

Representations of Surface Groups

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Abstract

We give several different descriptions of the space of surface group representations. This will lead to interesting relationships between gauge theories and moduli spaces of bundles as well as a non-Abelian Hodge theorem.

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1 Introduction

The underlying theme of this talk is the interplay between topology, differential geometry and algebraic geometry and the benefits of looking at a single object from all three points of view. The example taken below is the study of certain spaces associated to a topological, smooth or complex surface Σ . We will see that nearly all of the constructions we perform have descriptions in each of the three categories and that comparisons at each stage will lead to new information about our spaces, none of which can be obtained from the construction in a single category. Many of the objects discussed below are still not fully understood. Moreover, the constructions below, in their smooth and algebraic forms, play important rôles in the Geometric Langlands Conjecture and Mirror Symmetry.

2 The Case of a Compact Group

Denote by Σ a closed, oriented topological surface of genus g . We would like to associate some spaces to Σ . To do so, fix a connected compact Lie group G . Later we will try and generalize what follows to the case in which G is non-compact and reductive, i.e. the complexification of a compact group.

2.1 Topological Category

We can assign to Σ the space of representations of $\pi_1(\Sigma)$ in G ,

$$\mathcal{R}(\Sigma, G) = \text{Hom}(\pi_1(\Sigma), G)/G.$$

Here G acts on $\text{Hom}(\pi_1(\Sigma), G)$ by conjugation. Recall that we have

$$\pi_1(\Sigma) = \langle A_1, \dots, A_g, B_1, \dots, B_g \mid \prod_{i=1}^g [A_i, B_i] \rangle.$$

There is a natural inclusion $\text{Hom}(\pi_1(\Sigma), G) \hookrightarrow G^{2g}$ which we use to put a topology on $\text{Hom}(\pi_1(\Sigma), G)$. Then $\mathcal{R}(\Sigma, G)$ has the quotient topology.

Example. • If $\Sigma = S^2$ then $\mathcal{R}(S^2, G) = pt$ since S^2 is simply connected.

- For $G = U(1)$

$$\text{Hom}(\pi_1(\Sigma), U(1))/U(1) = \text{Hom}(\pi_1(\Sigma), U(1)) = (U(1))^{2g}$$

corresponding to the images of A_i, B_i in $U(1)$.

- Consider now $G = U(n)$ and $\Sigma = S^1 \times S^1$. Then $\pi_1(S^1 \times S^1) = \mathbb{Z} \oplus \mathbb{Z}$ with generators A and B . Denote the image of A (resp. B) in $U(n)$ under a representation by a (resp. b). Then we must have $ab = ba$ in $U(n)$. Since we are working up to conjugation we can assume that

$$a = \begin{pmatrix} e^{i\theta_1} & & \\ & \ddots & \\ & & e^{i\theta_n} \end{pmatrix}, \quad \theta_i \in \mathbb{R}.$$

If the eigenvalues of a are distinct (this is generically so) then b must be of the form

$$b = \begin{pmatrix} e^{i\phi_1} & & \\ & \ddots & \\ & & e^{i\phi_n} \end{pmatrix}, \quad \phi_i \in \mathbb{R}.$$

In other words, a and b are in the maximal torus T of $U(n)$. There is a residual conjugation action remaining, namely the Weyl group $W = N(T)/T \simeq S_n$. Hence we have

$$\text{Hom}(\pi_1(S^1 \times S^1), U(n))/U(n) \simeq (T \times T)/S_n.$$

If a is not generic the description remains the same, although now the Weyl group, which acts by permuting the diagonal entries, has a non-trivial stabilizer group and the quotient will contain singularities, although they are only of orbifold type.

2.2 Smooth Category

Now assume that Σ is given the structure of a smooth manifold. Such a choice is unique up to diffeomorphism. We can then assign to Σ the moduli space of flat connections on principal G -bundles on Σ , which we denote by $\mathcal{M}_{flat}(\Sigma, G)$. For simplicity we will assume the G bundles under consideration are trivizable. This is in fact guaranteed if $\pi_1(G) = 0$. Recall that after picking a trivialization a connection on a principal G -bundle $P \rightarrow \Sigma$ is a one form $A \in \Omega^1(\Sigma; \mathfrak{g})$. Its curvature is

$$F_A = dA + \frac{1}{2}[A \wedge A] \in \Omega^2(\Sigma, \text{ad}P).$$

The connection is said to be flat if $F_A = 0$. The group of gauge transformations \mathcal{G}_P is $\Omega^0(\Sigma, \text{Ad}P) \simeq C^\infty(\Sigma, G)$ and its Lie algebra is $C^\infty(\Sigma, \mathfrak{g})$. The gauge transformations parametrize trivializations of P . The group \mathcal{G}_P acts on the space of connections on P . The curvature transforms in the adjoint representation under the \mathcal{G}_P action so that a connection being flat is a gauge invariant property.

Example. Consider the case in which $G = U(1)$ and the principal bundle is trivial. Then a connection A is a differential form on Σ with curvature $F_A = dA$. Hence the space of flat connections is $\ker(d: \Omega^1(\Sigma) \rightarrow \Omega^2(\Sigma))$. An infinitesimal gauge transformation is a smooth function ϕ on Σ and acts on A by

$$A \mapsto A + d\phi.$$

In this case the property of being flat is a gauge invariant; it is just the statement that $d^2 = 0$. Hence we see that the space of flat connections modulo the image of the Lie algebra of gauge transformations is the de Rham cohomology group $H^1(\Sigma; \mathbb{R})$. To quotient by the whole group of gauge transformations we need to quotient $H^1(\Sigma; \mathbb{R})$ by $H^1(\Sigma; \mathbb{Z})$; the latter space is the obstruction to writing a map $\Sigma \rightarrow U(1)$ as the lift of a map $\Sigma \rightarrow \mathfrak{u}(1)$. Hence we see that

$$\mathcal{M}_{flat}(\Sigma, U(1)) = \mathbb{R}^{2g}/\mathbb{Z}^{2g} \simeq (U(1))^{2g}$$

which is homeomorphic to the space of representations constructed above.

The conclusion of the example is quite general. Given a connection A on a principal G -bundle $P \rightarrow \Sigma$ the holonomy is a map

$$\Omega(\Sigma, x_0) \rightarrow G$$

where we write $\Omega(\Sigma, x_0)$ for smooth loops in Σ based at $x_0 \in \Sigma$. If A is flat it is easy to check that the holonomy descends to a homomorphism $\pi_1(\Sigma, x_0) \rightarrow G$. We have the following result.

Proposition 2.1. *The holonomy map gives an bijection of sets*

$$\text{Hom}(\pi_1(\Sigma, x_0), G)/G \simeq \mathcal{M}_{flat}(\Sigma, G).$$

Sketch of proof: We have argued that a flat connection gives an element in $\text{Hom}(\pi_1(\Sigma, x_0), G)$. One then needs to check how the holonomy changes under gauge transformations.

In the opposite direction, given a homomorphism $\rho : \pi_1(\Sigma, x_0) \rightarrow G$ we construct a principal G -bundle with connection having holonomy given by ρ . Let $\tilde{\Sigma}$ be the universal cover of X and \tilde{x}_0 a point in the fibre above x_0 . Then $\pi_1(\Sigma, x_0)$ acts on $\tilde{\Sigma}$ by deck transformations. From this we get a principal G -bundle

$$P_\rho = \tilde{\Sigma} \times_\rho G \rightarrow \Sigma.$$

To get a connection, consider the trivial flat connection on $\tilde{\Sigma} \times G$. This descends to the quotient and one can check that its holonomy is given by ρ . □

2.3 Algebraic Category

Adding more structure again, endow Σ with a complex structure. In this case there is no longer a canonical choice as there are moduli of complex structures. For example, if $g \geq 2$ the moduli space of complex structures is the quotient of $\mathbb{C}^{3(g-1)}$ by a finite group. Equivalently, let Σ be a smooth projective algebraic curve over \mathbb{C} . Let $\mathcal{M}_{Dol}(\Sigma, G)$ be the moduli space of degree zero holomorphic principal $G^{\mathbb{C}}$ -bundles over Σ . We can also think of the objects parameterized by $\mathcal{M}_{Dol}(\Sigma, G)$ in terms of holomorphic vector bundles.

Example. • Say $G = U(n)$ so that $G^{\mathbb{C}} = GL(n, \mathbb{C})$. Then $\mathcal{M}_{Dol}(\Sigma, U(n))$ parametrizes isomorphism classes of holomorphic vector bundles on Σ of rank n and degree zero. Recall that the degree of a holomorphic vector bundle E is $\text{deg } E = c_1(\det E)$.

- Say $G = SU(n)$ so that $G^{\mathbb{C}} = SL(n, \mathbb{C})$. Then $\mathcal{M}_{Dol}(\Sigma, SU(n))$ parametrizes holomorphic rank n vector bundles on Σ with holomorphically trivial determinant bundle.

We need to introduce a stability condition in $\mathcal{M}_{Dol}(\Sigma, G)$. We say that a holomorphic vector bundle $E \rightarrow \Sigma$ is stable (resp. semi-stable) if for all holomorphic sub-bundles $F \subset E$ we have $\mu(F) < \mu(E)$ (resp. $\mu(F) \leq \mu(E)$). Here

$$\mu(E) = \frac{\deg E}{\text{rk } E}$$

is the slope of E . We say E is polystable if $E = \oplus_i E_i$ such that each E_i is stable and all E_i have the same slope. The construction of $\mathcal{M}_{Dol}^s(\Sigma, G)$ is through Geometric Invariant Theory. One finds that $\mathcal{M}_{Dol}^s(\Sigma, G)$ is a smooth Kähler manifold. However, it is non-compact. To solve this problem, one can consider the moduli space of semi-stable bundles, $\mathcal{M}_{Dol}^{ss}(\Sigma, G)$ which is a projective algebraic variety containing the stable moduli space as an dense (Zariski) open subset, i.e. $\mathcal{M}_{Dol}^{ss}(\Sigma, G)$ provides a natural compactification of $\mathcal{M}_{Dol}^s(\Sigma, G)$.

Theorem 2.2 (Narasimhan-Seshadri [9]). *There is a homeomorphism*

$$\mathcal{M}_{flat}(\Sigma, G) \simeq \mathcal{M}_{Dol}^{ps}(\Sigma, G)$$

where on the right hand side Σ is endowed with a complex structure. Restriction gives a homeomorphism

$$\mathcal{M}_{flat}^*(\Sigma, G) \simeq \mathcal{M}_{Dol}^s(\Sigma, G)$$

with $*$ denoting irreducible connections.

Note that the definition of the space $\mathcal{M}_{Dol}^{ps}(\Sigma, G)$ depends strongly on the algebraic structure on Σ . The theorem implies that its topological type is in fact independent of this structure.

We very briefly discuss why one might believe the theorem to be true. The key is in thinking about holomorphic structures on a smooth complex vector bundle $E \rightarrow \Sigma$ in terms of differential operators. If $E \rightarrow \Sigma$ is holomorphic there is a first order differential operator $\bar{\partial}_E : \Omega^0(\Sigma; E) \rightarrow \Omega^{0,1}(\Sigma, E)$ such that $\bar{\partial}_E^2 = 0$ (which in this case is trivial). In the other direction, if there exists a first order differential operator $\bar{\partial}_E$ on a smooth complex vector bundle E such that $\bar{\partial}_E^2 = 0$ we can use it to define a holomorphic structure on E . The holomorphic sections $s \in H^0(\Sigma, E)$ then satisfy $\bar{\partial}_E s = 0$. The group of endomorphisms of E , $\mathcal{G}^{\mathbb{C}}$, acts on the space of such $\bar{\partial}_E$ operators by conjugation and the quotient space is $\mathcal{M}_{Dol}^{ps}(\Sigma, G)$. To find a map $\mathcal{M}_{flat}(\Sigma, G) \rightarrow \mathcal{M}_{Dol}^{ps}(\Sigma, G)$ fix a Hermitian metric h on E and decompose d_A for a connection A into $d'_A + d''_A$. The desired map is $A \mapsto d''_A$. The key to the proof of the theorem from this gauge theoretic point of view is Uhlenbeck's compactness theorem.

Finally, we explain how to extend the discussion to bundles with non-zero degree d . In the algebraic case this is straightforward. What we need are constructions in the topological and smooth cases. Topologically, remove a small open disk $D \subset \Sigma$ and consider $\pi_1(\Sigma \setminus D)$. This is the free group on $2g$ generators A_i, B_i . Restrict attention to the case $G = U(n)$ and let the degree d be coprime with n .

$$\mathcal{R}_d(\Sigma, U(n)) = \left\{ \rho \in \text{Hom}(\pi_1(\Sigma \setminus D), U(n)) \mid \rho\left(\prod_{i=1}^g [A_i, B_i]\right) = e^{\frac{2\pi id}{n}} \mathbb{I} \right\} / U(n).$$

That is, $e^{\frac{2\pi id}{n}} \mathbb{I}$ generates the centre of $U(n)$. Alternatively, one can consider the universal central extension

$$0 \rightarrow \mathbb{Z} \rightarrow \Gamma \rightarrow \pi_1(\Sigma) \rightarrow 1$$

with the central element denoted by J . The space of representations to consider is then

$$\mathcal{R}_d(\Sigma, U(n)) = \left\{ \rho \in \text{Hom}(\Gamma, U(n)) \mid \rho(J) = e^{\frac{2\pi id}{n}} \mathbb{I} \right\} / U(n).$$

We can now state a generalized version of the theorem of Narasimhan and Seshadri.

Theorem 2.3 (Narasimhan-Seshadri [9]). *There is a homeomorphism*

$$\mathcal{R}_d(\Sigma, U(n)) \simeq \mathcal{M}_{Dol}^{ps}(\Sigma, U(n), d)$$

where on the right hand side Σ is endowed with a complex structure. Restriction gives a homeomorphism

$$\mathcal{R}_d^*(\Sigma, U(n)) \simeq \mathcal{M}_{Dol}^s(\Sigma, G, d).$$

From the description of $\mathcal{R}_d(\Sigma, U(n))$ it is clear what the corresponding smooth space should be: the moduli space of projectively flat connections. That is, with a fixed Hermitian metric on E the canonical connection A_h satisfies

$$F_A = -i\mu\mathbb{I}_E\omega$$

where ω is the Kähler form of Σ . A short computation shows that μ is the slope of E .

3 Cohomology of the Moduli Space

We begin with an application and some motivation. Let M be a closed, connected, oriented 3-manifold and $M = M_1 \cup_{\Sigma} M_2$ be a Heegaard decomposition. The pushout diagram of epimorphisms

$$\begin{array}{ccc} & \pi_1(M_1) & \\ \nearrow & & \searrow \\ \pi_1(\Sigma) & & \pi_1(M) \\ \searrow & & \nearrow \\ & \pi_1(M_2) & \end{array}$$

gives rise to a diagram of monomorphisms

$$\begin{array}{ccc} & \mathcal{R}(M_1, G) & \\ \nwarrow & & \swarrow \\ \mathcal{R}(\Sigma, G) & & \mathcal{R}(M, G) \\ \swarrow & & \nwarrow \\ & \mathcal{R}(M_2, G) & \end{array}$$

One can check¹ that $\mathcal{R}(M_1, G)$ and $\mathcal{R}(M_2, G)$ are Lagrangian submanifolds in $\mathcal{R}_G(\Sigma)$. The Casson invariant of M is

$$\lambda(M) \propto \mathcal{R}_G(M_1) \cap \mathcal{R}_G(M_2).$$

In particular, we are motivated to understand the intersection theory in the moduli spaces defined above.

We begin with a simpler task- identifying the generators of $H^*(\mathcal{M}(\Sigma, G))$. We remark that we will always take complex coefficients. Consider the case in which $G = U(n)$ with degree d coprime to n . In this case stability and semi-stability coincide and the moduli space of stable bundles is smooth and compact. Moreover, there exists a holomorphic universal bundle

$$\mathcal{U} \rightarrow \mathcal{M}(\Sigma, G) \times \Sigma$$

¹The following is not so straightforward; a number of subtle transversality questions must be answered. See [2] for details.

with fibre over $\{[E]\} \times \Sigma$ equal to E . This generalizes the Poincaré bundle in the case of Abelian varieties. The Chern classes of \mathcal{U} lie in $H^\bullet(\mathcal{M}(\Sigma, G) \times \Sigma)$. Let $1 \in H_0(\Sigma)$, $\alpha_i \in H_1(\Sigma)$ and $[\Sigma] \in H_2(\Sigma)$ denote homology generators. Put, for $2 \leq r \leq n$,

$$f_r = \int_{\Sigma} c_r(\mathcal{U}), \quad a_r^j = \int_{\alpha_j} c_r(\mathcal{U}), \quad b_r = \int_1 c_r(\mathcal{U}).$$

Theorem 3.1 (Atiyah-Bott [1]). *f_r , a_r^j and b_r generate the cohomology ring $H^\bullet(\mathcal{M}(\Sigma, G))$.*

The intersection theory of $\mathcal{M}(\Sigma, G)$ is considerably more difficult. It was understood in the case where $(n, d) = 1$ by Jeffrey and Kirwan [7] using non-Abelian localization [8]. Very roughly, this proceeds as follows. The space of connections \mathcal{A} is naturally symplectic and carries a Hamiltonian action of the group of gauge transformations \mathcal{G} . The moment map μ is the curvature, so that $\mathcal{M}_{flat}(\Sigma, G)$ is the symplectic reduction of \mathcal{A} . The inclusion $i_d : \mu^{-1}(d) \hookrightarrow \mathcal{A}$ gives a ring homomorphism in equivariant cohomology

$$i_d^* : H_{\mathcal{G}}^\bullet(\mathcal{A}) \rightarrow H_{\mathcal{G}}^\bullet(\mu^{-1}(d)).$$

If d is a regular value of μ and the action of \mathcal{G} on $\mu^{-1}(d)$ has finite isotropy groups then canonically $H_{\mathcal{G}}^\bullet(\mu^{-1}(d)) \simeq H^\bullet(\mathcal{M}_{flat}(\Sigma, G))$. Denote the induced ring homomorphism (often called the Kirwan map) by

$$\kappa_d : H_{\mathcal{G}}^\bullet(\mathcal{A}) \rightarrow H^\bullet(\mathcal{M}_{flat}(\Sigma, G)).$$

If κ_d were surjective² by studying the kernel of κ_d one can study the relations in the cohomology ring of $\mathcal{M}_{flat}(\Sigma, G)$. Using the tools of non-Abelian localization Jeffrey and Kirwan completed this task.

One major problem with the above sketch is that the spaces \mathcal{A} and \mathcal{G} are infinite dimensional, while non-Abelian localization and surjectivity of the Kirwan κ_d map hold for compact groups. To get around this problem one can reduce in stages. Alternatively, one can construct a finite dimensional extended moduli space that carries Hamiltonian $SU(n)$ action. The symplectic quotient then coincides with $\mathcal{M}_{flat}(\Sigma, G)$ and one can proceed using the above plan.

Example. *Say $G = SU(2)$ with $n = 2$ and $d = 1$. The generators of cohomology are*

$$f \in H^2(\mathcal{M}) \quad b \in H^4(\mathcal{M}).$$

One has

$$\int_{\mathcal{M}} b^j e^f = \frac{(-1)^j}{2^{2g-2} \pi^{2(g-1-j)}} (1 - 2^{2g-3-2j}) \zeta(2g - 2 - 2j).$$

Since the cohomology ring enjoys Poincaré duality the relations are determined by the intersection numbers of monomials in the generators. Hence this result determines the ring structure.

Finally, we remark the similarity between the above discussion and that occurring in many applications of physics to mathematics. In particular, the definition of topological invariants from gauge theories often proceeds as above. For example, Donaldson theory is effectively the study of the intersection theory on the moduli space of instantons. The classes f_r , a_r^j and b_r can be compared with those arising in Donaldson theory [4] (and indeed, their definition is completely analogous). Witten gave a physical derivation of the ring structure of $\mathcal{M}_{flat}(\Sigma, G)$ using two dimensional Yang-Mills theory [11], [12]. In fact, it is in this work that he discovered non-Abelian localization.

²Kirwan proved quite generally that κ_d is surjective in the case of finite group actions which is why one might expect the result to be true here

4 Reductive Groups

We would like to generalize the above work to the case in which G is a non-compact reductive group. That is, G is the complexification of a compact Lie group H . For example if $H = U(n)$ then $G = GL(n, \mathbb{C})$. The generalization of the space of representations and the moduli space of flat connections is straightforward. Moreover, the homeomorphism of the underlying topological spaces continues to hold (with the same proof). What is needed is the analogue of the complex space $\mathcal{M}_{Dol}(\Sigma, G)$, which for compact G was the moduli space of holomorphic principal $G^{\mathbb{C}}$ -bundles (often identified with certain holomorphic vector bundles).

We begin with the answer. First we need a definition. A (projectively) flat connection is said to be reductive if it arises from the class of a reductive representation of $\pi_1(\Sigma)$ in G . Note that for compact G all flat connections are reductive. Let $E \rightarrow \Sigma$ be a smooth complex vector bundle with a reductive, projectively flat connection D . Let h be a Hermitian metric on E and decompose the connection as

$$D = \nabla + \psi$$

where ∇ is a connection compatible with h and $\psi \in \Omega^1(\Sigma, iad E)$. We say that the metric h is harmonic if

$$\nabla^* \psi = 0.$$

Here ∇^* is a connection built from the Hermitian metric h and a Riemannian metric in the conformal class of Σ .

Theorem 4.1 (Donaldson [5], Corlette [3]). *Let (E, D) be a projectively flat connection on a smooth complex vector bundle $E \rightarrow \Sigma$. Then (E, D) admits a harmonic metric if and only if D is reductive. If D is reductive the harmonic metric is unique.*

This gives a kind of non-Abelian Hodge theorem. Indeed, (E, D) represents a point in the non-Abelian cohomology set³

$$H^1(\Sigma, G) = \text{Hom}(\pi_1(\Sigma), G)/G.$$

The theorem says that only certain classes (E, D) admit a harmonic representative.

We give here a second interpretation of a metric being harmonic, this time in the language of principal bundles. Let $P \rightarrow \Sigma$ be a principal G bundle and $H \subset G$ a maximal compact subgroup. A good example to keep in mind is $G = GL(n, \mathbb{C})$ and $H = U(n)$. A metric on P is a $\pi_1(\Sigma)$ -equivariant map

$$h : \tilde{\Sigma} \rightarrow G/H.$$

This is equivalent to a section of $P \times_G G/H \rightarrow \Sigma$. Note that in the case of $G = GL(n, \mathbb{C})$ and $H = U(n)$ a metric in this sense is simply a Hermitian metric, i.e. a reduction of the structure group from $GL(n, \mathbb{C})$ to $U(n)$. Then h is harmonic if and only if it is extremal for the energy functional

$$E(h) = \frac{1}{2} \int_{\tilde{\Sigma}} |dh|^2.$$

Here $dh \in \Omega^1(\tilde{\Sigma}, h^*T(G/H))$. The space G/H is Riemannian symmetric space (with metric given by the Killing form) which defines the norm $|\cdot|$. The connection ∇^* above is then interpreted as the pull-back of the corresponding Levi-Civita connection on G/H to $h^*T(G/H)$.

³On the left hand side G has the discrete topology.

Continuing, we still want to find an analogue of $\mathcal{M}_{Dol}(\Sigma, G)$ when G is, say, $GL(n, \mathbb{C})$. We start with a definition. A pair (E, ϕ) where $E \rightarrow \Sigma$ is a holomorphic vector bundle and $\phi \in H^0(\Sigma, \text{End}(E) \otimes K_\Sigma)$ is called a Higgs bundle. A Higgs bundle (E, ϕ) is stable if all ϕ -invariant holomorphic sub-bundles $F \subset E$ satisfy $\mu(F) < \mu(E)$. Definitions of semi-stability and poly-stability are similar. Denote by $\mathcal{M}_{Hit}(\Sigma, G)$ the moduli space of Higgs bundles.

As before, let $(E, D) \in H^1(\Sigma, G)$ and pick a Hermitian metric h on E . This gives rise to a decomposition $D = \nabla + \psi$.

Proposition 4.2. *If h is harmonic then the pair $(\nabla, \psi + \psi^*)$ is a Higgs bundle. If D is irreducible then $(\nabla, \psi + \psi^*)$ is stable.*

Corollary 4.3 (Hitchin [6], Simpson [10]). *There is a homeomorphism $\mathcal{R}_d^0(\Sigma, GL(n, \mathbb{C})) \simeq \mathcal{M}_{Hit}(\Sigma, U(n), d)$ that restricts to a homeomorphism $\mathcal{R}_d^*(\Sigma, GL(n, \mathbb{C})) \simeq \mathcal{M}_{Hit}^s(\Sigma, U(n), d)$. Here the superscript 0 (resp. *) mean reductive (resp. irreducible) representations.*

This gives another avatar of a non-Abelian Hodge theorem. Recall that the usual Hodge theorem states that for a compact Kähler manifold Σ

$$H^1(\Sigma, \mathbb{C}) = H^1(\Sigma, \mathcal{O}_\Sigma) \oplus H^0(\Sigma, K_\Sigma).$$

The corollary shows that a harmonic element $(E, D) \in H^1(\Sigma, G)$ is represented uniquely by a Higgs bundle (E, ϕ) precisely when D is reductive. That is, an element in $H^1(\Sigma, GL(n, \mathcal{O}_\Sigma)) \oplus H^0(\Sigma, \text{End}(E) \otimes K_\Sigma)$.

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