Supersymmetric σ-Models

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Abstract

The following are notes from talks given in the RTG seminar in Geometry and Physics at Stony Brook University, Sept. 15, 22 and 29, 2008. They are essentially a summary with comments of [1, 3] and contain no new material. They were written for the author's understanding.

1 Bosonic σ-Models

To define a σ-model, we need

1. X a Riemannian manifold with metric $\langle \cdot, \cdot \rangle$
2. $V : X \to \mathbb{R}$ a potential energy function.

The fields in the theory is the set of all maps $\mathcal{F} := \{\phi : M \to X\}$.

The Lagrangian density describing the system is a map $\mathcal{F} \to \mathrm{Dens}(M)$ given by

$$L = \frac{1}{2}|d\phi|^2 - \phi^* V |d^n x|.$$

Note that $d\phi \in \Omega^1(M, \phi^* TX)$. We use the notation $|d\phi|^2 = \langle d\phi \wedge * d\phi \rangle$, with the Hodge star using the metric from M and the angeled brackets the metric on X. In components

$$\langle d\phi \wedge * d\phi \rangle = \langle \partial^\mu \phi, \partial_\mu \phi \rangle |d^n x|.$$

Hence, both metrics are involved in the Lagrangian.

The most basic case occurs for $M = X = \mathbb{R}$ with the Euclidean metric; we interpret M as time. Then

$$L = \left(\frac{1}{2}\dot{\phi}^2 - V(\phi) \right) |dt|$$

is the Lagrangian for a classical particle in a potential V. More generally for $M = \mathbb{R}^n$ with metric η, we have

$$L = \left(\frac{1}{2}\eta^{ab}\partial_a \phi \partial_b \phi - V(\phi) \right) |d^n x|.$$

The choice $V(\phi) = \frac{m^2}{2}\phi^2$, for example, gives a free field of mass m. For X a general Riemannian manifold them Lagrangian in terms of components is

$$L = \left(\frac{1}{2} g_{\mu\nu} \eta^{ab} \partial_a \phi^\mu \partial_b \phi^\nu - V(\phi) \right) |d^n x|.$$

Continuing, we want to vary L to find the equations of motion. To do so we define the variation of a field $\phi \in \mathcal{F}$ to be $\delta\phi$, where we think of δ as a differential on $\mathcal{F}$. We want $D = \delta + d$ to be a differential on $\mathcal{F} \times M$, and hence require $\delta d = -d\delta$. The negative sign in this formula does not occur in typical physics textbooks; we correct it by adding an extra sign to the variational one-form below. We compute

$$\begin{aligned} \delta L &= \langle \delta_\nabla d\phi \wedge *d\phi \rangle - \langle \delta\phi, \phi^*\text{grad}V \rangle |d^n x| \\ &= -d\langle \delta\phi \wedge *d\phi \rangle - \langle \delta\phi \wedge d_\nabla * d\phi \rangle - \langle \delta\phi, \phi^*\text{grad}V \rangle |d^n x|. \end{aligned}$$

We define the variational one-form as $\gamma = \delta\phi \wedge *d\phi$. The equations of motion is the subset $\mathcal{M} \subset \mathcal{F}$ defined by

$$\delta L + d\gamma = 0.$$

Comparing with the usual derivation of the Euler-Lagrange equations, we see that γ acts as a tool to perform integration by parts with out actually integrating. Explicitly, the equations of motion are

$$\Box\phi + \phi^*\text{grad}V = 0$$

where $\Box = *d_\nabla * d$. For $M = \mathbb{R}$ this equation reduces to Newton's law, while for $M = \mathbb{R}^n$ with the Minkowski metric with $V(x) = \frac{m^2}{2}x^2$ this gives the Klein-Gordon equation. We note $\delta\gamma = \omega$ is the symplectic form for the solution space $\mathcal{M}$. The case of a classical particle, γ and ω reduce to the familiar form from symplectic geometry and classical mechanics.

From the Lagrangian we can compute from symmetries the corresponding conserved quantities. As a simple example, consider the classical particle with Lagrangian

$$L = \left(\frac{1}{2}\dot{\phi}^2 - V(\phi) \right) |dt|.$$

Consider now the vector field ∂_t on $\mathcal{F}$ that generates time translations. We have $\iota(\partial_t)\delta\phi = -\dot{\phi}$. We say that ∂_t generates a symmetry of the system if $\text{Lie}(\partial_t)L = d\alpha_t$ for some α. Indeed, in this case we compute

$$\text{Lie}(\partial_t)L = d\left(-\frac{1}{2}\dot{\phi}^2 - V(x) \right).$$

The associated current is

$$j_t = \iota(\partial_t)\gamma - \alpha_l = -\left(\frac{1}{2}\dot{\phi}^2 + V(x) \right) + C$$

which in this case is just (minus) the energy. On-shell we can check that the current is conserved, $dj_t = 0$. This is the statement that systems with time translational symmetry have conserved energy.

In the general σ-model we can check that a Killing field ξ generates a symmetry we associated current

$$j_\xi = \langle \xi, *d\phi \rangle.$$

One important use of σ-models is to describe Goldstone bosons, which occur when a symmetry is spontaneously broken. In particular, one often considers coset models, where X is a quotient of groups, related to symmetries and broken symmetries of the system. Mathematically, the (supersymmetric) σ-model is the starting point for Witten's approach to Morse theory.

2 Gauge Theory with Bosonic Matter

In this section we combine pure Yang-Mills theory with a σ-model, which is often called gauge theory with bosonic matter. The data for the theory are

1. G a Lie group, with Lie algebra $\mathfrak{g}$,

2. $\langle \cdot, \cdot \rangle$ a bi-invariant inner product on $\mathfrak{g}$,

3. X a Riemannian manifold on which G acts by isometries,

4. $V : X \to \mathbb{R}$ a G-invariant potential energy function.

The fields are a connection A on a principal G-bundle $P \to M$ and a section ϕ of the associated bundle $P \times_G X \to M$. Alternatively, ϕ can be viewed as a G-equivariant map $P \to X$. The Lagrangian is the natural generalization of both the Yang-Mills and σ-model Lagrangians:

$$L = \left(-\frac{1}{2}|F_A|^2 + \frac{1}{2}|d_A\phi|^2 - \phi^* V \right) |d^n x|.$$

The equations of motion are

$$\Box\phi + \phi^* \text{grad} V = 0, \quad d_A^* F_A = [d_A\phi, \phi],$$

where now $\Box$ is twisted by A.

One natural way to reach the combination of Yang-Mills and a σ-model is as follows. Let us begin with a simple σ-model with Lagrangian

$$L = \frac{1}{2}|d\phi|^2 - \phi^* V |d^n x|$$

where $\phi : M \to \mathbb{C}$ and V is a $U(1)$-invariant function. Then L is invariant under the global (i.e. constant) $U(1)$ transformation

$$\phi \mapsto e^{i\alpha}\phi, \quad \alpha \in \mathbb{R}.$$

However, L is not invariant under the local $U(1)$ transformation

$$\phi(x) \mapsto e^{i\alpha(x)}\phi(x), \quad \alpha : M \to \mathbb{R}.$$

In order to have these local transformations as a symmetry, we modify L to obtain the $U(1)$-invariant theory:

$$L' = \frac{1}{2}|d_A\phi|^2 - \phi^* V |d^n x|$$

where d_A is the $U(1)$-covariant derivative. In the Lagrangian L' the gauge field (connection) A entres only algebraically. Indeed, the equation of motion for the connection A_μ is

$$A_\mu = -\frac{i}{2} \sum_i \frac{\overline{\phi}_i \partial_\mu \phi_i - \phi_i \partial_\mu \overline{\phi}_i}{|\phi_i|^2}$$

To remedy this, we should add a kinetic term for the connection, which is precisely the Yang-Mills action. Adding this term to L' gives the gauged σ-model.

Let us briefly consider the case in which the potential vanishes, $V = 0$. The equations of motion then reduce to

$$\Box\phi = 0, \quad d_A^* F_A = [d_A\phi, \phi],$$

where we view ϕ as a section of some associated bundle with an adjoint action of G. Just as in pure Yang-Mills theory on a four manifold, where instead of studying the full equations of motion we looked for simpler solutions (self-dual and anti-self-dual connections), we can try a similar trick here, this time restricting our attention to three manifolds. Note that if we can solve

$$F_A = *d_A\phi$$

then we immediately have $\Box\phi = d_A F_A = 0$ by the Bianchi identity. In a straightforward calculation we can also verify that the second equation of motion is also satisfied. Hence solving the equation $F_A = *d_A\phi$ gives a solution to the gauged σ-model equations. This equation is known as the Bogomolny equation[1], and leads to some interesting mathematics and physics, but we do not discuss this here.

3 Superspace

We discuss here briefly the construction of superspace. We begin with Minkowski space $\check{M}^n$, with underlying vector space of translations V. Associated to V is a minimal real spin representation of $Spin(V)$, which we denote by S, with $\dim S = s$. Since V has signature $n-1$ there exists a symmetric equivariant pairing

$$\Gamma : S^* \otimes S^* \to V.$$

Choosing a basis $\{f_a\}$ of S^* and $\{e_\mu\}$ of V, we write $\Gamma(f_a, f_b) = \Gamma^\mu_{ab} e_\mu$. We assume that Γ is chosen so that $\Gamma^0_{aa} > 0$; this can always be done and has important implications to the quantum theory. We also have a pairing

$$\tilde{\Gamma} : S \otimes S \to V.$$

The pairings Γ and $\tilde{\Gamma}$ satisfy the Clifford relation

$$\Gamma^\mu_{ab}\tilde{\Gamma}^{\nu bc} + \Gamma^\nu_{ab}\tilde{\Gamma}^{\mu bc} = 2g^{\mu\nu}\delta^c_a.$$

The pairing Γ will be used to define superspace and appears (sometimes implicitly) in many superspace calculations.

Consider the super Lie algebra with even part $\mathcal{L}^0 = V$ and odd part $\mathcal{L}^1 = S^*$. We take V to be central and the non-trivial bracket to be

$$[f_a, f_b] = -2\Gamma(f_a, f_b).$$

Exponentiating $\mathcal{L}$ gives a super Lie group. Denote the underlying supermanifold $M^{n|s}$. This is our model of superspace with n spacetime dimensions and s supersymmetries. The group action on $\exp\mathcal{L}$ is easily found through the Hausdorff formula, which terminates quickly:

$$(e_1, f_1) \cdot (e_2, f_2) = \left(e_1 + e_2 + \frac{1}{2}\Gamma(f_1, f_2), f_1 + f_2\right).$$

Note that the structure of $\mathcal{L} = V \oplus S^*$ implies $M^{n|s} = \check{M}^n \times \Pi S^*$.

By choosing coordinates of V and S^*, we get coordinates of $M^{n|s}$, which we denote by $\{x^\mu\}$ and $\{\theta^a\}$. Instead of using the coordinate vector fields we use the framing of left invariant vector fields $\left\{\partial_\mu, D_a = \partial_a - \theta^b\Gamma^\mu_{ab}\partial_\mu\right\}$ to write Lagrangians. We also have the corresponding right invariant framing $\left\{\partial_\mu, \tau_a = \partial_a + \theta^b\Gamma^\mu_{ab}\partial_\mu\right\}$, which we use to write symmetries. Easy calculations show that the only non-zero brackets are

$$[D_a, D_b] = -2\Gamma^\mu_{ab}\partial_\mu, \quad [\tau_a, \tau_b] = 2\Gamma^\mu_{ab}\partial_\mu.$$

The super Poincaré group is defined by $P^{n|s} = Spin(V) \ltimes \exp\mathcal{L}$, while the super Poincaré algebra is the graded Lie algebra $\mathfrak{p}^{n|s} = (V \oplus \mathfrak{so}(V)) \oplus S^*$. Roughly, V generates translations, $\mathfrak{so}(V)$ generates Lorentz rotations and S^* the supersymmetries. The choice $\{\partial_a, D_a\}$ gives a representation of the super Poincaré algebra on superfields.

We now make a general remark on the choice of component fields. Roughly, a superfield $\Phi : M^{1|1} \to \mathbb{R}$ has a terminating Taylor series expansion

$$\Phi(x) = f_0(x) + f_1(x)\theta.$$

[1]We will see below the appearance of Bogomolny's work again, this time in the context of supersymmetry σ-models, were we study BPS states. Most of what is said there has analogues in this gauge theory context, and in fact was first discovered in the gauge theory context.

Then, to specify Φ it is sufficient to use as component fields f_0 and f_1; note that f_1 is odd. If we denote by $i : \check{M}^1 \hookrightarrow M^{1|1}$ the inclusion, whose pullback i^* sets $\theta = 0$, then

$$f_0 = i^*\Phi, \quad f_1 = i^*\partial_\theta\Phi.$$

We will do nearly the same thing below, but instead of expanding Φ in the coordinate basis, we will use the left invariant global framing discussed above. In general, it can be shown that the set (or category in the cause of gauge theory) of superfields is in bijection (or an equivalence of categories) with the supermanifold of component fields (multiplets). One manner of checking to see that the component fields are sufficient to describe all superfields is by looking at representations of the super Poincaré group $P^{n|s}$. Given an irreducible representation of $P^{n|s}$ we can restrict to the even part and get a representation of the Poincaré group P^n. However, this representation is reducible; its irreducible components correspond to the component fields that should be defined in the theory.

4 The Supersymmetric Particle

We present here our first example of a supersymmetric theory. Consider super Minkowski space $M^{1|1}$ with linear coordinates (t, θ). In one spacetime dimension $\mathrm{Spin}(V) \simeq \mathbb{Z}_2$ and S is one dimensional. The only component of the pairing Γ is $\Gamma^0_{11} = 1$. The vector field $D = \partial_\theta - \theta\partial_t$ is right invariant while the vector field $\tau = \partial_\theta + \theta\partial_t$ is left invariant. The only non-trivial brackets are

$$[D, D] = -2\partial_t, \quad [\tau, \tau] = 2\partial_t.$$

The first bracket says $-\partial_t = D^2$. In the quantum theory $-\partial_t$ becomes the Hamiltonian operator $\hat{H}$, while D becomes a self-adjoint operator. Hence the Hamiltonian is a square of a self-adjoint operator, and hence is positive. So, we see our assumption $\Gamma^0_{aa} > 0$ leads to positive energy in supersymmetric theories.

Let $\Phi : M^{1|1} \to \mathbb{R}$ be a superfield and define the component fields

$$\phi = i^*\Phi, \quad \psi = i^*D\Phi$$

where we have use the coordinates (t, θ) to define the inclusion $i : \check{M} \hookrightarrow M^{1|1}$. We note that ϕ is even while $\psi \in C^\infty(\check{M}, \phi^*\Pi T\mathbb{R})$ is odd.

We now make some general comments that apply to all spacetime dimensions. Let η^a be odd parameters. Using basis $\{f_a\}$ for S^* we get an induced action on component fields from the even vector field $\eta^a f_a$. This vector field generates a diffeomorphism $e^{\eta^a\tau_a}$ of $M^{n|s}$. Denote by $\hat{\xi}f$ the induced change on the component field f due to diffeomorphism acting on Φ by inverse pullback:

$$\hat{\xi}f := \frac{d}{dt}|_{t=0} i^*\mathcal{D}\left(e^{-t\eta\tau}\right)^*\Phi$$

where f is the component field $i^*\mathcal{D}\Phi$, with $\mathcal{D}$ being some string of differential operators. We compute

$$\begin{aligned} \hat{\xi}f &= \tfrac{d}{dt}|_{t=0} i^*\left(e^{-t\eta\tau}\right)^*\mathcal{D}\Phi, \qquad \text{since } [D,\tau]=0 \\ &= -\eta i^*\tau\mathcal{D}\Phi \\ &= -\eta i^*D\mathcal{D}\Phi \end{aligned}$$

since D and τ agree when restricted to $\check{M}$.

In particular, we find

$$\hat{\xi}\phi = -\eta\psi, \quad \hat{\xi}\psi = \eta\partial_t\phi$$

where we have used $D^2 = -\partial_t$. In a different notation,

$$\iota(\hat{\xi})\delta\phi = -\eta\psi, \quad \iota(\hat{\xi})\delta\psi = \eta\partial_t\phi.$$

This is a supersymmetry transformation-it exchanges bosons and fermions.

The Lagrangian density in terms of the superfield is[2]

$$L = -|dt|d\theta\frac{1}{2}D\Phi\partial_t\Phi|dt|d\theta = |dt|d\theta\ell.$$

The component Lagrangian[3] is $\check{L} = i^*D\ell$. A simple calculation gives

$$\check{L} = \left(\frac{1}{2}(\partial_t\phi)^2 + \frac{1}{2}\psi\partial_t\psi\right)|dt|.$$

The equations of motion are easily computed to be

$$\ddot{\phi} = 0, \quad \dot{\psi} = 0.$$

To check that $\check{L}$ is supersymmetric, we must show that $\hat{\xi}\check{L}$ is exact, giving rise to the supercurrent, and that the supersymmetry algebra is generated on-shell by the transformations above. We find

$$\hat{\xi}\check{L} = \partial_t(-\frac{1}{2}\eta\psi\dot{\phi})|dt|.$$

Continuing, we verify the supersymmetry algebra holds for ϕ:

$$[\hat{\xi}_1, \hat{\xi}_2]\phi = 2\eta_1^a\eta^2\partial_t\phi,$$

agreeing with the general result that must hold, $[\hat{\xi}_1, \hat{\xi}_2]f = 2\eta_1^a\eta^2\Gamma^\mu_{ab}\partial_\mu f$. We omit the check for the fermion term. Observe that we did not use the equations of motion to verify the supersymmetry algebra; in higher spacetime dimensions superfields will only be a representation of $\mathfrak{p}^{n|s}$ on-shell (unless auxiliary fields are included).

The bosonic component of the Lagrangian is $\check{L}_{bos} = \frac{1}{2}(\partial_t\phi)^2|dt|$, showing that the theory considered here is a supersymmetric extension of a free classical particle (in fact it is a supersymmetric σ-model!).

The variational form is computed to be $\gamma = \phi\delta\phi + \frac{1}{2}\psi\delta\psi$. With this we compute the current associated to $\hat{\xi}$:

$$\begin{aligned} j_{\hat{\xi}} &= \iota(\hat{\xi})\gamma - \alpha_\xi \\ &= \iota(\hat{\xi})\gamma + \tfrac{1}{2}\eta\psi\dot{\phi} \\ &= -\eta\dot{\phi}\psi. \end{aligned}$$

The associated supercharge is $Q_\xi = \dot{\phi}\psi$ and is the Noether current associated to $\hat{\xi}$. In the quantum theory there is a geometric interpretation of the supercharge. Under quantization ψ and $\dot{\phi}$ are promoted to Clifford multiplication on covariant derivative, respectively, so that Q_ξ becomes a Dirac operator.

We also note that time translations are a symmetry, and give rise to a current $H = \frac{1}{2}|\dot{\phi}|^2$ which is the energy. Note that the fermion terms do not contribute to the energy.

We remark that one can study the superparticle in curved space, $\Phi : M^{1|1} \to X$. In this case

$$\check{L} = \left(\frac{1}{2}|\dot{\phi}|^2 + \frac{1}{2}\langle\psi, \nabla_{\dot{\phi}}\psi\rangle\right)|dt|$$

and the equations of motion are

$$\nabla_{\dot{\phi}}\dot{\phi} = R(\psi,\psi)\dot{\phi}, \quad \nabla_{\dot{\phi}}\psi = 0$$

with R the curvature of ϕ^*TX.

[2]We will write the differentials in front of the functions in Lagrangian densities, since odd integration is actually like differentiation. Perhaps the notation

$$L = -d\theta\frac{1}{2}D\Phi\partial_t\Phi|dt|$$

would be better, but it's cumbersome.

[3]We discuss later how we choose the component Lagrangian

5 The Supersymmetric σ-Model in Three Dimensions

We now move on to σ-models in three spacetime dimensions. Our superspacetime is $M^{3|2}$ with underlying vector space V. In dimension three we have the exceptional isomorphism $Spin(1,2) \simeq SL(2,\mathbb{R})$. The minimal representation S of $Spin(V)$ is real, has dimension 2 and is self-dual: $S^* \simeq S$. Furthermore, we have an isomorphism $V \simeq \mathrm{Sym}^2 S^*$. We also fix an non-degenerate, skew-symmetric isomorphism

$$\epsilon : \Lambda^2 S^* \to \mathbb{R}$$

which we regard as a volume form on S. We take coordinates so that $\epsilon^{12} = 1$. The super Poincaré group $P^{3|2}$ acts on $M^{3|2}$. This model is the minimal supersymmetric model in three dimensions, and thus is sometimes called the $\mathcal{N} = 1$ supersymmetric σ-model in physics literature.[4] We put coordinates y^{11}, y^{12}, y^{22} on V and θ^1, θ^2 on ΠS^*. In terms of the usual coordinates x^i on V we have

$$y^{11} = \frac{x^0 + x^1}{2}, \quad y^{22} = \frac{x^0 - x^1}{2}, \quad y^{12} = y^{21} = \frac{x^2}{2}.$$

That these coordinates are defined as they are can been seen as follows. Consider the matrices that generate $\mathfrak{sl}(2,\mathbb{R})$:

$$\Sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Sigma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \Sigma^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The map $x^\mu \mapsto x^\mu \Sigma_\mu \in M_{2\times 2}(\mathbb{R})$ converts elements of V to 2×2 matrices. Furthermore, $\|x\| = \det(x^\mu \Sigma_\mu)$. We then see how the isomorphism $Spin(1,2) \simeq SL(2,\mathbb{R})$ may be constructed. In particular, we see

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \end{pmatrix} \mapsto \begin{pmatrix} x^0 + x^1 & -x^2 \\ x^2 & x^0 - x^1 \end{pmatrix}$$

which shows where the y^{ab} came from. The pairing $\Gamma : S \otimes S \to V$ can then be viewed as a vector of matrices $(\Sigma^0, \Sigma^1, \Sigma^3)$, i.e.

$$\Gamma(f,g) = \left(\Sigma^0_{ab} f^a g^b, \cdots\right) \in V, \quad f, g \in S.$$

While the choice of coordinates on V may seem odd, they simplify many terms involving the pairing Γ that follow below. The vector fields $\partial_{ab} = \frac{\partial}{\partial y^{ab}}$ are bi-invariant, while the vector fields

$$D_a = \frac{\partial}{\partial \theta^a} - \theta^b \partial_{ab}$$

are left invariant. Recall that in general, the left invariant vector field above is

$$D_a = \frac{\partial}{\partial \theta^a} - \theta^b \Gamma^\mu_{ab} \partial_\mu$$

where ∂_μ is a coordinate vector field. That is, $\Gamma^\mu_{ab}\partial_\mu = \partial_{ab}$, shows why we choose the coordinates y^{ij}. We also will need the following right invariant vector fields:

$$\tau_a = \frac{\partial}{\partial \theta^a} + \theta^b \partial_{ab}.$$

Similar to the case of the supersymmetric particle, we find

$$[D_a, D_b] = -2\partial_{ab}, \quad [\tau_a, \tau_b] = 2\partial_{ab}$$

[4]In physics the nomenclature of supersymmetric fields theory usually involves the $\mathcal{N}$ notation, which is dimension dependent. The $\mathcal{N}$ refers to the amount of supersymmetry, with $\mathcal{N} = 1$ referring to the minimal amount, $\mathcal{N} = 2$ twice that, and so on. In our notation, the superspacetime $M^{n|s}$ has n spacetime dimensions and s supersymmetries. So, for example $M^{4|4}$ and $\mathcal{N} = 1$ correspond, as do $M^{4|8}$ and $\mathcal{N} = 2$.

with all other brackets vanishing. We also again have a natural inclusion $i^* : \check{M}^3 \hookrightarrow M^{3|2}$.

Consider now a superfield $\Phi : M \to X$, where X is a Riemannian manifold. Define the component fields as

$$\phi = i^*\Phi, \quad \psi_a = i^* D_a \Phi, \quad F = -\frac{1}{2}\epsilon^{ab} i^* D_a D_b \Phi.$$

We make some comments here about the field content. The components ϕ and F are even, and $F \in C^\infty(\check{M}, \phi^* TX)$. The components ψ_a, $a = 1, 2$, are odd. We can combine ψ_1 and ψ_2 into a spinor field, $\psi = \psi_a f^a$, for $\{f^a\}$ a basis for S. Then $\psi \in C^\infty(\check{M}, \phi^*\Pi(TX \otimes S))$. We will see below that F entres only algebraically into the component Lagrangian, and hence we may replace it by its equations of motion. However, if we keep F, then the supersymmetry algebra closes off-shell, which is not true if we eliminate F. In fact, why may ask why F even appeared here. Indeed, from the representation theory of $P^{3|2}$ it is known that when a free theory is quantized the one-particle Hilbert space contains only the (massive or non-massive) multiplet $\{\phi, \psi\}$, with F nowhere to be found. We remark also that the amount of bosonic and fermionic degrees of freedom are the same off-shell.

We now study the supersymmetry transformations. Denote by $\hat{\xi}$ the even vector field induced by $\eta^a \tau_a$. We easily compute $\hat{\xi}\phi = -\eta^a \psi_a$. Also,

$$\begin{aligned} \hat{\xi}\psi_a &= \eta^b i^*(D_b D_a \Phi) \\ &= \eta^b i^*(\partial_{ab} - \epsilon_{ab} D^2)\Phi \\ &= \eta^b(\partial_{ab}\phi - \epsilon_{ab} F). \end{aligned}$$

We have used the definition $D^2 = \frac{1}{2}\epsilon^{ab} D_a D_b$ and identity $D_a D_b = -\left(\partial_{ab} - \epsilon_{ab} D^2\right)$. Continuing, in the case that X is flat, using the identity $D_a D^2 = -\epsilon^{bc}\partial_{ab} D_c$, we have

$$\begin{aligned} \hat{\xi}F &= -\eta^a i^*(D_a D^2 \Phi) \\ &= -\eta^a i^*(-\epsilon^{bc}\partial_{ab} D_c \Phi) \\ &= \eta^a \epsilon^{bc}\partial_{ab}\psi_c \\ &=: \eta^a (\not{D}\psi)_a. \end{aligned}$$

More generally, when X is curved

$$\hat{\xi}F = \eta^a \left((\not{D}\psi)_a + \frac{1}{3}\epsilon^{bc} R(\psi_a, \psi_b)\psi_c \right).$$

We remark that both the Dirac operator $\not{D}$ and the curvature R depend on ϕ (since the fields take values in $\phi^* TX$ and its variations).

The Lagrangian density in the superfield formalism reads

$$L = |d^3 x| d^2\theta \frac{1}{4}\epsilon^{ab}\langle D_a \Phi, D_b \Phi\rangle.$$

Here $\langle \cdot, \cdot \rangle$ denotes the metric on TX pulled back to $\Phi^* TX$.The density $|d^3x|d^2\theta$ is super Poincaré invariant. Also, since D are Poincaré left invariant and the Poincaré group acts from the left, the Lagrangian function ℓ is invariant. Hence, we conclude L is (manifestly) supersymmetric.

We now seek the component Lagrangian. From the product structure $M^{n|s} = \check{M}^n \times \Pi S^*$ we have a natural projection $\pi : M^{n|s} \to \check{M}^n$, and hence an integration $\pi_* : \mathrm{Dens}(M^{n|s}) \to \mathrm{Dens}(\check{M}^n)$ from which we could obtain a component Lagrangian. We, however, will take a different approach. We will instead look for a string of derivatives, denoted by $\overline{D}$, such that there exists a super-Poincaré invariant differential operator Δ on $\check{M}$ so that

$$\pi_* L = \left(i^* \overline{D}\ell + \Delta i^* \ell\right) |d^n x|.$$

We then take the component Lagrangian on $\check{M}$ as $\check{L} = i^* \overline{D}\ell |d^n x|$, which differs from $\pi_* L$ only by an exact term. We will choose the operator $\overline{D}$ so that $\check{L}$ contains only first order derivatives in the component fields. While L was manifestly supersymmetric, in general $\check{L}$ is not; the associated Noether current for the supersymmetry transformation gives rise to the supercurrent.

Returning to the case at hand, we take $\overline{D} = -\frac{1}{2}\epsilon^{ab} D_a D_b$, which in this case agrees with $\int d^2\theta$, i.e. $\Delta = 0$. This gives the component Lagrangian

$$\check{L} = \left(\frac{1}{2}|d\phi|^2 + \frac{1}{2}\langle\psi \not{D}\psi\rangle + \frac{1}{12}\epsilon^{ab}\epsilon^{cd}\langle\psi_a, R(\psi_b,\psi_c)\psi_d\rangle + \frac{1}{2}|F|^2\right)|d^3x|.$$

Note that the Dirac operator uses the covariant derivative,

$$\left(\not{D}\psi\right)_a = -\epsilon^{bc}\nabla_{ab}\psi_c.$$

As promised, F entres the Lagrangian only algebraically. We can thus eliminate F using its equation of motion, $F = 0$. The resulting Lagrangian is still invariant (up to an exact term) with respect to the transformations

$$\hat{\xi}\phi = -\eta^a\psi_a, \quad \hat{\xi}\psi_a = \eta^b\partial_{ab}\phi.$$

However, this transformation is only a supersymmetry transformation on shell, i.e. only after using the equations of motion for ϕ and ψ are these transformations a representation of the super Poincaré algebra. This contrasts the case of the superparticle, where the representation of $P^{1|1}$ on superfields held globally, i.e. off-shell. The case for ϕ ie easy; we don't use the equations of motion. For ψ_a, we compute

$$[\hat{\xi}_1,\hat{\xi}_2]\psi_a = \left(\eta_1^c\eta_2^b + \eta_1^b\eta_2^c\right)\partial_{ab}\psi_c.$$

We want to show

$$[\hat{\xi}_1,\hat{\xi}_2]\psi_a = 2\eta_1^b\eta_2^c\partial_{bc}\psi_a.$$

An exercise shows that using the equations of motion

$$\partial_{22}\psi_1 = \partial_{12}\psi_2, \quad \partial_{11}\psi_2 = \partial_{12}\psi_1$$

this is the case.

We can check that $\check{L}$ is supersymmetric. In the case of flat X we find

$$\hat{\xi}\check{L} = -\partial_\nu\frac{1}{2}\eta^c\left(\tilde{\Gamma}^{\mu ba}\Gamma^\nu_{cb}\eta^c\partial_\mu\phi\right).$$

The variational one-form is

$$\gamma = -\iota(D_a)|d^3x|d^2\theta\frac{1}{2}\epsilon^{ab}\langle D_b\Phi, \delta\Phi\rangle.$$

We can then compute the supercurrent, with $F = 0$:

$$j_a = \iota(\partial_{cd})|d^3x|\left(\epsilon^{cb}\epsilon^{de}\langle\partial_{ae}\phi, \psi_b\rangle\right).$$

For example, the $a = 1$ supercharge is then

$$Q_1 = \int_{x^0=0}|dx^1dx^2|\left\{\langle\partial_{11}\phi,\psi_1\rangle + \langle\partial_{12}\phi,\psi_2\rangle\right\}.$$

Note the similarity to the supercharge obtained for the superparticle.

In the case of flat X, the equations of motion are found to be

$$\Delta\phi = 0, \quad \not{D}\psi = 0.$$

In coordinates, these are

$$g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = 0, \quad \tilde{\Gamma}^\mu_{ba}\partial_\mu\psi_a = 0.$$

The bosonic component of the Lagrangian is $\check{L}_{bos} = \frac{1}{2}|d\phi|^2|d^3x|$, and shows that this model indeed is a supersymmetric extension of the bosonic σ-model.

5.1 Adding a Superpotential

Like in the bosonic case, we can also consider what happens when we add a potential term to the Lagrangian. In the bosonic case, we chose any potential function $V : X \to \mathbb{R}$. In the supersymmetric case we do not have such freedome, as we will see below. We begin by considering a superpotential $h : X \to \mathbb{R}$ and consider the new Lagrangian density

$$L' = |d^3x|d^2\theta \left(\frac{1}{4}\epsilon^{ab}\langle D_a\Phi, D_b\Phi\rangle + \Phi^* h \right).$$

The contribution to the component Lagrangian is

$$\ell' = -i^* D^2 \Phi^* h.$$

In components this is

$$\begin{aligned} \ell' &= -\tfrac{1}{2} i^* \epsilon^{ab} D_a D_b \Phi^* h \\ &= -\tfrac{1}{2} i^* \epsilon^{ab} D_a \iota(D_b\Phi) dh \\ &= -\tfrac{1}{2} i^* \epsilon^{ab} \left(\iota(\nabla_a D_b \Phi) dh - \iota(D_b\Phi) D_a(\Phi^* dh) \right) \\ &= \iota(F)\phi^* dh - \tfrac{1}{2}\epsilon^{ab}\phi^*(\text{Hess}h)(\psi_a, \psi_b) \\ &= \langle F, \phi^*\text{grad}h\rangle - \tfrac{1}{2}\epsilon^{ab}\phi^*(\text{Hess}h)(\psi_a, \psi_b). \end{aligned}$$

We have defined the covariant Hessian of h, $\text{Hess}h = \nabla dh$, which is a symmetric tensor on X. In local coordinates $\{x^\mu\}$, this is

$$\text{Hess}h = \left(\frac{\partial^2 h}{\partial x^\mu \partial x^\nu} - \Gamma^\epsilon_{\mu\nu} \frac{\partial h}{\partial x^\epsilon} \right) dx^\mu \otimes dx^\nu.$$

We can eliminate F using its equation of motion $F + \phi^*\text{grad}h = 0$. The Lagrangian density is then

$$\check{L} = \left(\frac{1}{2}|d\phi|^2 + \frac{1}{2}\langle\psi \not{D}\psi\rangle + \frac{1}{12}\epsilon^{ab}\epsilon^{cd}\langle\psi_a, R(\psi_b,\psi_c)\psi_d\rangle - \frac{1}{2}|\phi^*\text{grad}h|^2 + \frac{1}{2}\epsilon^{ab}\phi^*(\text{Hess}h)(\psi_a,\psi_b) \right) |d^3x|.$$

We see that through adding a superpotential h we have gained a potential term for the scalar field,

$$V = \frac{1}{2}|\phi^*\text{grad}h|^2,$$

so that restricting to the bosonic case ($\psi_a = 0$) we have a non-linear σ-model with potential. As claimed above, we no longer have a generic potential energy function. In particular, in order for our Lagrangian to be supersymmetric, the potential energy for the scalar field must be non-negative. We also see that there is a fermion mass term, namely $\frac{1}{2}\epsilon^{ab}\phi^*(\text{Hess}h)(\psi_a,\psi_b)$. As an example, say

$$h(x) = \frac{1}{2}m^2x^2 + \frac{1}{6}\lambda x^3.$$

This gives a mass term, Yukawa coupling and cubic interactions.

In the superfield formalism, the equations of motion are

$$-\frac{1}{2}\epsilon^{ab}\mathcal{D}_a D_b\Phi = \Phi^*\text{grad}h.$$

In components this gives $F + \phi^*\text{grad}h = 0$, along with

$$\left(\not{D}\psi\right)_a := -\epsilon^{bc}\phi^*\nabla_{ab}\psi_c = -\frac{1}{3}\epsilon^{bc}\phi^* R(\psi_a,\psi_b)\psi_c + \phi^*(\nabla\text{grad}h)\psi_a$$

and

$$\Box\phi := \nabla_{11}\partial_{22} - \nabla_{12}\partial_{12} = -\frac{1}{2}\phi^*\text{grad}|dh|^2 + (\text{Terms that vanish at } \psi = 0 \text{ involving derivatives of } h \text{ and } R).$$

In particular, setting $\psi_a = 0$ we get Newton's law. All this for Newton's law ?!?! The supercurrent in this case reads

$$j_a = \iota(\partial_{cd})|d^3x| \left(\epsilon^{cb}\epsilon^{de}\langle\partial_{ae}\phi, \psi_b\rangle - \epsilon^{bd}\delta^c_a\langle F, \psi_b\rangle \right).$$

The supercharge is found directly from j_a. For example, after using the equation of motion for F,

$$Q_1 = \int_{x^0=0} |dx^1 dx^2| \left\{ \langle\partial_{11}\phi, \psi_1\rangle + \langle\partial_{12}\phi, \psi\rangle - \langle\phi^*\text{grad}h, \psi_2\rangle \right\}.$$

5.2 Vacuum Solutions and Supersymmetry Breaking

The vacuum solutions are $\psi = 0$ and $\phi = \phi_{crit}$ a constant so that the potential energy $V = \frac{1}{2}|\phi^*\text{grad}h|^2$ is minimized. The latter condition is that ϕ_{crit} be a critical point of h. Such states are supersymmetric, i.e. the vector field generating the supersymmetry $\hat{\xi}$ vanishes:

$$\hat{\xi}\phi = -\eta^a\psi_a = 0, \quad \hat{\xi}\psi_a = \eta^b\left(\partial_{ab}\phi_{crit} + \epsilon_{ab}\phi^*_{crit}\text{grad}h\right) = 0.$$

In this case we say the supersymmetry is unbroken. Say, for example, that h is a generalized Morse function. That is, h is smooth, and the critical set of h is a closed submanifold of X. Furthermore, the Hessian of h is non-degenerate in the directions normal to the critical set. Note that if the critical points of h are isolated, then h is a usual Morse function. For ϕ_{crit} a critical point of h, there may or may not be massless scalar and fermion fields associated, depending on whether or not the Hessian of h has a kernel (which is necessarily along the tangent to the critical manifold). In particular, if h is Morse, there are no massless modes. In any case, the masses of the scalar particles and fermions are the same. Indeed, the masses of scalar particles are the eigenvalues of the Hessian of V, which in the case that ϕ_{crit} is critical, is just the square of the Hessian of h:

$$\phi^*_{crit}\text{Hess}(V) = \phi^*_{crit}(\text{Hess}h)^2.$$

Similarly, the mass matrix for fermions is simply the term in the Lagrangian quadratic in ϕ_a, which is $\phi^*_{crit}\text{Hess}h$. Summarizing, we see that if supersymmetry is unbroken, then fermions and scalar particles have the same masses and come in multiplets. This is somewhat of a problem, since in nature we don't observe bosons with partner fermions having the same mass. Hence, we should expect that supersymmetry is broken.

Say now that ϕ_{loc} is a local minima of V, but not a global minimum, $V(\phi_{loc}) \neq 0$. Hence, we see ϕ_{loc} is not a critical point of h. Note in this case the supersymmetry is broken:

$$\hat{\xi}\psi_a = -\eta^b\epsilon_{ab}\phi^*_{loc}\text{grad}h \neq 0.$$

So this state is not supersymmetric. Since $dV = \langle \text{grad}h, \nabla\text{grad}h\rangle$ and $\phi^*_{loc}dV = 0$, we see that in this case Hess$h$ must have a kernel. Let us assume that ϕ_{loc} is an isolated local minima, in which case there may be no massless scalar fields. However, we necessarily have a massless fermion (since Hess$h$ has a kernel). Hence, when supersymmetry is broken we no longer have equality of masses between scalars and fermions. We call the massless fermions encountered here Goldstone fermions. This term is in analogy with Goldston bosons, which are massless scalar fields that arise when an even symmetry is broken.

Example (O'Reaighferteigh Models)**.**

5.3 Dimensional Reduction to 2 Dimensions

We briefly discuss the technique of dimensional reduction, where by a model in a given spacetime dimension is reduced to give a model in lower dimensions. Our attention is restricted to σ-models; we will see that it provides a simple manner to get theories with non-minimal supersymmetry.

To begin, say we are given a supersymmetric σ-model on $M^{n|s}$ with minimal supersymmetry. The component fields of the theory are then maps from n-dimensional Minkowsi space M^n to some other space, which we generically denote by X. We obtain a supersymmetric theory on $n-1$-dimensional Minkowski space M^{n-1} by considering only maps

$$M^n/\mathbb{R}v \to X$$

where v is some non-zero spacelike direction in M^n. Physically, we are restricting to fields that are independent of the v direction. The super Poincaré group acting on $M^n/\mathbb{R}v$ is

$$P_v = N_{P^{n|s}}(\mathbb{R}v)/\mathbb{R}v$$

where $P^{n|s}$ is the super Poincaré group acting on $M^{n|s}$ and N_P denotes the normalizer in P. Note that while we have reduced the symmetries on the spacetime, we have not reduced the amount of supersymmetry, since all supersymmetry generators normalize ∂_v, the spatial translation generator in direction v. Hence, the theory on M^{n-1} still have s supercharges, which in general will not be the minimal amount in $n-1$-dimensions.

Using this method, we can obtain the following dimensional reductions (amongst others):

$$\begin{array}{ccccc} P^{1|2} & \hookrightarrow & P^{2|(1,1)} & \hookrightarrow & P^{3|2} \\ \mathcal{N}=2 & & \mathcal{N}=1 & & \mathcal{N}=1 \end{array}$$

and

$$\begin{array}{ccccccc} P^{1|4} & \hookrightarrow & P^{2|(2,2)} & \hookrightarrow & P^{3|4} & \hookrightarrow & P^{4|4} \\ \mathcal{N}=4 & & \mathcal{N}=2 & & \mathcal{N}=2 & & \mathcal{N}=1 \end{array}$$

Let's now look at the specific case $P^{2|(1,1)} \hookrightarrow P^{3|2}$ which will give the $\mathcal{N}=1$ σ-model in two dimensions. Recall that for spacetime dimension 2 we have $Spin(1,1) \simeq \mathbb{R}^{\geq 0} \times \mathbb{Z}_2$ and the one dimensional, minimal real spin representation S decomposes into one dimensional complex half-spin representations $S^{\pm}$. We also have $V \simeq (S^{+*})^{\otimes 2} \oplus (S^{-*})^{\otimes 2}$, i.e. the vector representation is reducible. In physics, sections of $S^{\pm}$ are called Majorana-Weyl spinors; Marjorana because they are real, and Weyl because they are elements of the half-spin representation. Proceeding with the reduction procedure outlined above, consider only fields independent of the spatial direction $y^{12} = \frac{x^2}{2}$:

$$\partial_{12} f = 0.$$

Motivated by the half-spin decomposition of S, we label spinors by $+$ and $-$, corresponding to 1 and 2. Also, chose lightcone coordinates on $\check{M}^2$,

$$\partial_+ = \frac{\partial_0 + \partial_1}{2}, \quad \partial_- = \frac{\partial_0 - \partial_1}{2}.$$

The on-shell ($F = \phi^* \mathrm{grad} h$) supersymmety transformations then reduce to

$$\hat{\xi}\phi = -(\eta^+ \psi_+ + \eta^- \psi_-), \quad \hat{\xi}\psi_\pm = \eta^\pm \partial_\pm \phi \pm \eta^\mp \phi^* \mathrm{grad} h.$$

The supercharge listed above then becomes

$$Q_+ = \int_{x^0=0} |dx^1 dx^2| \left\{ \langle \partial_+ \phi, \psi_+ \rangle - \langle \phi^* \mathrm{grad} h, \psi_- \rangle \right\}.$$

We will also need the second supercharge:

$$Q_- = \int_{x^0=0} |dx^1 dx^2| \left\{ \langle \partial_- \phi, \psi_- \rangle + \langle \phi^* \mathrm{grad} h, \psi_+ \rangle \right\}.$$

Note the similarity to the supercharges in the case of the superparticle, where we found $Q_{1 \text{ dim}} = \langle \partial_t \phi, \psi \rangle$, which we should really view as the integral over a zero dimensional manifold. **Atiyah Singer Index?**

Alternatively, we note that we could have reduced the model in superspace to obtain a superspace formulation of the σ-model on $M^{2|(2,2)}$.

5.4 BPS Instantons

Continuing with the σ-model on $M^{2|(1,1)}$ with potential, we consider the underlying bosonic theory. We look for static solutions, $\phi = \phi(x^1)$, of minimal energy. The energy density for these fields is

$$\mathcal{E} = \left(\frac{1}{2}|\partial_1\phi|^2 + \frac{1}{2}\phi^*|\text{grad}h|^2\right)|dx^1|$$

which can be rewritten as (depending on the sign of grad$h$)

$$\mathcal{E} = \frac{1}{2}|\partial_1\phi \pm \phi^*\text{grad}h|^2|dx^1| \mp d\left(\phi^* h\right).$$

We will restrict our attention to configurations of finite energy,

$$E = \int_{\mathbb{R}} \mathcal{E} < \infty.$$

This implies $\lim_{x^1\to\pm\infty}\partial_1\phi = 0$, which allows us to define the limiting values

$$\phi_\pm = \lim_{x^1\to\pm\infty}\phi(x^1) \in X.$$

We also see that $\phi_\pm$ must be critical points of h, since

$$0 = \lim_{x^1\to\pm\infty}\phi^*\text{grad}h = \phi_\pm^*\text{grad}h.$$

We can then label the finite energy configurations by components of the critical point set of h. That is, the finite energy configurations are given by an element in $\pi_0(\text{crit}(h)) \times \pi_0(\text{crit}(h))$, corresponding to the values $\phi_\pm$. Note that h is constant on each component of crit(h). This product structure of finite energy configurations is special to two dimensions, since in this case spatial infinity is disconnected.

Example. *Say X is a compact Riemannian manifold and h is a Morse function. Then $\pi_0(crit(h))$ is finite.*

Example. *Consider the function $h : \mathbb{R} \to \mathbb{R}$ given by*

$$h(x) = \frac{1}{2}x^3 - a^2x, \quad a > 0.$$

Note that h is Morse. The potential induced by h is

$$V(\phi) = \frac{1}{2}\left(\phi^2 - a^2\right)^2.$$

The minima are at $\phi = \pm a^2$, whence the finite energy configurations are labeled by an element of $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Consider the diagonal component $\Delta \subset \pi_0(\text{crit}(h)) \times \pi_0(\text{crit}(h))$. In this case, the energy of the field is

$$E = \frac{1}{2}\int_{\mathbb{R}} |\partial_1\phi \pm \phi^*\text{grad}h|^2 dx^1.$$

It is only in the diagonal set Δ that zero-energy configurations exists. Such a configuration exists precisely when ϕ satisfies the gradient flow equation

$$\partial_1\phi = \mp\phi^*\text{grad}h.$$

In particular, if ϕ is a constant with values in the critical point set of h this equation is satisfied. Such configurations ϕ make u the classical vacua.

Continuing, outside the diagonal Δ, i.e. where $\phi_+ \neq \phi_-$, the field configurations necessarily have positive energy:

$$E = \int_{\mathbb{R}} \mathcal{E} \geq |\int_{\mathbb{R}} d(\phi^* h)| = |Z|$$

where we have defined $Z = h(\phi_+) - h(\phi_-)$. Z is called the central charge of the theory. In this sector, minimal energy configurations must satisfy the gradient flow equations (since ϕ can no longer be constant as in the case above). The inequality $E \geq |Z|$ is called the BPS inequality. Configurations that saturate the inequality are called BPS solitons. In the quantum theory, because of the BPS inequality, we expect the quantized solitons to be massive particles.

The gradient flow equations, which we derived using supersymmetry, are first order. However, differentiating we obtain the second order equations of motion for ϕ. So, we see that supersymmetry, even in the classical case, has affects on the bosonic theory.

Having dealt so far only with bosons, we would like to see what fermions have to do with the picture. We begin with a computation of Poisson brackets. We have

$$\begin{aligned} \{Q_+, Q_-\} &= \int_{x^0=0} \{j_+, j_-\}\, dx^1 \\ &= \int_{x^0=0} \left(-\langle \partial_- \phi, \phi^* \mathrm{grad} h\rangle + \langle \partial_+ \phi, \phi^* \mathrm{grad} h\rangle\right) dx^1 \\ &= 2\int_{x^0=0} \langle \partial_1 \phi, \phi^* \mathrm{grad} h\rangle dx^1 \\ &= 2\int_{x^0=0} \partial_1 (h(\phi)) dx^1 \\ &= 2Z. \end{aligned}$$

Hence, we have a classical extension of the supersymmetry algebra by Z:

$$\{Q_+, Q_-\} = 2Z.$$

This explains the terminology introduced above for Z.

What happens if we extend the quantum supersymmetry algebra, i.e. centrally extend $\mathfrak{p}^{2|(1,1)}$? That is, say the supersymmetry generators now satisfy

$$[Q_+, Q_+] = -2\partial_+, \quad [Q_-, Q_-] = -2\partial_-, \quad [Q_+, Q_-] = 2Z.$$

If we view our theory in two dimensions as the dimensional reduction of a theory on $M^{3|2}$, we get some idea where Z is coming from. The supersymmetry generators $Q_\pm$, viewed as generators in $P^{3|2}$, satisfy

$$[Q_+, Q_-] = -2\partial_{12}.$$

So we see we have a correspondence $\partial_{12} \sim -Z$. The coincidence that central charges can be identified in this way with eliminated spatial symmetries is in fact quite general.

Working with our centrally extended algebra $\mathfrak{p}_c^{2|(1,1)}$, we have

$$\begin{aligned} [Q_+ \pm Q_-, Q_+ \pm Q_-] &= [Q_+, Q_+] + [Q_-, Q_-] \pm [Q_+, Q_-] \pm [Q_-, Q_+] \\ &= -2\partial_+ - 2\partial_- \pm 4Z \end{aligned}$$

so that

$$\frac{1}{4}[Q_+ \pm Q_-, Q_+ \pm Q_-] = -\partial_t \pm Z.$$

In the quantum theory, $\frac{1}{2}(Q_+ \pm Q_-)^2$ is quantized to $\hat{H} \pm \hat{Z}$, where $\hat{H}$ is the Hamiltonian and $\hat{Z}$ is some locally constant operator on the Hilbert space. Since under quantization $\frac{1}{2}(Q_+ \pm Q_-)^2$ is positive, we arrive at the quantum BPS inequality:

$$\hat{H} \geq |\hat{Z}|.$$

In the case of a stationary state, the energy (Hamiltonian) is just the mass of that state, so that we get from the BPS inequality a lower bound on the mass of the quantum state, as suggested above in our analysis of the classical BPS inequality.

Motivated by the calculation (before central extension) that showed $(Q_+ \pm Q_-)^2 = \partial_t$, since we have been looking for static solitons, we can look for partially supersymmetric solutions, i.e. configurations that satisfy one of

$(Q_+ + Q_-)\Phi$ or $(Q_+ - Q_-)\Phi$ vanish. To find the conditions on Φ, we can work in components, were these equations are equivalent to[5] $\hat{\xi}\phi = 0$ and $\hat{\xi}\psi_\pm = 0$. Explicitly,

$$\hat{\xi}\phi = -\eta^+(\psi_+ \pm \psi_-), \quad \hat{\xi}\psi_+ = \eta^+(\partial_+\phi \pm \phi^*\text{grad}h), \quad \hat{\xi}\psi_- = \eta^+(\pm\partial_-\phi - \phi^*\text{grad}h).$$

These equations imply

$$\psi_+ \pm \psi_- = 0, \quad \partial_0\phi = 0, \quad \mp\partial_1\phi + \phi^*\text{grad}h = 0.$$

We see that ϕ must be constant in time, and also that ϕ satisfies the gradient flow equation. These are precisely the equations we found above by requiring ϕ to locally minimize the energy. We now have a derivation of this fact by requiring the configuration to be partially supersymmetric. To see what happens to the fermions, we must look at their equations of motion:

$$\partial_+\psi_- = -R(\psi_+, \psi_-)\psi_+ - \phi^*(\nabla\text{grad}h)\psi_+, \quad \partial_-\psi_+ = R(\psi_-, \psi_+)\psi_- + \phi^*(\nabla\text{grad}h)\psi_-.$$

Combining these equations with the condition $\psi_+ \pm \psi_- = 0$ we find

$$\partial_0\psi_+ = 0, \quad \pm\partial_1\psi_+ + \phi^*(\nabla\text{grad}h)\psi_+ = 0.$$

To interpret these equations, note that they are simply the variations (linearizations) of the equations above for ϕ in the direction ψ_+. In other words, the fermion is an odd tangent vector to the manifold of flow lines. In particular, the equations satisfied by the fermions are implied by the equations for ϕ. This gives a manner in which we can visualize (in some sense) fermions.

Check We close with some remarks on the quantum theory. In the case of the superparticle, where the fields in question are maps $\Phi : M^{1|1} \to X$, the symplectic space we would like to quantize is $\phi^*\Pi TX \to TX$. In the case of the σ-model discussed above, denote by F the manifold of flow lines related to the manifold X. The fermions being tangent vectors to the manifold of flows lines implies the to quantize this model is equivalent to quantizing the symplectic space

$$\pi^*\Pi TN \to TN.$$

That is, the quantum theory of BPS states is equivalent to doing supersymmetric quantum mechanics, where the fields take values in the manifold of flow lines.

6 The Supersymmetric σ-Model in Four Dimensions

We work in $M^{4|4}$. In four dimensions we have the exceptional isomorphism $Spin(1,3) \simeq SL(2,\mathbb{C})$, and hence the minimal spin representation is four dimensional. After complexification[6] it decomposes as

$$S_{\mathbb{C}} \simeq S' \oplus S''$$

with $S' \simeq \overline{S''}$. This is the half-spin decomposition, which exists in all even dimensions. Spinors with values in $S_{\mathbb{C}}$ are Dirac spinors, while those with values in S' and S'' are called Weyl spinors. Both S' and S'' have complex dimension

[5]Recall that the action of $\hat{\xi}$ on component fields is induced by the supersymmetry $\eta^a\tau_a$, so that invariance under $Q_+ \pm Q_-$ is equivalent to the action of $\hat{\xi}$ vanishing on component fields for $\eta^+ = \pm\eta^-$.

[6]One may ask why we are complexifying our spin representation, when in all the previous cases we stuck with the real representations. From a physics point of view, the opposite question is in fact appropriate, since a physicist often constructs all the minimal complex representations (Dirac spinors) and only then may reduce to the real representations (Majorana spinors). Our motivation is that we have an isomorphism

$$V_{\mathbb{C}} \simeq S'^* \otimes S''^*$$

which allows the definition of convenient (complex) coordinates on V.

two and have invariant complex volume forms ϵ, which we assume are normalized. Fix linear complex coordinates θ^1 and θ^2 on $\Pi S'^*$; these induce conjugate coordinates $\overline{\theta}^{\dot{1}}$ and $\overline{\theta}^{\dot{2}}$ on $\Pi S''^*$. We fix coordinates on $V_{\mathbb{C}}$ after describing the map $\Gamma : S \otimes S \to V$. As in the three dimensional case, we construct Γ by considering the usual proof that $Spin(1,3) \simeq SL(2,\mathbb{C})$. In the four dimensional case define a map

$$x^\mu \mapsto x^\mu \sigma_\mu$$

where

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

This gives a bijection between elements of $\mathbb{C}^4$, view as complexified coordinates on spacetime, with the set of all 2×2 Hermitian matrices. We then define

$$\Gamma^\mu_{a\dot{b}} = \sigma^\mu_{a\dot{b}}$$

which has the desired properties. This then gives a natural choice for coordinates on $V_{\mathbb{C}}$:

$$y^{1\dot{1}} = \frac{x^0 + x^1}{2}, \quad y^{2\dot{2}} = \frac{x^0 - x^1}{2}, \quad y^{1\dot{2}} = \frac{x^2 + ix^3}{2}, \quad y^{1\dot{1}} = \frac{x^2 - ix^3}{2}.$$

The global framing of left invariant vector fields corresponding then take the simple form

$$\partial_{a\dot{b}}, \quad D_a = \frac{\partial}{\partial \theta^a} - \overline{\theta}^{\dot{b}} \partial_{a\dot{b}}, \quad \overline{D}_{\dot{a}} = \frac{\partial}{\partial \overline{\theta}^{\dot{a}}} - \theta^b \partial_{b\dot{a}}.$$

The only non-vanishing bracket is $[D_a, \overline{D}_{\dot{b}}] = -2\partial_{a\dot{b}}$.

6.1 The Linear σ-Model

We begin by studying the linear σ-model. Unlike the σ-model in three spacetime dimensions, here we must consider complex superfields $\Phi : M^{4|4} \to \mathbb{C}$. That real superfields are insufficient can be seen from the representation theory of $P^{4|4}$, which says that in the minimal multiplet there should be two real scalar fields and a single spinor field. However, if Φ were real valued we would only get a single scalar field: $\phi = i^*\Phi$. Assuming that Φ is complex valued, we then see that if we were to consider all components of Φ, we would get four independent spinor components

$$\psi_a = i^* D_a \Phi, \quad \psi'_{\dot{a}} = i^* \overline{D}_{\dot{a}} \Phi$$

giving a spinor in each of $\Pi S'$ and $\Pi S''$. A guess to constrain the fields is that Φ be chiral: $\overline{D}_{\dot{a}}\Phi = 0$. Its conjugate is then anti-chiral: $D_a \overline{\Phi}$. From above, we see that this constraint removes two spinor components and hence we have the correct number of spinor degrees of freedom. Chirality is a super-analogue to holomorphicity of complex functions. In particular, chiral fields form a ring.

We define the component fields as

$$\phi = i^*\Phi, \quad \psi_a = \frac{1}{\sqrt{2}} i^* D_a \Phi, \quad F = -\frac{1}{2}\epsilon^{ab} i^* D_a D_b \Phi$$

with corresponding conjugate fields. The set $\{\phi, \psi, F\}$ is called a chiral multiplet. The Lagrangian density for the linear σ-model is

$$L = |d^4x| d^4\theta \frac{1}{4} \overline{\Phi}\Phi.$$

Instead of straightforward Berezin integration we use

$$\int d^4\theta + \Box i^* = \frac{1}{2} i^* \left(D^2 \overline{D}^2 + \overline{D}^2 D^2 \right).$$

This allows us to compute the component Lagrangian function:

$$\check{L} = \left(\frac{1}{2}\langle d\overline{\phi}, d\phi\rangle + \frac{1}{2}\left(\overline{\psi}\not{D}\psi + \psi\not{D}\overline{\psi}\right) + \langle\overline{F}, F\rangle\right)|d^4x|.$$

As written the spinor term is real. We could however rewrite it a single complex term since

$$\frac{1}{2}\left(\overline{\psi}\not{D}\psi + \psi\not{D}\overline{\psi}\right) = \overline{\psi}\not{D}\psi + \frac{1}{2}\partial_\mu(\overline{\psi}\Gamma^\mu\psi),$$

and exact terms do not contribute to the equations of motion. As in the three dimensional case, the field F is auxiliary and entres only algebraically, with equation of motion $F = 0$.

Denote by $\hat{\xi}$ the vector field on chiral multiplets induced by the even vector field $\eta^a Q_a + \overline{\eta}^{\dot{a}}\overline{Q}_{\dot{a}}$ on $M^{4|4}$. Note that our odd parameters are now complex valued. Easy computations show

$$\hat{\xi}\phi = \sqrt{2}\eta^a\psi_a, \quad \hat{\xi}\psi_a = \sqrt{2}(\overline{\eta}^{\dot{a}}\partial_{a\dot{b}}\phi - \eta^b\epsilon_{ab}F), \quad \hat{\xi}F = \sqrt{2}\overline{\eta}^{\dot{a}}(\not{D}\psi)_{\dot{a}}.$$

Playing the same games as above, we can show $\check{L}$ is supersymmetric (up to exact terms), compute the associated supercharges, etc.

6.2 The Non-linear σ-model

We have seen above that to define the linear σ-model in four dimensions the target manifold X must be restricted. We see below that in the non-linear case the geometric restrictions on X are even more stringent. In fact, we will see that X must have the structure of a Kähler manifold. Below we give two suggestive arguments of this.

We begin by giving the standard physics argument that X must be Kähler. In the linear case we considered the superspace Lagrangian

$$L = |d^4x|d^4\theta\frac{1}{4}\overline{\Phi}\Phi.$$

The natural generalization, for X curved, is to define

$$L = |d^4x|d^4\theta\frac{1}{2}K(\overline{\Phi}, \Phi)$$

for some complex valued function K. Reducing to components we find that the term quadratic in the scalar fields is

$$\check{L} = -\frac{1}{2}\frac{\partial^2 K}{\partial\Phi^i\partial\overline{\Phi}^{\overline{j}}}\partial^\mu\overline{\phi}^{\overline{j}}\partial_\mu\phi^i + \cdots$$

where the indices label coordinates on X, not spinor indices. This is the kinetic term for the scalar field ϕ. Comparing with what we expect, we see the metric on X should have the form

$$g_{i\overline{j}} = \frac{\partial^2 K}{\partial\Phi^i\partial\overline{\Phi}^{\overline{j}}}.$$

This is precisely the condition for X to be Kähler, with Kähler potential K.

Another way of seeing that X should be Kähler is as follows. We take the point of view that initially X is just a Riemannian manifold. Let us assume that that our supersymmetry transformation of ϕ is still of the form

$$\hat{\xi}\phi = -\eta^a\psi_a$$

where, as stated above, we should now view η as an odd, complex valued parameter. Since $\psi_a : \check{M}^4 \to \phi^*\Pi TX$, we see that we must be able to multiply tangent vectors in X by complex numbers. That is, TX should have an almost complex structure, which we denote by J. The metric on X defines a connection ∇ on TX with a real structure. For

∇ to be compatible with the complex structure J we require $\nabla J = 0$. But, it is known that $\nabla J = 0$ is equivalent to J being integrable and X being is Kähler.

Having established that X must Kähler, we see what this implies for the theory. The Kähler form ω can locally be written in terms of a Kähler potential K,

$$\omega = i\partial\overline{\partial}K.$$

We emphasize that this is only a local description; there is an ambiguity in the choice of K, in that $\tilde{K} = K + \Lambda + \overline{\Lambda}$ for Λ a holomorphic function is again a Kähler potential for ω. This ambiguity implies that the choice of Lagrangian above,

$$L = |d^4x|d^4\theta\frac{1}{2}K(\overline{\Phi}, \Phi)$$

is only a local choice. However, we note that the component Lagrangian is globally well-defined. This is because Lagrangian density on spacetime,

$$\int_{\theta^i, \overline{\theta}^{\overline{i}}} L,$$

depends only on ω. That this is so follows from the Berezinian integration formula, which shows that $\int d^4\theta$ annihilates holomorphic and antiholomorphic functions.

As in the linear case, we must restrict our space of fields. We say that a superfield $\Phi : M^{4|4} \to X$ is said to be chiral if $D_a\Phi$ is a vector field of type $(1,0)$ on X. This is equivalent to requiring that $\Phi^*f : M^{4|4} \to \mathbb{C}$ be chiral (as a complex valued superfield) for any holomorphic function $f : X \to \mathbb{C}$. With the Lagrangian density above, we can perform the usual analysis. Reducing to components we find all the features of the dimensional case; we do not repeat them here. Instead, we will restrict our attention to further constraining the geometry of X to get ride of the locality of L. Recall that Hodge manifold X is a Kähler manifold such that its Kähler form satisfies $\omega \in H^{1,1}(X, \mathbb{Z})$. In this case, there exists a Hermitian line bundle over X, $L \to X$, whose curvature is $i\omega$. For topological reasons[7] we can lift any chiral superfield $\Phi : M^{4|4} \to X$ to a non-zero, global chiral section $\tilde{\Phi} : M^{4|4} \to \Phi^*L$. We can then show that $i\omega = \overline{\partial}\partial\log\|\tilde{\Phi}\|^2$, so that $\tilde{\Phi}$ provides a global Kähler potential for X; here the norm is taken using the pulled-back metric on Φ^*L. We may then define, globally, the action for the σ-model by

$$L = |d^4x|d^4\theta\frac{1}{2}\log\|\tilde{\Phi}\|^2.$$

It can be shown that L is independent of the choice of lift $\tilde{\Phi}$, as is $\check{L}$. Indeed, say $f \cdot \tilde{\Phi}$ is another lift, with f holomorphic. Then

$$\log\|f \cdot \tilde{\Phi}\|^2 = \log\|\tilde{\Phi}\|^2 + \log f + \log\overline{f}$$

from which the claim follows.

In summary, we have seed a marked difference between the three and four dimensional σ-models. From one point of view, this change can be seen to originate from the fact that the odd parameters η were taken to be complex in the four dimensional case; in the three dimensional case they were real, and clearly the tangent space TX already has a real structure. As a final remark, we note that in the six dimensional case we have $Spin(1,5) \simeq SL(2,\mathbb{H})$, and analogous arguments show that TX should have a left (say) $\mathbb{H}$-structure. This leads to requiring X to be hyper-Kähler.

[7] Specifically, Minkowski space is homotopic to a point, and if $\mathcal{O}^*$ denotes the sheaf of non-zero chiral superfields $M^{4|4} \to X$, then $H^1(M^{4|4}, \mathcal{O}^*) = 0$.

6.3 Adding a Superpotential

We continue to assume that X is Hodge, for simplicity. To add a superpotential we consider a global holomorphic function $W : X \to \mathbb{C}$. Since Φ is assumed chiral, so too is Φ^*W. Now, consider the superspace[8] $M^{4|2_{\mathbb{C}}} \simeq \check{M}^4 \times \Pi S'^*_{\mathbb{C}}$. Then $|d^4x|d^2\theta$ is super Poincaré invariant. Since $\int d^2\theta = -\frac{1}{2}\epsilon^{ab}i^* D_a D_b$ the density

$$|d^4x|d^2\theta\Phi^*W$$

is supersymmetric, and we can consider it as (part of) a potential term. Indeed, the Lagrangian density for the non-linear σ-model with superpotential on a Hodge manifold X is

$$L = |d^4x| \left(d^4\theta\frac{1}{2}\log\|\tilde{\Phi}\| + d^2\theta\frac{1}{2}\Phi^*W + d^2\overline{\theta}\frac{1}{2}\overline{\Phi^*W} \right).$$

With this action the auxiliary field has the equation of motion $F = -\phi^*\text{grad}W$. The component Lagrangian in this case has mainly the same terms that occur in the three dimensional case. As a final remark, we note that to define L we have used three different superspaces, something we have not seen before.

6.4 Dimensional Reduction from $M^{4|4}$

We make only rough comments here, at least to show the reader that it is worth the effort to study the dimensional reduction of the σ-model in four dimensions. When we reduce to spacetime dimensions $n < 4$ we will still have $s = 4$ supersymmetries; the non-minimal supersymmetry is what makes these reductions mathematically mathematical. For example, in the case of reduction to $M^{2|(2,2)}$, the classical BPS instantons lead to holomorphic Morse theory. In the reduction to $M^{1|4}$, the quantum Hilbert space is the space of holomorphic forms on a Kähler manifold X, $\Omega^{\bullet,\bullet}(X)$. The four supercharges Q_i act as ∂, $\overline{\partial}$ and their adjoints under quantization. The action of the R-symmetry group ($SU(2)$ in this case) on $\Omega^{\bullet,\bullet}(X)$ leads to a supersymmetric interpretation of the Kähler identities.

References

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[3] P. Deligne and D. Freed. *Supersolutions*, Quantum Fields and Strings: A Course for Mathematicians, *loc. cit.*, pp. 227-355. American Mathematical Society, 1999.

[4] P. Deligne, P. Etingof, et al, (eds.). *Quantum Fields and Strings: A Course for Mathematicians*, 2 volumes, American Mathematical Society, Providence, R.I., 1999.

[8]We finesse some issues here. What we are using is that $M^{4|4}$ is a split complex super manifold with an odd two dimensional complex tangent bundle sitting over it. In fact, there are two such manifolds, corresponding to a choice of real structure of representation S. The canonical densities on these manifolds are precisely $d^2\theta$ and $d^2\overline{\theta}$.