. For example, if we tried to use our new rule to find the total cost when we parked 3 hours, we would get

$$y=12+2(3-4)=12+2(-1)=\$10$$

Yet we know that it only costs \$9 to park for three hours and therefore the rule in equation (2) doesn’t work for 3 hours. So, the rule that we use to find the total cost depends on how many hours we parked. Equation (2) only works when $x > 4$, so we should rewrite this rule as

$$\text{Eq. (3)} \qquad y = 12 + 2(x - 4) \ \text{ if } x > 4$$

Okay, so we have two rules; one that works when we park between 0 and 4 hours (Eq. (2)), and one that works when we park for more than 4 hours (Eq. (3)). Mathematicians combine the two rules together in what is known as a *piecewise function*. Let's now think of y (cost) as a function of x, the total time that we were parked in the garage, so $y = f(x)$. The following is how we write the function:

$$\text{Eq. (4)} \qquad f(x) = \begin{cases} 3x & \text{if } 0 \le x \le 4 \\ 12 + 2(x - 4) & \text{if } x > 4 \end{cases}$$

This kind of function is called a piecewise function because it is made up of two "rules". One rule (let's call it the top rule) is used only when the number of hours we've parked is between 0 and 4. Another rule (let's call it the bottom rule) is used only when the number of hours we've parked is greater than 4. The "if" statements after the rule tell us when to use which rule.

Granted this analysis of a simple situation seems to make something simple way more complicated. However, this analysis is not meant to explain the parking situation mathematically. Rather, it is meant to explain the idea of a piecewise function by showing you where one exists in real life. In fact, one might even claim that piecewise functions are used more often in real life than any other kind of function. For example, you might spend \$200 on a shopping spree only if you made \$1000 or more for the week. Otherwise, you might only spend \$50. The piecewise function below gives the amount of the shopping spree, S, as a function of x, the amount of money you made that week:

$$S(x) = \begin{cases} 50 & \text{if } 0 \le x < 1000 \\ 200 & \text{if } x \ge 1000 \end{cases}$$

Anyway, that's enough motivating for now. Let's see what we can do with these piecewise functions.

Evaluating Piecewise Functions

Example 1. If $f(x)=\begin{cases}3x & \text{if } 0\le x\le 4\\ 12+2(x-4) & \text{if } x>4\end{cases}$, find

(a) $f(3)$
(b) $f(4)$
(c) $f(9.5)$
(d) $f(-2)$

Solution.

(a) First, we see that $x = 3$. We then look to the "if" statements and see which "if" statement 3 fits into. Since $0\le 3\le 4$, we use the rule to the left of that "if" statement, which is $f(x)=3x$. So to find $f(3)$, we simply would write

$$f(3)=3(3)=9$$

(b) First, we see that $x = 4$. We then look to the "if" statements and see which "if" statement 4 fits into. Since $0\le 4\le 4$, we use the rule to the left of that "if" statement, which is $f(x)=3x$. So to find $f(4)$, we simply would write

$$f(4)=3(4)=12$$

(c) First, we see that $x = 9.5$. We then look to the "if" statements and see which "if" statement 9.5 fits into. Since $9.5>4$, we use the rule to the left of that "if" statement, which is $f(x)=12+2(x-4)$. So to find $f(9.5)$, we simply would write

$$f(9.5)=12+2(9.5-4)=12+2(5.5)=12+11=23$$

(d) First, we see that $x = -2$. We then look to the "if" statements and see which "if" statement -2 fits into. It doesn't fit into any of them. Another way to say this is that there's no rule that helps us find $f(-2)$, and we can say that $f(-2)$ is *undefined*.

One doesn't have to have just two pieces to a piecewise function; one may have three or more, as illustrated in the next example.

Example 2. If $g(x) = \begin{cases} x-2 & \text{if } x < -3 \\ 3x+5 & \text{if } -3 \le x < 4 \\ x^2 & \text{if } x > 4 \end{cases}$, find

(a) $g(-6)$
(b) $g(4)$
(c) $g(\pi)$

Solution.
(a) First we see that –6 < –3, so we shall use the "top" rule to find $g(-6)$:

$$g(-6) = -6 - 2 = -8$$

(b) We look to see if 4 satisfies any of the "if" statements, and it doesn't. Therefore $g(4)$ is undefined.

(c) We look so see which "if" statement π satisfies; it makes the middle statement true since $-3 \le \pi < 4$. Therefore to find $g(\pi)$, we will use the "middle" rule:

$$g(\pi) = 3\pi + 5$$

Graphing Piecewise Functions

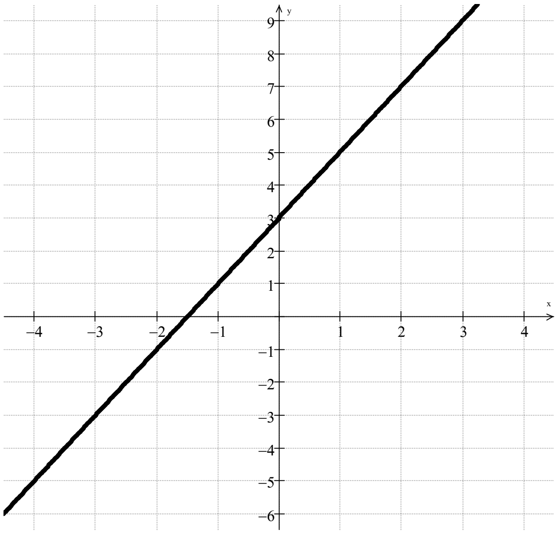
If we know what the graphs of the individual pieces of a piecewise function, then it's possible to graph a piecewise function.

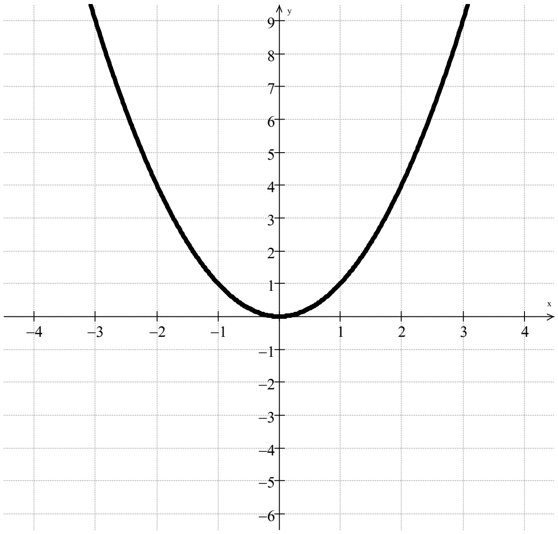
Example 3. Graph the function $f(x) = \begin{cases} 2x+3 & \text{if } x < 1 \\ x^2 & \text{if } x \ge 1 \end{cases}$.

Solution. First we should recall the graphs of the individual pieces. The graph of $y = 2x + 3$ is simply a line with slope equal to 2 and has a *y*-intercept of

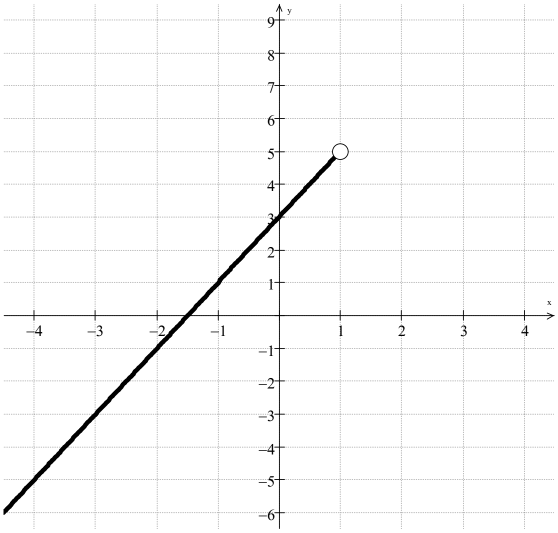
(0, 3).

The graph of $y = x^2$ is the standard parabola found in section 4.3 of your textbook (you should know its graph by heart). Below are the two graphs:

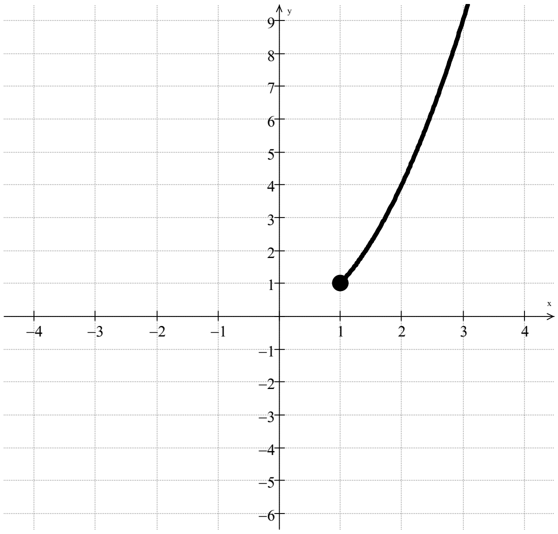




We want to graph $f(x) = \begin{cases} 2x+3 & \text{if } x < 1 \\ x^2 & \text{if } x \geq 1 \end{cases}$. So the graph of f is the line above when $x < 1$. At 1 and when $x > 1$, the graph of f is the parabola above. Hence we have the pieces below, which we will eventually need to consolidate into one graph:

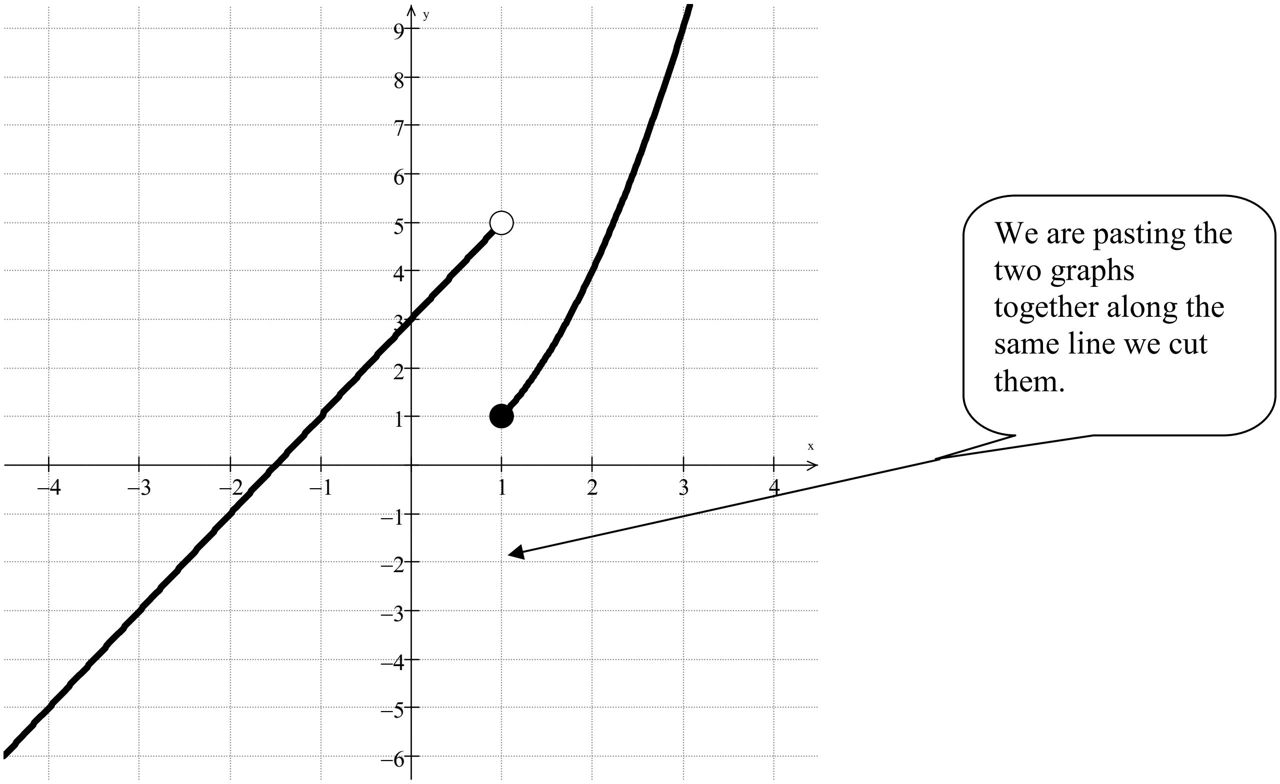


$y = 2x + 3$ if $x < 1$



$y = x^2$ if $x \geq 1$

Note that we are using an open circle in the left graph since the graph of *f* is equal to that graph when *x* is **strictly** less than 1; we use a filled-in circle for the right part of the graph since this is what *f* looks like when *x* is greater than **or equal** to 1. Finally, in order to produce one graph of the function, we will "cut" the graphs at the line $x = 1$ and "paste" them together along this line:



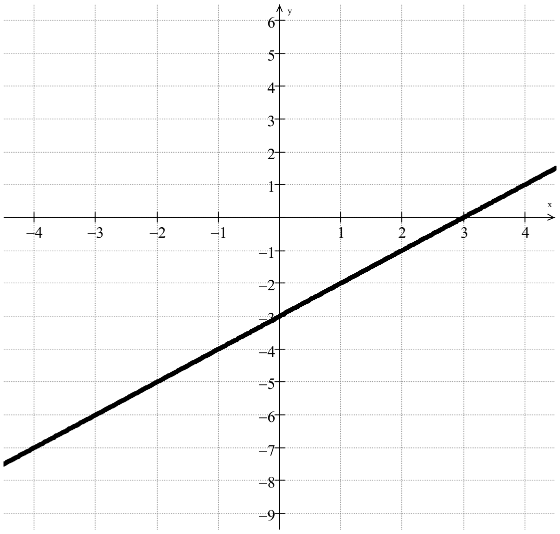
We are pasting the two graphs together along the same line we cut them.

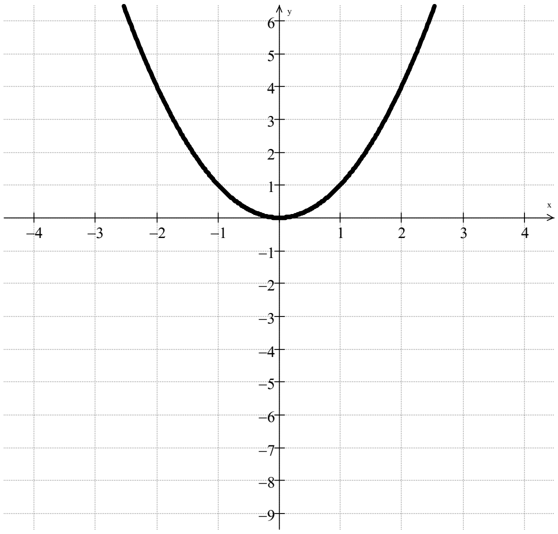
Example 4. Graph the function $g(x) = \begin{cases} x - 3 \text{ if } x \le -1 \\ x^2 \quad \text{ if } -1 < x \le 2 \\ 5 \quad \text{ if } x > 2 \end{cases}$.

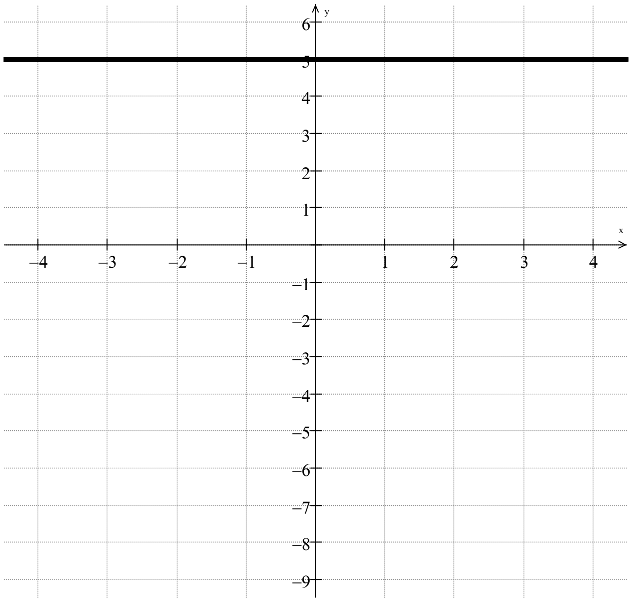
Solution. First, start by graphing the individual pieces you see. We see three functions:

1. $y = x - 3$, which is a line with slope 1 and *y*-intercept (0, –3).
2. $y = x^2$, the standard parabola.
3. $y = 5$, a horizontal line at height 5.

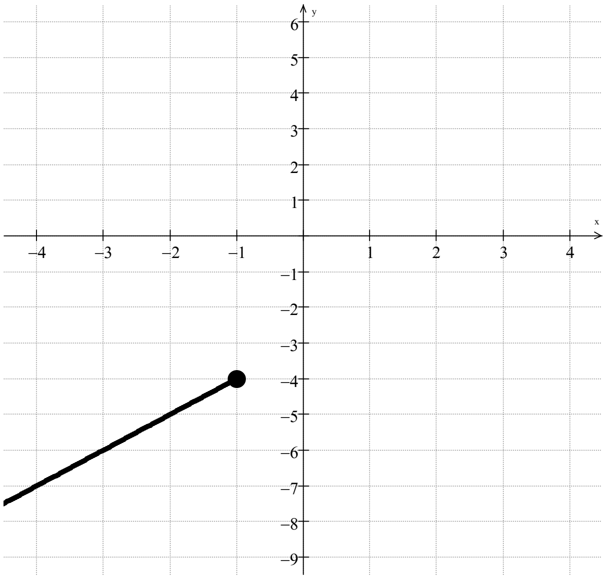
The graphs of each are below:



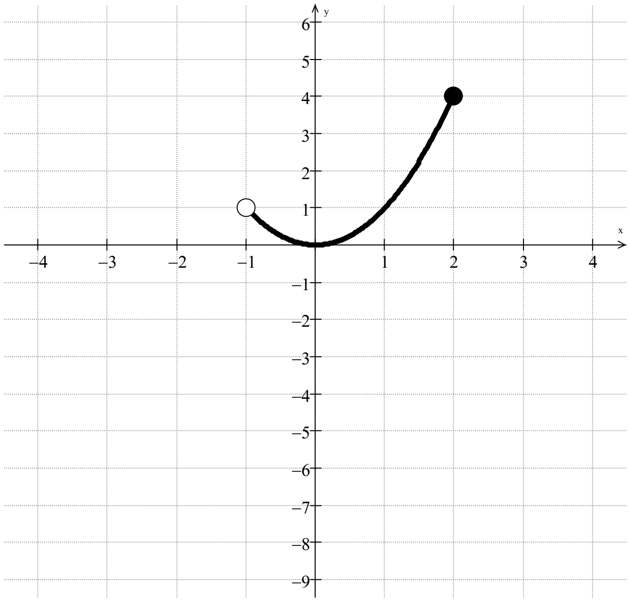




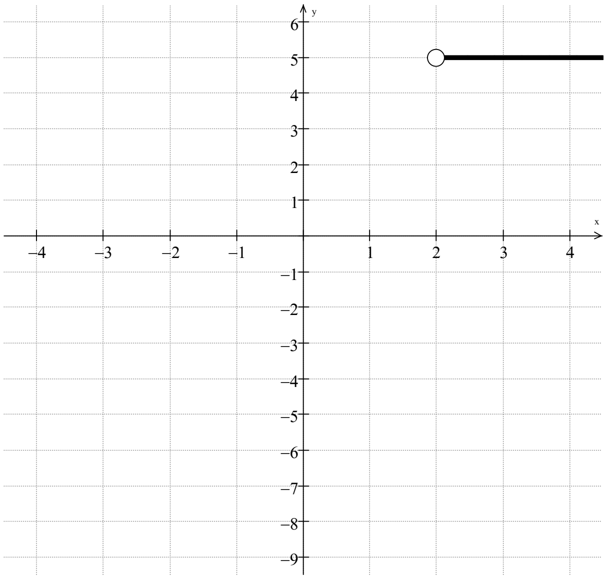
Now we need to figure out how to “cut” each graph. The graph of *g* is the same as the first graph above when $x \le -1$, so we will “cut” this graph on the line $x = -1$ and throw away the part to the right. We will fill in the circle at $x = -1$ because the inequality associated with this rule in our piecewise function is a “less than **or equal”** kind.



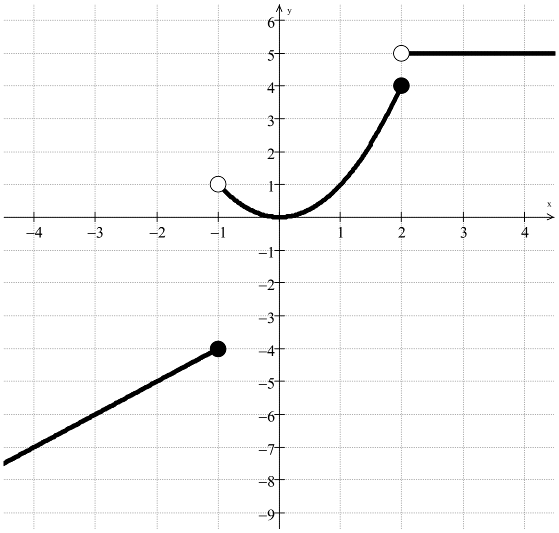
As for the graph of $y = x^2$, the graph of g is the same as the graph of $y = x^2$ when $-1 < x \le 2$, so we need to cut at the lines $x = -1$ and $x = 2$ and keep the middle piece. We will need an open circle on the left and a closed one on the right due to the strict inequality (<) on the left and to the less than **or equal** type ($\le$) on the right:



Lastly, we will cut the graph of $y = 5$ at $x = 2$ and keep the right portion. We will use an open circle on the line $x = 2$ due to the strict inequality given to us in the original piecewise function:



Finally, to form the graph of g, we will bring our three pieces together into one graph:



The Absolute Value Function

Believe it or not, you know a function which can be viewed quite nicely as a piecewise function: $f(x) = |x|$. You know that the absolute value of x is just a number's distance away from 0. For example, $|5| = 5, |7.2| = 7.2, |-3| = 3$, and $|-45\pi| = 45\pi$. These examples are illustrative of what the absolute value does.

If the value of x is positive or equal to 0 ($x \geq 0$), then $|x| = x$. A positive number's absolute value is just the number itself (and this also includes 0).

However, if the number is negative ($x < 0$), we simply remove the negative sign. Removing a negative sign is equivalent to taking the opposite of the number, and the opposite of x is just $-x$.

Therefore, $|x|$ can be broken down into two simple rules or pieces, depending on whether or not x is positive or negative. We can write down the function as a piecewise function:

$$|x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$$

Example 5 – Using the rule to compute absolute values. Find the values of the following using the piecewise function above for $|x|$:

(a) $|4|$

(b) $|-5.6|$

Solution.

(a) We look to see which if statement 4 makes true. Since $4 \geq 0$, we will use the "bottom" rule to determine $|4|$. The value of $|4|$ = 4, since the bottom rule just tells us to give back the value inside the absolute value bars.

(b) We look to see which if statement –5.6 makes true. Since $-5.6 < 0$, we will use the "top" rule to determine $|-5.6|$. The value of $|-5.6|$ = –(–5.6) = 5.6, since the top rule just tells us to negate the value inside the absolute value bars.

While it may seem to you that this piecewise definition of the absolute value is unnecessarily complicated, you will see that in later courses, viewing absolute value in this way is quite useful. As one last example, let's try to write the function $f(x) = |2x-1|$ as a piecewise function.

Example 6. Write the function $f(x) = |2x-1|$ as a piecewise function made up of two simple rules.

Solution. If the quantity inside the absolute value bars is positive or 0, then f does not change it at all. If it is negative, we simply take that quantity's opposite. The quantity inside the absolute value bars is $2x - 1$. So our piecewise function would look something like this:

$$|2x-1| = \begin{cases} -(2x-1) & \text{if } 2x-1<0 \\ 2x-1 & \text{if } 2x-1 \geq 0 \end{cases}$$

We wouldn't usually keep the if statements like this. We want to solve those if statements for x so that given a value for x, we can easily tell which rule to use. So let's solve the inequalities for x:

$$\begin{aligned} &2x-1<0 \\ &\quad\;\; +1 \quad +1 \\ &2x \qquad <1 \\ &\frac{2x}{2}<\frac{1}{2} \\ &x<\frac{1}{2} \end{aligned}$$

A very similar procedure will lead to the second inequality being written as $x \geq \frac{1}{2}$. So, our piecewise function may simply be written as

$$|2x-1| = \begin{cases} -(2x-1) & \text{if } x<\dfrac{1}{2} \\ 2x-1 & \text{if } x \geq \dfrac{1}{2} \end{cases}$$

Exercises.

Exercises 1-12: Let

$$f(x)=\begin{cases} 3x-2 & \text{if } x<-2 \\ x^2-x+4 & \text{if } x\ge -2 \end{cases} \quad \text{and} \quad g(x)=\begin{cases} x-x^2 & \text{if } x\le 0 \\ \sqrt{x} & \text{if } 0<x\le 4 \\ \dfrac{1}{x+4} & \text{if } x>4 \end{cases}$$

Find the following function values:

1. $f(0)$
2. $g(0)$
3. $f(-2)$
4. $g(1)$
5. $f(-6)$
6. $g(6)$
7. $f(0)+g(4)$
8. $f(-1)-g(-1)$
9. $f(k)$ if $k>1$
10. $g(z)$ if $z>10$
11. $g(t+1)$ if $t+1>9$
12. $f(x+h)$ if $x+h<-10$

Exercises 13-18: Graph each piecewise function below.

13. $f(x)=\begin{cases} x-2 & \text{if } x<-2 \\ -x+4 & \text{if } x\ge -2 \end{cases}$

14. $g(x)=\begin{cases} 2-x & \text{if } x\le 1 \\ -2x+4 & \text{if } x>1 \end{cases}$

15. $h(x)=\begin{cases} 3x-2 & \text{if } x\le -1 \\ 6 & \text{if } -1<x\le 4 \\ -x & \text{if } x>4 \end{cases}$

16. $F(x)=\begin{cases} 3x+3 & \text{if } x\le 0 \\ -x+4 & \text{if } 0<x<3 \\ x^2 & \text{if } x\ge 3 \end{cases}$

17. $G(x)=\begin{cases} x-2 & \text{if } x<-2 \\ x^3 & \text{if } -2<x\le 2 \\ 3x-5 & \text{if } x>2 \end{cases}$

18. $H(x)=\begin{cases} 3x-2 & \text{if } x<-2 \\ -1 & \text{if } -2<x<3 \\ x-5 & \text{if } 3<x<4 \\ 0 & \text{if } x\ge 4 \end{cases}$

Exercises 19-22. Write each function below as a piecewise function consisting of only two rules (see example 6).

19. $|x+5|$ 20. $|2x-3|$ 21. $|3x+2|$ 22. $|2-x|$

23. You are a buyer for a grocery store and you are asked to purchase potatoes for the grocery store. The distributor of potatoes tells you that if you buy up to 50 bushels of potatoes, you will pay $40 per bushel; and for each bushel you purchase above 50 bushels, you will pay $30 per bushel.
    (a) How much will your grocery store pay in total if you decide to purchase 40 bushels? 60 bushels? 100 bushels?
    (b) Write a function which has as its input values (*x*-values) the number of bushels of potatoes purchased and outputs the total amount of money that your grocery store will pay for the potatoes.

24. Some parking garages don't work quite like the one that we explored in the introduction of these notes. For example, a parking garage in Manhattan charges in the following way:

    For each hour **or part of an hour**, the garage charges $10 per hour, with a daily maximum of $50 per day.

    (a) How much will a customer pay if he/she parks 2 hours? 3 ½ hours? 4 hours and 20 minutes? 10 hours?
    (b) Write a piecewise function that has as its input the number of hours parked and outputs the total price paid by the customer.