Equivariant Cohomology and Localization

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Abstract

When a smooth manifold is acted on by a compact Lie group the equivariant cohomology gives a meaningful generalization of the ordinary cohomology of M/G when the G-action is not free. We explain how to represent equivariant cohomology using equivariant differential forms and prove a basic localization theorem in this context. We also give a number of applications. These are notes from a talk in the Graduate Student Seminar at SUNY Stony Brook, Sept. 23, 2009.

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1 Introduction

Let M be smooth, compact manifold and G a compact Lie group. Assume G acts smoothly on M. That is, for each $g \in G$ there is a smooth map $\phi_g : M \to M$ such that $\phi_e = id_M$ and $\phi_{g_1} \circ \phi_{g_2} = \phi_{g_1 g_2}$. If the action of G on M is free, the quotient space M/G is in a natural way a smooth manifold. Hence we can use the same tools (say, differential forms) to study M/G as we could to study M. However, if the G action is not free, it is not clear how to proceed (at least if we want to have non-singular spaces, etc.) Equivariant cohomology was invented to solve this problem, that is, to define some type of cohomology that is defined for G-spaces in which not all points have trivial stabilizers. We give two approaches to equivariant cohomology below, one more topological (due to Borel) and one more algebraic (due to Cartan).

2 Equivariant Cohomology

2.1 The Borel Construction

Let M and G be as above. That G is compact guarantees the existence of a universal G-bundle, which we denote by $EG \to BG$. EG is characterized up to homotopy equivalence by being a contractible free G-space. For example $G = S^1$ acts freely on $S^\infty \subset \mathbb{C}^\infty$, which is contractible. Hence $ES^1 = S^\infty$, from which it follows that $BS^1 = \mathbb{CP}^\infty$. Note that in this case EG and BG are not finite dimensional (in fact this is quite generally true). However, they may be endowed with the structure of a CW complex. More generally, since any compact Lie group has a faithful finite dimensional representation (which is necessarily unitary), it can be embedded in $U(N)$ for some large N. By restriction of the $U(N)$ action on $EU(N)$ (which we may take as an infinite Steifel manifold) we get a model for EG.

In any case, assuming EG is given, we define the homotopy quotient $M_G := EG \times_G M$, where we view $EG \times M$ as a right G-space with action

$$(p, m) \cdot g = (p \cdot g, g^{-1} \cdot m).$$

Note that since G acts freely on EG it also acts freely on $EG \times M$ (regardless of if the G action on M is free). Hence the quotient M_G is non-singular. We proceed to define the G-equivariant cohomology of M as the (singular, say) cohomology of M_G,

$$H_G^\bullet(M) := H^\bullet(M_G).$$

In this paper we will always assume complex coefficients. The first thing we check is that $H_G^\bullet$ reduces to what we expect when M is a free G-space. Indeed, the projection $EG \times M \to M$ induces a fibration $M_G \to M/G$ with fibre EG. Since EG is contractible the long long exact sequence for homotopy groups implies that $EG \times_G M \to M/G$ is a (weak) homotopy equivalence and so induces an isomorphism in cohomology. Note also that when $M = pt$ the equivariant cohomology reduces to the group cohomology $H^\bullet(BG)$, while if $G = pt$ it is just the ordinary cohomology.

Equivariant cohomology enjoys many of the same properties as does usual cohomology. For example, $H_G^\bullet$ is a contravariant functor from the category of G-spaces to the category of abelian groups and is invariant under G-equivariant homotopies. Note however, that $H_G^\bullet$ is not technically a cohomology theory (although it is a *generalized* cohomology theory) since the equivariant cohomology of a point is not just the coefficient field. For example, if $G = \mathbb{T}$ is a torus, $H_{\mathbb{T}}^\bullet = \mathbb{C}[x_1, \ldots, x_n]$, corresponding to the homotopy equivalence $B\mathbb{T} \simeq (\mathbb{CP}^\infty)^n$. An important difference between ordinary cohomology and equivariant cohomology is the following. The G-equivariant map $\pi : M \to pt$ induces a ring homomorphism

$$\pi^* : H_G^\bullet \to H_G^\bullet(M)$$

which endows $H_G^\bullet(M)$ with the structure of a $H_G^\bullet$-module. Effectively, this allows us to take the coefficient ring to be $H_G^\bullet(M)$. Note that for $G = \mathbb{T}^n$ with $n \geq 2$, $H_G^\bullet$ is not a PID. Hence, $H_G^\bullet$-modules are not of the simple form $Tor \oplus Free$, as are $\mathbb{Z}$-modules.

2.1.1 $H_{S^1}^\bullet(S^2)$

As an example we compute the equivariant cohomology for the S^1 action on S^2 given by rotating S^2 about the z-axis. We will compute the ring structure of $H_{S^1}^\bullet(S^2)$ using the (equivariant) Mayer-Vietoris sequence.

Theorem 2.1 (Mayer-Vietoris)**.** *Let $U, V \subset M$ be an excisive pair such that each is G-invariant. Then the induced sequence in equivariant cohomology*

$$\cdots \to H_G^k(U \cup V) \to H_G^k(U) \oplus H_G^k(V) \to H_G^k(U \cap V) \to H_G^{k+1}(U \cup V) \to \cdots$$

is exact.

To study the example at hand let $U = S^2 \backslash p_+$ be S^2 with the north pole removed and $U = S^2 \backslash p_-$ be S^2 with the south pole removed. Note that $p_\pm$ are the only fixed points of the action and that both U and V are S^1-invariant. Furthermore, $U \cap V$ equivariantly retracts to the equator $S^1 \hookrightarrow S^2$ on which S^1 acts freely. Hence

$$H^\bullet_{S^1}(U \cap V) \simeq H^\bullet_{S^1}(S^1) \simeq H^\bullet(S^1/S^1)$$

which is $\mathbb{C}$ in degree zero and zero elsewhere. Also

$$H^\bullet_{S^1}(U) \simeq H^\bullet_{S^1}(p_-) = H^\bullet(BS^1) = \mathbb{C}\left[x_-\right]$$

with x_- having degree 2, and similarly for V we have $H^\bullet_{S^1}(V) \simeq \mathbb{C}\left[x_+\right]$. Hence Mayer-Vietoris gives

$$H^\bullet_{S^1}(S^2) \simeq \mathbb{C}\left[x_+, x_-\right]/(x_+ x_-).$$

Remark 2.1. *Our computation of $H^\bullet_{S^1}(S^2)$ hints at a general phenomenon in equivariant cohomology that has no analogue in ordinary cohomology.*

Theorem 2.2 (Borel Localization)**.** *Let $i : F \to M$ denote the inclusion of the G-fixed point set of M. Then*

$$i^* : H^\bullet_G(M) \to H^\bullet_G(F) \simeq H^\bullet(F) \otimes H^\bullet_G$$

is an isomorphism modulo $H^\bullet_G$-torsion.

Hence we see that we can learn much about the structure of $H^\bullet_G M$ from the fixed point set of the G-action. Our main localization theorem (which is not unrelated to this one) will be another realization of this phenomenon.

2.2 The Cartan Construction

We now take a different approach, developing an analogue of the de Rham approach to the cohomology of a smooth manifold. The action of G on M induces an action of G on $C^\infty(M)$ by

$$(g \cdot \phi)(x) = \phi(g^{-1} \cdot x) \quad \forall g \in G, \ \ x \in M.$$

Note the presence of g^{-1}; this is required for the action to be a group action.

Denote by $\mathfrak{g}$ the Lie algebra of G. There is an induced action of $\mathfrak{g}$ on $C^\infty(M)$ given by

$$(X \cdot \phi)(x) = \frac{d}{dt}|_{t=0} \phi(e^{-tX} \cdot x) \quad \forall X \in \mathfrak{g}.$$

This gives a Lie algebra homomorphism $\mathfrak{g} \to \Gamma(M, TM)$. Similarly, G (and so $\mathfrak{g}$) act on the algebra of differential forms $\Omega^\bullet(M)$ and vector fields $\Gamma(M, TM)$.

Let $\mathbb{C}\left[\mathfrak{g}\right]$ denote the algebra of complex polynomials on $\mathfrak{g}$. We regard $\mathbb{C}\left[\mathfrak{g}\right] \otimes \Omega^\bullet(M)$ as the algebra of polynomial maps $\mathfrak{g} \to \Omega^\bullet(M)$. The group G acts on $\mathbb{C}\left[\mathfrak{g}\right] \otimes \Omega^\bullet(M)$ by

$$(g \cdot \alpha)(X) = g \cdot (\alpha(g^{-1} \cdot X)) \quad \forall g \in G, \ \ X \in \mathfrak{g}, \ \ \alpha \in \mathbb{C}\left[\mathfrak{g}\right] \otimes \Omega^\bullet(M).$$

Let $\Omega^\bullet_G(M) = \left(\mathbb{C}\left[\mathfrak{g}\right] \otimes \Omega^\bullet(M)\right)^G$ be the subalgebra of G-invariant forms; this is the algebra of equivariant differential forms on M. Explicitly, if $\alpha \in \Omega^\bullet_G(M)$ then

$$g \cdot \alpha(X) = \alpha(g \cdot X) \quad \forall g \in G, \ \ X \in \mathfrak{g}$$

which explains the reasoning for calling elements of $\Omega^\bullet_G(M)$ equivariant. The algebra $\mathbb{C}\left[\mathfrak{g}\right] \otimes \Omega^\bullet(M)$ has natural $\mathbb{Z}$ grading given by

$$\deg(P \otimes \omega) = 2\deg(P) + \deg(\omega) \quad P \in \mathbb{C}\left[\mathfrak{g}\right], \ \ \omega \in \Omega^\bullet(M)$$

and induces a $\mathbb{Z}$-grading on equivariant differential forms.

Define the G-equivariant exterior differential $d_{\mathfrak{g}} : \mathbb{C}[\mathfrak{g}] \otimes \Omega^\bullet(M) \to \mathbb{C}[\mathfrak{g}] \otimes \Omega^\bullet(M)$ by

$$(d_{\mathfrak{g}}\alpha)(X) = d(\alpha(X)) - \iota(X)(\alpha(X)) \quad \forall X \in \mathfrak{g}.$$

With the described grading it is clear that $d_{\mathfrak{g}}$ has degree 1.

Lemma 2.3. *$d_{\mathfrak{g}}$ preserves $\Omega_G^\bullet(M)$.*

Proof. Let $\alpha \in \Omega_G^\bullet(M)$ and $X \in \mathfrak{g}$. Then

$$\begin{aligned} g \cdot (d_{\mathfrak{g}}\alpha)(X) &= g \cdot \left(d\alpha(g^{-1} \cdot X) - \iota(g^{-1} \cdot X)\alpha(g^{-1} \cdot X)\right) \\ &= d\left(g \cdot \alpha(g^{-1} \cdot X)\right) - g \cdot \iota(g^{-1} \cdot X)(g^{-1} \cdot \alpha(X)) \\ &= d\alpha(X) - \iota(X)\alpha(X) \\ &= d_{\mathfrak{g}}\alpha(X) \end{aligned}$$

where in the second last step we used that α is G-equivariant and that $\iota(g^{-1} \cdot X) = g^{-1}\iota(X)g$. □

The homotopy identity shows that $d_{\mathfrak{g}}^2\alpha(X) = -\mathcal{L}(X)\alpha(X)$. Hence, while $d_{\mathfrak{g}}$ does not square to zero on $\mathbb{C}[\mathfrak{g}] \otimes \Omega^\bullet(M)$, it does when restricted to $\Omega_G^\bullet(M)$. Indeed, if $\alpha \in \Omega_G^\bullet(M)$ then

$$\begin{aligned} \mathcal{L}(X)\alpha(X) &= \tfrac{d}{dt}_{|t=0}\left(e^{tX} \cdot \alpha\right)\left(X\right) \\ &= \tfrac{d}{dt}_{|t=0}\alpha\left(e^{tX} \cdot X\right) \\ &= \tfrac{d}{dt}_{|t=0}\alpha(X) \end{aligned}$$

since $[X, X] = 0$. We summarize our results so far.

Proposition 2.4. *$(\Omega_G^\bullet(M), d_{\mathfrak{g}})$ is a complex.*

We may thus form the cohomology $H^\bullet\left(\{\Omega_G^\bullet(M), d_{\mathfrak{g}}\}\right)$. The following fundamental result is due to Cartan.

Theorem 2.5 (Equivariant de Rham Theorem)**.** *Let a compact, connected Lie group G act on a smooth, compact manifold M. Then $H_G^\bullet(M) \simeq H^\bullet\left(\{\Omega_G^\bullet(M), d_{\mathfrak{g}}\}\right)$.*

We end with a few concrete interpretations of equivariant cohomology classes in low degrees. If $f \in \Omega_G^0(M)$ then f is a smooth G-invariant function on M. We see that f $d_{\mathfrak{g}}$-closed if and only if it is d-closed. Hence, $H_G^0(M) \simeq \mathbb{C}^k$ where $\pi_0(M/G) = \mathbb{Z}^k$.

If $\eta \in \Omega_G^1(M)$, then η is a G-invariant one-form on M. The $d_{\mathfrak{g}}$-closed condition is

$$d\eta - \iota(X)\eta = 0$$

which says η is d-closed and horizontal. Exact elements of $\Omega_G^1(M)$ are d-exact basic elements. One can show $H_G^1(M) \simeq H^1(M/G)$ where the cohomology of M/G is the singular or Čech cohomology.

An element in $\Omega_G^2(M)$ is of the form $\omega + \alpha$ for ω a G-invariant two-form on M and α an invariant linear function on $\mathfrak{g}$ with values in $C^\infty(M)$. The $d_{\mathfrak{g}}$-closed condition implies $d\omega = 0$ and $\iota(X)\omega = d\alpha(X)$ for all $X \in \mathfrak{g}$.

Remark 2.2. *The results in this section can be repeated almost verbatim in the more general case where we consider a $G^\star$-modules instead of the algebra $\Omega^\bullet(M)$. The most often occurring examples of $G^\star$-modules are $\Omega^\bullet(M)$ for a G-space M or $\Omega_G^\bullet(M)$ for a $G \times K$ space M. For details see [5].*

3 Localization

The idea of localization shows up in many areas of mathematics. Perhaps the most familiar case arises in complex analysis. Indeed, consider a function $f : U \to \mathbb{C}$, with $U \subset \mathbb{C}$ a simply connected open set, that is holomorphic on $U \backslash \{a_1, \ldots, a_n\}$. Then for a loop γ in U, not meeting any a_i, the Residue Theorem asserts

$$\oint_\gamma f(z)dz = 2\pi i \sum_{i=1}^{n} \mathrm{Res}(f; a_i).$$

Hence the integral here has localized in the sense that it can be evaluated as a finite sum. The result we prove below is completely analogous in nature (and in fact the proofs will be reminiscent, too!).

3.1 Proof in the Cartan Model

We largely follow [2]. In this section we make the further assumption that M is oriented. We denote the dimension of M by n. An orientation of M induces a map, integration,

$$\int_M : \Omega_G^\bullet(M) \to \mathbb{C}\left[\mathfrak{g}\right]^G$$

given by $\left(\int_M \alpha\right)(X) = \int_M \alpha(X)$. Implicitly in this formula we consider only the forms of top degree on M.

Lemma 3.1. *Suppose $\alpha \in \Omega_G^\bullet(M)$ is $d_\mathfrak{g}$-exact, $d_\mathfrak{g}\eta = \alpha$. Then $\alpha(X)_{[n]}$ is d-exact for all $X \in \mathfrak{g}$.*

Proof. We have $d_\mathfrak{g}\eta(X) = d\eta(X) - \iota(X)\eta(X)$. In this equation only $d\left(\eta(X)_{[n-1]}\right)$ has degree n, whence $\alpha(X)_{[n]} = d\left(\eta(X)_{[n-1]}\right)$. □

Corollary 3.2. *If $\alpha = d_\mathfrak{g}\eta$, then $\int_M \alpha = 0$. Hence integration descends to a map*

$$\int_M : H_G^\bullet(M) \to \mathbb{C}\left[\mathfrak{g}\right]^G .$$

In fact, integration in equivariant cohomology is predicted on functorial grounds; it is the equivariant push-forward of $\pi : M \to pt$.

We now establish an important tool used during the proof of the localization theorem.

Proposition 3.3. *Let M and G be as above. Then there exists a G-invariant Riemannian metric on M.*

Proof. A partition of unity argument gives a Riemannian metric $\langle\cdot,\cdot\rangle$ on M. Denoting by dg the right-invariant Haar measure on G, we define

$$(v, w) = \frac{1}{\mathrm{vol}(G)} \int_G \langle g\cdot w, g\cdot v\rangle dg \quad \forall v, w \in T_pM.$$

It is a simple matter to check that $(\cdot,\cdot)$ is Riemannian metric on M. That it is G-invariant follows from the right-invariance of the Haar measure. □

It is the following lemma that first suggests the existence of a localization result.

Lemma 3.4. *Suppose $\alpha \in \Omega_G^\bullet(M)$ is $d_\mathfrak{g}$-closed. Let $X \in \mathfrak{g}$ be such that the zero set $M_0(X)$ is finite. Then $\alpha(X)_{[n]}$ is d-exact on $M\backslash M_0(X)$.*

Proof. Let $d_X = d - \iota(X)$. Define $\theta \in \Omega^1(M)$ to be the dual of X in the G-invariant metric $(\cdot,\cdot)$,

$$\theta(\xi) = (X, \xi) \quad \forall \xi \in \Gamma(M, TM).$$

We compute

$$\begin{aligned}
(\mathcal{L}(X)\theta)(\xi) &= \tfrac{d}{dt}|_{t=0}\left(e^{tX}\cdot\theta\right)(\xi) \\
&= \tfrac{d}{dt}|_{t=0}\theta(e^{tX}\cdot\xi) \\
&= \tfrac{d}{dt}|_{t=0}(X, e^{tX}\cdot\xi) \\
&= \tfrac{d}{dt}|_{t=0}(e^{-tX}\cdot X, \xi) \quad (G\text{-invariance of the metric}) \\
&= 0.
\end{aligned}$$

Hence θ is X-invariant. We also see $d_X\theta = d\theta - |X|^2$. Hence, off $M_0(X)$ $d_X\theta$ is invertible in $\Omega_G^\bullet(M)$ (because its zero degree component is non-zero, while the rest of it is nilpotent). So on $M\backslash M_0(X)$ we have

$$\alpha(X)_{[n]} = d\left(\frac{\theta\wedge\alpha(X)}{d_X\theta}\right)_{[n-1]}.$$

Indeed,

$$\begin{aligned}
d_X\left(\tfrac{\theta\wedge\alpha(X)}{d_X\theta}\right) &= \tfrac{d_X\theta\wedge\alpha(X)}{d_X\theta} - \tfrac{\theta\wedge d_X\alpha(X)}{d_X} \pm (\cdots)d_X^2\theta \\
&= \alpha(X)
\end{aligned}$$

since α is assumed $d_{\mathfrak{g}}$-closed and θ is X-invariant. Taking the top degree proves the lemma. ☐

Continuing with the assumption that $M_0(X)$ is finite, fix $p\in M_0(X)$. Then the Lie action $\mathcal{L}(X)(\xi) = [X,\xi]$ for $\xi\in\Gamma(M,TM)$ gives rise to an endomorphism L_p of T_pM.

Lemma 3.5. *L_p is invertible and has only imaginary eigenvalues.*

Proof. Say $\xi\in T_pM$ is in the kernel of L_p. Denoting by $(\cdot,\cdot)$ a G-invariant metric on M, consider the curve $\gamma(t) = \exp_p(t\xi)$. Then

$$\exp(sX)\cdot\gamma(t) = \gamma(t) \quad \forall s,t$$

because $[X,\xi]_p = 0$. Hence the image of γ is fixed (pointwise) by G, contradicting the finiteness of $M_0(X)$.

To prove the second claim it suffice to show that L_p is skew-symmetric. To see this note that because the metric $(\cdot,\cdot)$ is G-invariant the vector field (induced by) X is a Killing field. Hence

$$X(v,w) = ([X,v],w) + (v,[X,w]) = (L_pv,w) + (v,L_pw) \quad \forall v,w\in T_pM.$$

Since $X|_p = 0$ the result follows ☐

From the lemma we see that dim$M = n$ is even and so we write $n = 2l$. Indeed, the characteristic polynomial for L_p is real and has only imaginary eigenvalues. Hence these eigenvalues occur in complex conjugate pairs. Alternatively, if M were odd dimensional, the characteristic polynomial would have to have at least one root, by the Intermediate Value Theorem.

We may pick an oriented $\mathbb{R}$-basis $e_1,\cdots,e_n$ of T_pM such that

$$L_pe_{2i-1} = \lambda_i e_{2i}, \quad L_pe_{2i} = \lambda_i e_{2i-1}$$

with $\lambda_i > 0$. Note that with this notation the eigenvalues of L_p are $\pm\sqrt{-1}\lambda_i$. It follows that $\det L_p = \lambda_1^2\cdots\lambda_l^2$. Using the orientation of M it is natural to choose $\det^{\frac{1}{2}}L_p = \lambda_1\cdots\lambda_l$.

We now state our main result.

Theorem 3.6. *Let M be a smooth, compact, oriented manifold acted on by a compact, connected Lie group G. Let $\alpha\in\Omega_G^\bullet(M)$ be $d_{\mathfrak{g}}$-closed and $X\in\mathfrak{g}$ with $M_0(X)$ finite. Then*

$$\int_M \alpha(X) = (-2\pi)^l \sum_{p\in M_0(X)} \frac{\alpha(X)_{[0]}(p)}{det^{\frac{1}{2}}L_p}.$$

Proof. Pick $p \in M_0(X)$. Replacing G by the closed Lie group generated by X we may assume G is abelian. We continute to denote by $(\cdot,\cdot)$ a G-invariant metric on M. Using this and the induced exponential map we pick a neighbourhood U_p of p with coordinates $(x_1,\dots,x_n)$ centred at p such that

$$X = \lambda_1(x_2\partial_{x_1} - x_1\partial_{x_2}) + \cdots + \lambda_l(x_{n-1}\partial_{x_n} - x_n\partial_{x_{n-1}}).$$

Also on U_p define the one-form

$$\theta^p = \frac{1}{\lambda_1}(x_2dx_1 - x_1dx_2) + \cdots + \frac{1}{\lambda_l}(x_{n-1}dx_n - x_ndx_{n-1}).$$

Then we check $\theta^p(X) = \sum_i x_i^2 = \|x\|^2$. A computation shows that $\mathcal{L}(X)\theta^p = 0$.

Let $\{\rho_i\}$ be a partition of unity subordinate to $\bigcup_{p\in M_0(X)} U_p \bigcup (M\backslash M_0(X))$; by averaging over G we may assume each ρ_i is G-invariant. Gluing together the local forms θ^p gives a global form $\theta \in \Omega^1(M)$ such that

1. $\theta|_{U_p} = \theta^p \quad \forall p \in M_0(X)$
2. $\mathcal{L}(X)\theta = 0$
3. $d_X\theta$ is invertible outside of $M_0(X)$

Note that 3) follows because in U_p, away from p (which corresponds in local coordinates to $x = 0$) we have $\|x\|^2 > 0$, so that the zeroth component of $d_X\theta$ is non-zero. Hence $(d_X\theta)^{-1} \in \Omega_G^\bullet(M)$.

With this, we compute

$$\begin{aligned}\int_M \alpha(X) &= \lim_{\epsilon\to 0}\int_{M\backslash\cup_{p\in M_0(X)}B(p,\epsilon)} \alpha(X)\\ &= \lim_{\epsilon\to 0}\int_{M\backslash\cup_{p\in M_0(X)}B(p,\epsilon)} d\left(\frac{\theta\wedge\alpha(X)}{d_X\theta}\right)\\ &= -\sum_{p\in M_0(X)}\lim_{\epsilon\to 0}\int_{\partial B(p,\epsilon)}\frac{\theta\wedge\alpha(X)}{d_X\theta}.\end{aligned}$$

In the last integral we may take $\theta = \theta^p$. Rescale the local coordinates x by a factor of $\epsilon^{\frac{1}{2}}$, so that $\partial B(p,\epsilon)$ becomes the unit $(n-1)$-sphere centred on p. Observe that, begin homogeneous, $\frac{\theta}{d_X\theta}$ is not affected by the scaling. Hence

$$\int_{\partial B(p,\epsilon)}\frac{\theta\wedge\alpha(X)}{d_X\theta} = \int_{S^{n-1}}\frac{\theta\wedge\alpha_\epsilon(X)}{d_X\theta}.$$

Here $\alpha_\epsilon(x,dx) = \alpha(\sqrt{\epsilon}x, \sqrt{\epsilon}dx)$. We have $\lim_{\epsilon\to 0}\alpha_\epsilon(X) = \alpha(X)(p)$. It remains to evaluate $\int_{S^{n-1}}\frac{\theta}{d_X\theta}$. Near p we have $d_X\theta = d\theta - |x|^2$ so that on S^{n-1}

$$\frac{\theta}{d_X\theta} = \theta(d\theta - 1)^{-1}.$$

By Stokes Theorem we see

$$\int_{S^{n-1}}\frac{\theta}{d_X\theta} = -\int_{B^n}(d\theta)^l.$$

We easily compute $(d\theta)^l = \frac{(-2)^l l!}{\lambda_1\cdots\lambda_l}$ Using that $\mathrm{vol}(B^{2l}) = \frac{\pi^l}{l!}$ we conclude

$$-\int_{\partial B(p,\epsilon)}\frac{\theta\wedge\alpha(X)}{d_X\theta} = (-2\pi)^l\frac{\alpha(X)_{[0]}(p)}{\det^{\frac{1}{2}}L_p}$$

which completes the proof. □

Remark 3.1. *We have restricted our attention to the case where $M_0(X)$ is finite for simplicity. For extensions to the case where $M_0(X)$ is a smooth manifold (as it necessarily must be) see [2]. For a functorial proof, we suggest [1]. The reader may wonder why in, say, [1], the theorem restricts attention to tori. In fact, the results here too are essentially 'abelian' results, since we restrict attention to a single form $\alpha(X)$ throughout. Hence we really only see information about the torus generated by X. In other words, for a given (possibly non-abelian) group G with maximal torus $\mathbb{T}$ we can apply the map $H_G^\bullet(M) \to H_{\mathbb{T}}^\bullet(M)$ and then apply the localization theorem. There are, however, non-abelian generalizations of localization. See [6], for example.*

4 Applications

4.1 Duistermaat-Heckman

Let (M, Ω) be a compact symplectic manifold of dimension $n = 2l$. Let G be a compact group acting on M by Hamiltonian transformations. That is, for each $X \in \mathfrak{g}$ there exists $\mu(X) \in C^\infty(M)$ such that

1. the map $X \mapsto \mu(X)$ is $\mathbb{R}$-linear
2. $d\mu(X) = \iota(X)\Omega$, i.e. X is the Hamiltonian vector field corresponding to $\mu(X)$
3. μ is G-equivariant.

Furthermore, $\mathcal{L}(X)\omega = 0$ since G acts by symplectomorphisms. Because of 2) and the non-degeneracy of Ω we see

$$M_0(X) = \{p \in M \mid d\mu(X)(p) = 0\}.$$

Finally, recall that the Liouville form on M is

$$d\beta = e^{\frac{\Omega}{2\pi}}_{[n]} = \frac{\Omega^l}{(2\pi)^l l!}$$

and determines a volume form for the canonical orientation of M.

Theorem 4.1 (Duistermaat-Heckman)**.** *With the notation above, let $X \in \mathfrak{g}$ be such that $M_0(X)$ is finite. Then*

$$\int_M e^{i\mu(X)} d\beta = i^l \sum_{p \in M_0(X)} \frac{e^{i\mu(X)}}{det^{\frac{1}{2}} L_p}.$$

Proof. Define the equivariant symplectic form $\Omega_{\mathfrak{g}} = \Omega + \mu$. It is easy to check that $d_{\mathfrak{g}}\Omega_{\mathfrak{g}} = 0$ and so defines a class in $H_G^\bullet(M)$. Hence $e^{i\Omega_{\mathfrak{g}}}$ is also $d_{\mathfrak{g}}$-closed. We see

$$\int_M e^{i\mu(X)} d\beta = (2\pi i)^{-l} \int_M e^{i\Omega_{\mathfrak{g}}}$$

which by the localization theorem is equal to the sum in the statement. □

The Duistermaat-Heckman Theorem is often stated in the following form. Recall that if $\phi \in C^\infty(M)$ is Morse then the large t asymptotics of the function

$$\Phi(t) = \int_M e^{it\phi} dx$$

give rise to the stationary phase approximation,

$$\Phi(t) = \sum_{\{p \in M \mid df_p = 0\}} \left(\frac{2\pi}{t}\right)^{\frac{n}{2}} e^{\frac{i\pi\sigma(H_p)}{4}} \frac{e^{it\phi}}{\det^{\frac{1}{2}} H_p} + O(t^{-\frac{n}{2}-1}) \qquad \text{as } t \to \infty.$$

Roughly, this says that the large t asymptotics are determined by the behaviour of f around its stationary points. The Duistermaat-Heckman Theorem then gives the following criterion for the stationary phase approximation to be exact.

Corollary 4.2. *Let (M,Ω) be a compact symplectic manfiold with Hamiltonian vector field X such that the flow of X is periodic. Denote by ϕ the Hamiltonian of X. Assume the fixed point set $M_0(X)$ is finite. Then the stationary phase approximation to*

$$\int_M e^{it\phi} d\beta$$

is exact.

4.1.1 A Simple Example

We consider the action of S^1 on S^2 considered above. Explicitly, if $\xi \in \mathrm{Lie}(S^1)$ then

$$e^{i\xi}\left(\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi\right) = \left(\cos(\theta+\xi)\sin\phi, \sin(\theta+\xi)\sin\phi, \cos\phi\right).$$

The induced vector field at $(\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$ is

$$\xi_M = (\sin\theta\sin\phi, -\cos\theta\sin\phi, 0) = -\partial_\theta.$$

Note that the only fixed points of the action are at the points $\phi = 0$ and $\phi = \pi$. Let $\omega = \sin\phi d\phi d\theta$ be the symplectic form on S^2 and $h(\theta,\phi) = \cos\phi$ be the height function. Clearly S^1 acts by symplectomorphisms. It is easy to check that the moment map for the S^1 action is determined by its value on a generator to be $-h$, extended $\mathbb{R}$-linearly. The Duistermaat-Heckman theorem then implies

$$\int_{S^2} e^{ith}\sin\phi d\phi d\theta = -\frac{2\pi}{it}\left(e^{-it} - e^{it}\right) = 4\pi\frac{\sin t}{t}.$$

That this result is correct can be computed directly.

Remark 4.1. *The exactness stationary phase has proven very useful in physics, particularly topological field theories. Roughly, the idea is as follows. In physical theories there is often some coupling constant, say t, on which integrals of interest depend. In certain topological theories one can argue that the integrals in question are in fact independent of t. One can then take the limit $t \to \infty$ and examine the stationary phase approximation. For many models one can identify the integrand with an equivariantly closed form, and hence the stationary phase approximation should be exact. Of course, in practice many of the encountered integrals are over infinite dimensional spaces, so while Duistermaat-Heckman is strictly speaking not applicable, the results are quite suggestive. For example, one uses Duistermaat-Heckman to localize correlation functions in the A-model to integrals over the finite dimensional moduli space of holomorphic curves.*

4.2 Characteristic Numbers

We prove here a classic theorem of Bott on the characteristic numbers of a smooth manifold. Bott was originally motived by the Lefschetz Fixed Point Theorem. We will see below that the Localization Theorem gives a quick an easy proof of this result.

We recall some definitions. Let $\phi \in \mathbb{C}\left[\mathfrak{so}(n)\right]^{O(N)}$ be an invariant polynomial, with $n = 2l$. ϕ is then uniquely determined by its restriction to the Cartan subalgebra $\mathfrak{t}$ which we take to consist of matrices X such that

$$Xe_{2i-1} = \lambda_i e_{2i}, \quad Xe_{2i} = -\lambda_i e_{2i-1}.$$

Now, ϕ restricted to $\mathfrak{t}$ is a symmetric function of λ_i and so can be viewed as an element of the ring of symmetric polynomials, $\mathbb{C}\left[\sigma_1,\ldots,\sigma_l\right]$. This shows $\mathbb{C}\left[\mathfrak{so}(n)\right]^{O(N)} = \mathbb{C}\left[\sigma_1,\ldots,\sigma_l\right]$.

Now, let M be a oriented, compact Riemannian manifold of dimension $n = 2l$. Let ∇ be the Levi-Civitá connection and $R_\nabla \in \Omega^2(M; \mathfrak{so}(n))$ its curvature. Chern-Weil theory gives a homomorphism $\mathbb{C}[\mathfrak{so}(n)]^{SO(n)} \to H^{2\bullet}(M)$ that maps ϕ to the class of $\phi(F_\nabla)$. The number

$$\phi(M) = (-2\pi)^{-l} \int_M \phi(F_\nabla)$$

is then called a characteristic number of M.

If M admits a circle action with $M_0(X)$ finite the Lie action on T_pM for $p \in M_0(X)$ determines (as before) an endomorphism L_p of T_pM such that for an oriented basis

$$L_p e_{2i-1} = -\lambda e_{2i}, \quad L_p e_{2i} = \lambda e_{2i-1}.$$

Then $\det^{\frac{1}{2}} L_p = \lambda_1 \cdots \lambda_l$.

Theorem 4.3 (Bott)**.** *If $\phi \in \mathbb{C}[\mathfrak{so}(n)]^{SO(n)}$ is homogeneous of degree k then*

$$\sum_{p \in M_0(X)} \frac{\phi(L_p)}{det^{\frac{1}{2}} L_p} = \begin{cases} \Phi(M), & \textit{if } k = l, \\ 0, & \textit{if } k < l. \end{cases}$$

Proof. Let G be the closed Lie group generated by X and identify $\mathfrak{g} = \{aX \mid a \in \mathbb{R}\}$. If $(\cdot, \cdot)$ is a G-invariant metric on M, the equivariant curvature of ∇ is

$$a\mu^M(X) + F_\nabla.$$

where $\mu^M(X) = -\nabla X$. Now, the map $a \mapsto \phi\left(a\mu^M(X) + F_\nabla\right)$ is polynomial in a and is d_X-closed. The Localization Theorem then gives

$$\begin{aligned} (-2\pi)^{-l} \int_M \phi\left(a\mu^M(X) + F_\nabla\right) &= \sum_{p \in M_0(X)} \frac{\phi(aM)}{\det^{\frac{1}{2}}(aL_p)} \\ &= a^{-l} \sum_{p \in M_0(X)} \frac{\phi(aM)}{\det^{\frac{1}{2}}(L_p)}. \end{aligned}$$

Comparing both sides in cases gives the result. ☐

Remark 4.2. *One area where Bott's formula for characteristic numbers has been quite fruitful is in enumerative geometry. For example, consider the problem of counting lines in a generic quintic threefold $V \subset \mathbb{P}^4$. Let $\gamma \to Gr(2,5)$ be the tautological rank 2 vector bundle over the complex Grassmannian. A defining equation for V determines a section of the bundle $Sym^5 U^*$. One can then show that the number of lines in question is equal to*

$$\int_{G(2,5)} c_6(Sym^5 U^*).$$

There is a natural action of $\mathbb{T}^5$ on $\mathbb{P}^4$ which allows the use of Bott's formula to reduce the integral to a sum over finitely many fixed points of the torus action. For details, see [4]. The computations shows that, generically, there are 2785 such lines.

More ambitiously, one can use localization to compute more general Gromov-Witten invariants. One must first learn how to apply the localization theorem to more singular spaces. However, these difficulties can be overcome and yield impressive results. See again [4] or [7] for details.

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