Chern-Simons Theory, Knots and Moduli Spaces of Connections

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Abstract

After reviewing the basics of classical Chern-Simons theory I will explain how its quantum version can be used to give an interpretation of certain knot invariants. I will also discuss the Hamiltonian approach, where moduli spaces of connections on surfaces are of central importance.

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1 Classical Chern-Simons Theory

Let G be a compact, connected Lie group. The Lie algebra $\mathfrak{g}$ is the vector space of left-invariant vector fields on G together with the Lie bracket. There is a canonical Lie algebra valued one-form on G, the Maurer-Cartan form, which assigns to each vector its left-invariant extension. We denote this by $\theta \in \Omega^1(G;\mathfrak{g})$. It satisfies

$$L_g^*\theta = \theta, \quad R_g^*\theta = Ad_{g^{-1}}\theta \quad \forall g \in G.$$

Here L_g (resp. R_g) is left (resp. right) multiplication by $g \in G$ and $Ad_g = L_g \circ R_{g^{-1}}$. The Maurer-Cartan form also satisfies the Maurer-Cartan equation

$$d\theta + \frac{1}{2}[\theta \wedge \theta] = 0.$$

Here, for vector fields X and Y on G

$$\frac{1}{2}[\theta \wedge \theta](X,Y) = [\theta(X), \theta(Y)].$$

Now let $\pi : P \to X$ be smooth principal G-bundle. We take P to be a right G-space and have $X = P/G$. For $x \in X$ let $P_x = \pi^{-1}(x)$. For each $p \in P_x$ we have an isomorphism of right G-spaces

$$\begin{array}{rcl} \psi_p : G & \to & P_x \\ g & \mapsto & p \cdot g. \end{array}$$

Given any other point $q \in P_x$ there is a unique $g \in G$ such that $q = p \cdot g$. Then $\psi_q^{-1}\psi_p = L_g$. In particular, this allows the identification $T_pP_x \simeq \mathfrak{g}$ for any $p \in P_x$. Pulling back the Maurer-Cartan form via the inclusion $i_x : P_x \hookrightarrow P$ gives $\theta_x \in \Omega^1(P_x;\mathfrak{g})$ such that

$$R_g^*\theta_x = Ad_{g^{-1}}\theta_x, \quad d\theta_x + \frac{1}{2}[\theta_x \wedge \theta_x] = 0.$$

A connection on $P \to X$ is a differential form $A \in \Omega^1(P;\mathfrak{g})$ such that

$$R_g^*A = Ad_{g^{-1}}A \quad \forall g \in G$$

and

$$i_x^*A = \theta_x \quad \forall x \in X.$$

From P we can form the associated bundle of Lie algebras, $\mathfrak{g}_P = P \times_G \mathfrak{g}$, where G acts on $\mathfrak{g}$ via the adjoint action.

Proposition 1.1. *Any smooth principal G-bundle $\pi : P \to X$ has a connection. Moreover, the space of all connections on P, denoted by $\mathcal{A}_P$, is an affine space over $\Omega^1(X; \mathfrak{g}_P)$.*

Proof. If $P = X \times G$ is the trivial G-bundle we can take the product connection. For arbitrary $P \to X$, let $\{U_\alpha\}$ be an open cover of X so that $P|_{U_\alpha}$ is trivial. Put the product connection A_α on each $P|_{U_\alpha}$. Let $\{\lambda_\alpha\}$ be a partition of unity subordinate to the chosen open cover. Then it is easy to check that

$$A = \sum_\alpha \lambda_\alpha A_\alpha$$

is a connection on P.

If A, A' are two connections on P, then $i_x^*(A - A') = 0$ and

$$R_g^*(A - A') = Ad_{g^{-1}}(A - A') \quad \forall g \in G.$$

This is precisely the statement that $A - A' \in \Omega^1(X; \mathfrak{g}_P)$. □

The curvature of a connection $A \in \mathcal{A}_P$ is

$$F_A = dA + \frac{1}{2}[A \wedge A].$$

A connection A is flat if $F_A = 0$. Note that while by definition $F_A \in \Omega^2(P; \mathfrak{g})$, the Maurer-Cartan equation implies that $i_x^* F_A = 0$ for all $x \in X$. Similarly, $R_g^* F_A = Ad_{g^{-1}} F_A$, so that the curvature is image of $\pi^* : \Omega^\bullet(X; \mathfrak{g}_P) \to \Omega^\bullet(P; \mathfrak{g})$. We will often think of the curvature as a differential form on X while still using the notation F_A.

Proposition 1.2. *The Bianchi identity holds:*

$$dF_A + [A \wedge F_A] = 0.$$

Proof. This follows from differentiating the definition of F_A, while being careful to include an extra minus sign due to the skew-symmetry of the Lie bracket. □

If we define the A-twisted differential by $d_A = d + [A \wedge -]$, which acts on $\Omega^\bullet(X; \mathfrak{g}_P)$, then the Bianchi identity reads $d_A F_A = 0$.

Denote by $\mathcal{G}_P = Aut(P)$ the set of G-equivariant bundle isomorphisms of P. Then $\mathcal{G}_P$ carries a natural structure of an infinite dimensional Lie group under composition. We call $\mathcal{G}_P$ the group of gauge transformations. Note that for each $\phi \in \mathcal{G}_P$ we can produce a G-equivariant map $g_\phi : P \to G$, where G is viewed as a G-space by the adjoint action. In fact, these two spaces are in bijection. Concretely,

$$\phi(p) = p \cdot g_\phi(p) \quad \forall p \in P$$

Hence $\mathcal{G}_P \simeq C^\infty(P, G)^G$. The group $\mathcal{G}_P$ acts on $\mathcal{A}_P$ by

$$\phi^* A = Ad_{g_\phi^{-1}} A + \theta_\phi, \qquad \phi \in \mathcal{G}_p, \ \ A \in \mathcal{A}_P$$

where θ_ϕ is the pull-back of the Maurer-Cartan form on G via $g_\phi : P \to G$. From this transformation formula we find that

$$\phi^* F_A = Ad_{g_\phi^{-1}} F_A.$$

In particular, the subspace of $\mathcal{A}_P$ consisting of flat connections is gauge invariant.

Let $\langle \cdot, \cdot \rangle$ denote a symmetric, non-degenerate, bilinear Ad-invariant form on $\mathfrak{g}$. If G is simple then by Schur's lemma this is unique up to normalization. The bilinear form on $\mathfrak{g}$ induces a map $\mathfrak{g}_P \otimes \mathfrak{g}_P \to \mathbb{R}$, which we again denote by $\langle \cdot, \cdot \rangle$. We restrict our attention now to the case that X is a closed, oriented 3-manifold. For $A \in \mathcal{A}_P$ we define the Chern-Simons 3-form [6] by

$$\alpha(A) = \langle A \wedge F_A \rangle - \frac{1}{6} \langle A \wedge [A \wedge A] \rangle \in \Omega^3(P; \mathbb{R}).$$

Using our formula for the curvature we can also write this in a more familiar form,

$$\alpha(A) = \langle A \wedge dA \rangle + \frac{1}{3} \langle A \wedge [A \wedge A] \rangle.$$

We remark that $\alpha(A)$ is a differential form on P and not on the base space X.

Proposition 1.3. *The Chern-Simons form enjoys the following properties:*

- $d\alpha(A) = \langle F_A \wedge F_A \rangle$
- $R_g^* \alpha(A) = \alpha(A) \qquad \forall g \in G$
- *Let* $\phi \in \mathcal{G}_P$. *Then*
$$\phi^* \alpha(A) = \alpha(A) - \frac{1}{6} \langle \theta_\phi \wedge [\theta_\phi \wedge \theta_\phi] \rangle + d \langle Ad_{g_\phi^{-1}} A \wedge \theta_\phi \rangle.$$

Proof. The only non-trivial statement is the last one, which is a direct calculation. □

Say B is a compact oriented 4-manifold that bounds X; that such a B exists follows from a theorem of Thom. Let $P' \to B$ be a principal G-bundle that restricts to $P \to X$. Let A' be a connection on P' and A the induced connection on P. Then, with $\langle \cdot, \cdot \rangle$ suitably normalized,

$$\alpha(A) = \int_B \langle F_{A'} \wedge F_{A'} \rangle \quad (mod\ \mathbb{Z}).$$

That the equality only holds in modulo the integers is due to the ambiguity in the choice of B.

As a further restriction we now impose the condition that G be simply connected. The purpose for this is the following result.

Proposition 1.4. *Let X be a closed manifold of dimension at most* 3. *Then any principal G bundle over X is trivializable.*

Proof. It is a classical result that $\pi_2(G) = 0$. Hence the classifying space BG is 3-connected. Cellular approximation then shows that any G-bundle on X is trivializable. □

With a fixed choice of trivialization of $P \to X$ many of the formulae above simplify. For example, the adjoint bundle $\mathfrak{g}_P$ is trivial, so that the space of connections is affine over $\Omega^1(X; \mathfrak{g})$. Also, we would have $\mathcal{G}_P \simeq C^\infty(X, G)$. Recall that $P \to X$ is trivializable if and only if it has a global section $s : X \to P$. Pick any such section s and define

$$S_X(s, A) = \int_X s^* \alpha(A).$$

Let $\phi \in \mathcal{G}_P$ be a gauge transformation and put $g = g_\phi \circ s : X \to G$ and $\theta_g = g^* \theta$ the pull-back of the Maurer-Cartan form to X. Then we have

$$S_X(\phi \cdot s, A) = S_X(s, \phi^* A) = S_X(s, A) + \int_X -\frac{1}{6} \langle \theta_g \wedge [\theta_g \wedge \theta_g] \rangle.$$

Given a section $s : X \to P$, any other section can be obtained by acting on s with gauge transformations. Hence, if S_X is to be an invariant of a connection A on P, the dependence on the choice of section must be removed.

It is well-known that the cohomology class determined by $-\frac{1}{6}\langle \theta \wedge [\theta \wedge \theta]\rangle$ is a generator of $H^3(G;\mathbb{R})$. Moreover, if we normalize the bilinear form $\langle\cdot,\cdot\rangle$ so that $\langle h_\alpha, h_\alpha\rangle = 2$ for all long roots h_α then the class determined by $-\frac{1}{6}\langle \theta \wedge [\theta \wedge \theta]\rangle$ is integral and is in fact (the image of) a generator of $H^3(G,\mathbb{Z})$ in de Rham cohomology. With such a normalization the action $S_X(s, A)$ is independent of the choice of section s, modulo the integers. For our purposes, it will be convenient to write $\langle\cdot,\cdot\rangle = k\langle\cdot,\cdot\rangle'$, where $\langle\cdot,\cdot\rangle'$ is the minimal[1] integral bilinear form and $k \in \mathbb{Z}$. The integer k is called the level and should be thought of as a class in $H^4(BG,\mathbb{Z}) \simeq \mathbb{Z}$, which parameterizes integral bilinear forms on $\mathfrak{g}$. It suffices to take k to be positive; negative k simply corresponds to reversing the orientation of X.

Summarizing, we have defined map

$$e^{2\pi i k S_X(-)} : \mathcal{A}_P/\mathcal{G}_P \to U(1) \subset \mathbb{C}.$$

This is the Chern-Simons action at level $k \in \mathbb{N}$. Observe that $e^{2\pi i k S_X(-)}$ is multiplicative under disjoint unions of 3-manifolds.

We also need to compute the critical points of the Chern-Simons action. To do this, let $A_t = A + ta$ be a path of connections in $\mathcal{A}_P$, so that $a \in \Omega^1(X;\mathfrak{g}_P)$. We have

$$F_{A_t} = F_A + t d_A a + t^2[a \wedge a].$$

Then (omitting the dependence on the choice of section, where here is irrelevant)

$$S_X(A_t) = S_X(A) + 2t\int_X \langle a \wedge F_A\rangle + t^2\int_X \langle a \wedge d_A a\rangle + O(t^3).$$

Hence

$$\frac{d}{dt}|_{t=0} S_X(A_t) = 2\int_X \langle a \wedge F_A\rangle$$

which vanishes for all $a \in \Omega^1(X;\mathfrak{g}_P)$ if and only if A is flat. Hence the equation of motion for the connection A, which is the basic field of the theory, is $F_A = 0$. Note that the equation of motion is gauge invariant. Indeed, it must be, as the action from which it was derived is gauge invariant.

While there is much more to be said about classical Chern-Simons theory, we will stop here. Indeed, our goal is to study the quantum Chern-Simons theory. The functorial properties of the above constructions can all be spelled out explicitly and turned into a theorem expressing that S_X is the action of a local Lagrangian topological field theory on X. The concept of locality here is that the fields (connections) can be glued from local data. This allows one to cut and paste, sometimes breaking up computations from complicated three manifolds into computations on simpler three manifolds, possibly with boundary, a process familiar form topological quantum field theory. Of course, we have said nothing above about the case of manifolds with boundary but we will naturally be led to this below. For a full exposition of the ideas above and further extensions see [8].

2 Quantum Chern-Simons Theory

In this section we discuss the quantum Chern-Simons theory, taking the Lagrangian approach [16]. That is, we use throughout Feynman path integrals. Of course these integrals are not well defined, making most of this section

[1] Since $\mathfrak{g}$ is simple the notion of minimal here makes sense.

rather heuristic. However if we proceed formally, assuming that if such integrals can in some way be defined then they would share many of the properties that usual integrals enjoy, we find some suggestive consequences. Later we will consider the Hamiltonian approach which is defined on 3-manifolds of the form $[0,1]\times\Sigma$ for some compact 2-manifold Σ. Much rigorous work can be done in this situation. We will also explain the relationship between the Lagrangian and Hamiltonian view points.

We continue with the assumption that X is a compact, oriented 3-manifold without boundary. $P \to X$ is a principal G-bundle over X, G being a compact, simple, simply connected Lie group. Define the partition function at level $k \in \mathbb{N}$ as the path integral

$$Z(X) = \int_{\mathcal{A}_P} \mathcal{D}A\, e^{2\pi i k S_X(A)}.$$

Here $\mathcal{D}A$ is some postulated measure on the infinite dimensional space of connections $\mathcal{A}_P$. In fact, as we have seen above the integrand is gauge invariant. Any good definition of $\mathcal{D}A$ should share this property so that we could instead take as our definition of $Z(X)$ an integral over the moduli space of connections, $\mathcal{A}_P/\mathcal{G}_P$. However, topologically the moduli space is quite complicated, so integrating over the affine space $\mathcal{A}_P$ is slightly easier.

Let $C : S^1 \to X$ be an oriented knot and R an irreducible representation of G. An important case later will be $X = S^3$, $G = SU(N)$ and R the fundamental representation. For each connection $A \in \mathcal{A}_P$ we get a map, the holonomy

$$Hol_C : \mathcal{A}_P \to G/G$$

that assigns to A the holonomy conjugacy class in G. Specifically, fix a basepoint $x \in X$ and pick $p \in P_x$. Using the connection lift C to a horizontal curve $\tilde{C}$ in P starting at p. Then there exists a unque $g \in G$ such that $\tilde{C}(1) = \tilde{C}(0)\cdot g$. By varying the choice of $p \in P_x$ this gives a conjugacy class in G. The Wilson loop operator is then

$$W_A(C;R) = \mathrm{Tr}_R Hol_C(A).$$

Note this is well defined since Tr_R is adjoint invariant. We will be interested in computing the correlation functions of Wilson loops,

$$\langle W(C;R)\rangle = \int_{\mathcal{A}} \mathcal{D}A\, W_A(C;R) e^{2\pi i k S_X(A)}.$$

More generally, if L is an oriented link with components C_i labeled by irreducible representations R_i we can form

$$\langle W(\{C_i\};\{R_i\})\rangle = \int_{\mathcal{A}} \mathcal{D}A\, \prod_i W_A(C_i;R_i) e^{2\pi i k S_X(A)}.$$

We remark that we have not introduced any geometric structures in defining the correlation functions. This suggests that the quantities $Z(X) = \langle 1\rangle$ and $\langle W(\{C_i\};\{R_i\})\rangle$ may be topological invariants. We now turn to this point.

First consider the case in which there are no knots, so that we study $Z(X)$. We will study the asymptotics $k \to \infty$. This should be thought of as a semi-classical limit $\hbar \to 0$ with k playing the rôle of $\hbar^{-1}$. Recall that if $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth function with finitely many non-degenerate critical points x_α then the method of stationary phase gives

$$\int_{\mathbb{R}^n} e^{ikf(x)} d^n x \sim \sum_\alpha \frac{\pi^{\frac{n}{2}} e^{ikf(x_\alpha)}}{k^{\frac{n}{2}}\sqrt{|\det Hess(f,x_\alpha)|}} e^{\frac{i\pi}{4} sgn\; Hess(f,x_\alpha)} \qquad \text{as } k \to \infty.$$

We would like to apply this to the path integral $Z(X)$ as $k \to \infty$. Let us assume here that all flat connections are isolated and that there are only finitely many. Fix a Riemannian metric g on X and let $*$ be the associated Hodge star. This induces an inner product on the algebra of differential forms $\Omega^\bullet(X;\mathfrak{g}_P)$,

$$(a,b) = \int_X \langle a \wedge *b\rangle \quad \forall a,b \in \Omega^\bullet(X;\mathfrak{g}_P).$$

From the formula for F_{A+ta} we see that the Hessian of the Chern-Simons functional at a flat connection A is $Q_A(a) = *d_A$ for $a \in T_A\mathcal{A}_P$. Note that because of the gauge symmetry the Hessian is degenerate. The Hodge star gives rise to an orthogonal decomposition

$$\Omega^1(X;\mathfrak{g}_P) = \mathrm{Im} d_A \oplus \ker d_A^*.$$

The kernel of Q_A is then $\mathrm{Im} d_a$. Hence, viewed as a map $\ker d_A^* \to \ker d_A^*$, the Hessian is non-degenerate. To compute the determinant[2] of Q_A we consider an auxiliary operator. Let $L_A : \Omega^0(X;\mathfrak{g}_P)\oplus\Omega^1(X;\mathfrak{g}_P) \to \Omega^0(X;\mathfrak{g}_P)\oplus\Omega^1(X;\mathfrak{g}_P)$ given by

$$L_A = \begin{pmatrix} 0 & -d_A^* \\ -d_A & *d_A \end{pmatrix}$$

It follows that L_A self-adjoint and elliptic. Its square is $\Delta_A^0 \oplus \Delta_A^1$, where Δ_A^i is the A-twisted Laplacian on i-forms with values in $\mathfrak{g}_P$. Hence $\det L_A^2 = \det \Delta_A^0 \cdot \det \Delta_A^1$. Under the decomposition $\Omega^1(X;\mathfrak{g}_P) = \mathrm{Im} d_A \oplus \ker d_A^*$ we have

$$L_A = Q_A \oplus S_a$$

where now

$$S_A = \begin{pmatrix} 0 & -d_A \\ -d_A^* & 0 \end{pmatrix} : \Omega^0(X;\mathfrak{g}_P) \oplus \mathrm{Im} d_A \to \Omega^0(X;\mathfrak{g}_P) \oplus \mathrm{Im} d_A$$

It follows that $|\det L_A| = |\det Q_A| \det \Delta_A^0$. Hence

$$|\det Q_A| = \left(\frac{\det \Delta_A^1}{\det \Delta_A^0} \right)^{\frac{1}{2}}.$$

Continuing, we can write

$$\frac{\sqrt{\det \Delta_A^0}}{\sqrt{|\det Q_A|}} = T_A^{-\frac{1}{2}}$$

where T_A is the Ray-Singer torsion of the local system determined by the flat connection A. It was shown [15] that the absolute value of the Ray-Singer torsion is a topological invariant, independent of the metric chosen in its definition. However, we must worry about the signs involved in choosing the root that appears above.

We remark that while the stationary phase formula above should certainly motivate the study of $|\det Q_A|^{-\frac{1}{2}}$, the appearance of the factor $\sqrt{\det \Delta_A^0}$ may be a mystery. Physically, this term arises as part of the standard gauge fixing procedure. Mathematically, one can understand it as follows. Consider the situation above used to describe the stationary phase approximation. Additionally, suppose a compact Lie group acts and that f is invariant under this action. In our setting the group is the group of gauge transformations $\mathcal{G}_P$ (which is not compact!) and f is the action S_X. The map $d_A : \Omega^0(X;\mathfrak{g}_P) \to \Omega^1(X;\mathfrak{g}_P)$ is identified with the infinitesimal action of the group of gauge transformations at A. Hence the determinant $\sqrt{\det \Delta_A^0} = \sqrt{\det d_A^* d_A}$ is nothing but the Jacobian for the local slice of the $\mathcal{G}$ action at A.

Continuing, we move on to study the phase ambiguity. We need to understand the signature of Q_A. Since $L_A = Q_A \oplus S_A$ and clearly $\mathrm{sgn}\, S_A = 0$ we have

$$\mathrm{sgn}\, L_A = \mathrm{sgn}\, Q_A.$$

Denote by λ the eigenvalues of Q_A. We put

$$\eta_A(s) = \sum_{\lambda \neq 0} |\lambda|^{-s} \mathrm{sgn}\, \lambda \quad s >> 0.$$

[2] All determinants are regularized.

Then η_A extends by analytic continuation to a function on $\mathbb{C}$ with no pole at $s = 0$. Morally $\eta_A(0)$ is the difference between the number of positive and negative eigenvalues of Q_A. Put $\operatorname{sgn} Q_A = \eta_A(0)$. Denote by ∇ the trivial connection on $P \to X$ and put

$$\tilde{\eta}_A = \eta_A(0) - \eta_\nabla(0).$$

By a theorem of Atiyah-Patodi-Singer we can write

$$\tilde{\eta}_A = \frac{4}{\pi} c_2(G) S_X(A).$$

Here $c_s(G)$ is the value of the quadratic Casimir in the adjoint representation. So we can write

$$Z(X) \sim e^{\frac{\pi i \eta_\nabla(0)}{2}} \sum_{\{A \in \mathcal{A} \mid F_A = 0\}} e^{i(k + \frac{c_2(G)}{2}) S_X(A)} \sqrt{T_A}.$$

The only non-topological term in this expression is $\eta_\nabla(0)$. Since ∇ is the trivial connection we have $\eta_\nabla(0) = \dim G \cdot \eta_{\nabla_X}(0)$ where ∇_X is the Levi-Cività connection on the spin bundle of X. Picking a trivialization s of the spin structure we can define the Chern-Simons invariant of ∇_X, which we call $CS(X, s)$. Again using the Atiyah-Patodi-Singer theorem, we have that

$$\frac{1}{2}\eta_{\nabla_X}(0) + \frac{1}{12}\frac{CS(X,s)}{2\pi}$$

is a topological invariant. Hence if we replace $\frac{1}{2}\eta_{\nabla_X}(0)$ with the above expression, the asymptotic expansion of the path integral becomes

$$Z(X) \sim e^{\pi i \left(\frac{\eta_\nabla(0)}{2} + \frac{1}{12}\frac{CS(X,s)}{2\pi}\right)} \sum_{\{A \in \mathcal{A} \mid F_A = 0\}} e^{i(k + \frac{c_2(G)}{2}) S_X(A)} \sqrt{T_A}.$$

which is topological. We have already seen that the Chern-Simons invariant changes in a tame way under different trivializations. Hence, while the asymptotics of $Z(X)$ isn't a topological invariant of X, it is an invariant of X together with a choice of framing. Moreover, under changes in framing $Z(X)$ transforms in a completely calculable manner.

We now briefly discuss in what sense Wilson loop correlation functions $\langle W(\{C_i\}; \{R_i\}) \rangle$ are knot invariants. For simplicity, consider the case in which the link has two components C_1 and C_2. Let $G = U(1)$ for simplicity and label the representations for each link C_i by an integer n_i. Let $P \to S^3 = X$ be the trivial $U(1)$-bundle. Then a connection A on P is just an imaginary one-form on S^3. The correlation function in question is

$$\int_{\mathcal{A}} \exp\left(2\pi i k S_X(A) + n_1 \int_{C_1} A + n_2 \int_{C_2} A\right).$$

As was shown in a previous seminar this is (up to some factors) an exponentiated sum of Gauss linking numbers,

$$\langle C_1 C_2 \rangle \sim \exp\left(\frac{1}{k}\sum_{i,j} n_i n_j I(C_i, C_j)\right)$$

where

$$I(C_i, C_j) = \int_{x \in C_i,\ y \in C_j} \ell(x - y)$$

and

$$\ell(x) = \frac{x_1 dx_2 \wedge dx_3 + x_3 dx_1 \wedge dx_2 + x_2 dx_3 \wedge dx_1}{4\pi |x|^3}.$$

The term in which $i = j$ however presents a problem as the integral in singular. What this number should represent is the self-linking number of a knot. It is well-known that in general this cannot be done canonically. However, what needs to be done is clear: one would like to perturb the knot C_i to a new knot C_i' and then take the linking number $I(C_i, C_i')$. So, if we consider framed knots, that is, knots together with a non-zero normal vector field, we can flow the knot along the framing to obtain the new knot C_i'. One can think of the framing as defining an oriented ribbon, the self-linking number being the number of twists of the ribbon. With such a definition of self-linking number, it is clear that the asymptotics of the Wilson loop correlation function $\langle W(\{C_i\}\,;\{R_i\})\rangle$ are well-defined. However, it is now an invariant of framed, oriented links. Note that any two framings of a knot C_i differ by an integer, the relative number of twists of the corresponding ribbons. If this relative number of twists is δ then it is clear that the correlation functions change as

$$\langle W(C_i; R_i)\rangle \mapsto e^{2\pi i\delta\frac{n_i^2}{k}}\langle W(C_i; R_i)\rangle.$$

So, while it is necessary to introduce extra data to the knots in order to compute the Wilson loop correlation functions, the dependence on the choice is completely calculable as in the case of the partition function, so that the final result is nearly as good as an invariant of a knot itself. For more on this and related topics see [4, 5].

2.1 Some Formal Properties and Consequences

Proposition 2.1. *Let X_1 and X_2 be closed, oriented 3-manifolds and $X = X_1\#X_2$ their connected sum. Let $\tilde{X}_i$ be X_i with the closed three ball removed and S^2 its boundary. Then*

$$Z(X)Z(S^3) = Z(X_1)Z(X_2).$$

Proof. Let $\mathcal{H}_{S^2}$ be the Hilbert space associated to the two sphere; this is one-dimensional as will be explained below. Put $v = Z(\tilde{X}_1) \in \mathcal{H}_{S^2}$ and $v^* = Z(\tilde{M}_2) \in \mathcal{H}_{S^2}^\vee$. We have $Z(X) = (v^*, v)$, with parentheses denoting the pairing between a vector space and its dual.. Also, write $S^3 = D_1^3 \cup_{S^2} D_2^3$ and put $u = Z(D_1^3) \in \mathcal{H}_{S^2}$ and $u^* = Z(D_2^3) \in \mathcal{H}_{S^2}^\vee$. Then $Z(S^3) = (u^*, u)$. $\mathcal{H}_{S^2}$ being one dimensional there exist $\lambda, \mu \in \mathbb{C}$ such that

$$u = \lambda v, \quad u^* = \mu v^*.$$

It is elementary to check then that the desired relation holds. □

Note that we can rewrite this relation as

$$\frac{Z(X)}{Z(S^3)} = \frac{Z(X_1)}{Z(S^3)}\frac{Z(X_2)}{Z(S^3)}.$$

Then the normalized correlation functions behave functorialy.

Continuing, let $L = \coprod_i C_i$ be an oriented link such that each component is the unknot and no components are linked. By applying the previous argument we have

$$\frac{\langle C_1, \ldots, C_n\rangle}{Z(S^3)} = \prod_i \frac{\langle C_i\rangle}{Z(S^3)}.$$

Assume now that $G = SU(N)$ and $X = S^3$. Consider a single knot C labeled with the fundamental representation R. Cut out a 3-ball around a complicated portion of the knot, so that the boundary of the corresponding three manifolds is S^2 with four marked points, two labeled with R and two with $R^\vee$, corresponding to the reversed orientation of the points. The Hilbert space corresponding to this marked S^2, $\mathcal{H}_{S^2,p_i,R_i}$, is two dimensional[3], since the multiplicity of the trivial representation in $R \otimes R \otimes R^\vee \otimes R^\vee$ is two. Let X_+ be the 3-ball with the complicated part and X' the other part. Then $Z(X') \in \mathcal{H}_{S^2,p_i,R_i}$ and $Z(X_+) \in \mathcal{H}^\vee_{S^2,p_i,R_i}$. We have

$$Z(C) = (Z(X_+), Z(X'))\,.$$

Performing half monodromies on the knot in X_+ we obtain two different configurations, called X_0 and X_+. Hence we have three vectors

$$Z(X_-), Z(X_0), Z(X_+) \in \mathcal{H}^\vee_{S^2,p_i,R_i}$$

which, since the vector space is two dimensional, obey some linear relation

$$aZ(X_-) + bZ(X_0) + cZ(X_+) = 0 \qquad a, b, c \in \mathbb{C}.$$

The constants a, b, c will depend on the parameters k, N used to define the quantum field theory. Pairing this linear relation with $Z(X')$ gives a generalized skein relation. It is known that once the value of a knot invariant is fixed for the unknot $\bigcirc$, the skein relations determine its value for any other knot. In terms of the constants a, b, c, it is easy to check that the invariant for the unknot is

$$\langle \bigcirc \rangle = -\frac{a + c}{b}.$$

What remains is to determine the actual values of the constants $a, b, c \in \mathbb{C}$. This is a problem in conformal field theory tantamount to understanding the braid group action on the space of conformal blocks and has been solved by Moore and Seiberg. We quote their results:

$$a = -e^{\frac{2\pi i}{N(N+k)}}, \quad b = -e^{\frac{\pi i(2-N-N^2)}{N(N+k)}} + e^{\frac{\pi i(2+N-N^2)}{N(N+k)}}, \quad c = e^{\frac{2\pi i(1-N^2)}{N(N+k)}}.$$

The skein relations now read

$$-q^{\frac{N}{2}} Z(X_-) + (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) Z(X_0) + q^{-\frac{N}{2}} Z(X_+) = 0$$

where $q = e^{\frac{2\pi i}{N+k}}$. If we take $G = SU(2)$ the skein relations reduced to those of the Jones polynomial. Also, we have for the unknot

$$\langle \bigcirc \rangle = \frac{q^{\frac{N}{2}} - q^{-\frac{N}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}.$$

Thus we have succeeded in showing that for $G = SU(2)$ Chern-Simons theory gives rise to the Jones polynomial.

We can also understand from this viewpoint how the correlation functions behave under change in framing. This time however the calculations are exact, whereas previously they were completed only for large k. Cut the 3-manifold X, containing a Wilson loop C, along an embedded surface Σ that acquires a marked point p corresponding to an intersection point with the knot. Associated to the marked surface is a Hilbert space $\mathcal{H}_{\Sigma,p}$. We will see that we can view the Hilbert space as any one of the fibres of a projectively flat vector bundle $\mathcal{H}$ over Teichmüller space. From this point of view $\mathcal{H}_{\Sigma,p}$ becomes a representation of the mapping class group of Σ. For a positive integer δ consider the δ-fold Dehn twist about p. Regluing the 3-manifolds along the Dehn twist gives back X, but the framing of the knot is shifted by δ. Hence, to understand the change in framing it suffices to understand the action of the mapping class group, or more specifically the action of particular Dehn twists, on the Hilbert spaces in question. These actions have been well studied in conformal field theory. The result is that the δ-fold Dehn twists act on the Hilbert space by multiplication by $e^{2\pi i \delta h_R}$, where h_R is the conformal weight of the primary field in the representation R.

[3]This argument works only in the large k limit; in general the Hilbert space $\mathcal{H}^k_{S^2,p_i,R_i}$ is contained in the space of invariants of $\otimes_i R_i$

3 Moduli Spaces of Flat Connections

We now consider the Hamiltonian view point of Chern-Simons theory. Let as usual X be a compact oriented 3-manifold without boundary. Cut X along an embedded surface Σ. Locally around the cut X will look like $\Sigma \times [0,\infty)$. We would like to understand how to solve Chern-Simons on $\Sigma \times [0,\infty)$ with hopes that by cutting and gluing we will be able to extend a solution to all 3-manifolds. It is easy to check that given any connection on $P \to \Sigma \times [0,\infty)$ one can find a gauge transformation so that the dt component vanishes, t being the coordinate on $[0,\infty)$. The Chern-Simons equations of motion are first order and hence one need specify only initial conditions on $\Sigma \times \{0\}$ to determine the solution on all of $P \to \Sigma \times [0,\infty)$. Precisely we have the following.

Proposition 3.1. *Let $i : \Sigma \times \{0\} \hookrightarrow \Sigma \times [0,\infty)$ be the inclusion. The restriction map $i^* : \mathcal{A}_P \to \mathcal{A}_{P|_{\Sigma\times\{0\}}}$ gives an isomorphism of the moduli spaces of flat connections on P and that on $P|_{\Sigma\times\{0\}}$.*

So, let Σ be a compact oriented 2-manifold without boundary of genus g. As proved above, any principal G-bundle Q on Σ is trivializable if G is simply connected, which we assume. We will give here a symplectic construction of the moduli space of flat connections on Q, following Atiyah and Bott [2]. To begin, recall that the space of connections $\mathcal{A}_Q$ on Q is affine over $\Omega^1(\Sigma;\mathfrak{g})$ after picking a trivialization of Q. This gives an identification $T_A\mathcal{A}_Q = \Omega^1(\Sigma;\mathfrak{g})$. Define an anti-symmetric bilinear form on $T_A\mathcal{A}_Q$ by

$$\omega_A(a,b) = \int_\Sigma \langle a \wedge b \rangle \quad a,b \in \Omega^1(\Sigma;\mathfrak{g}).$$

Then ω defines a 2-form on $\mathcal{A}_Q$. It is easy to check that ω is closed (as it is translation invariant) and non-degenerate, the latter following from the non-degeneracy of the bilinear from on $\mathfrak{g}$. This gives $\mathcal{A}_Q$ the structure of an infinite dimensional symplectic affine space. Moreover, the action of the gauge group $\mathcal{G}_Q$ on $\mathcal{A}_Q$ is by symplectomorphisms. In fact, more is true: the $\mathcal{G}_Q$ action is Hamiltonian. To see this, we find the moment map. Recall this is a $\mathcal{G}_Q$-equivariant map

$$\mu : \mathcal{A}_Q \to (\mathrm{Lie}(\mathcal{G}_Q))^*$$

such that

$$d\mu(\epsilon) = \iota_\epsilon \omega \quad \forall \epsilon \in \mathrm{Lie}(\mathcal{G}_Q).$$

We have $\mathrm{Lie}(\mathcal{G}_Q) \simeq \Omega^0(\Sigma;\mathfrak{g})$ so that integration over Σ together with the bilinear form on $\mathfrak{g}$ gives an isomorphism $(\mathrm{Lie}(\mathcal{G}_Q))^* \simeq \Omega^2(\Sigma;\mathfrak{g})$. Hence, the curvature gives an equivariant map $\mu : \mathcal{A}_Q \to (\mathrm{Lie}(\mathcal{G}_Q))^*$. If $a \in T_A\mathcal{A}_Q$ then

$$d\mu|_A(a) = d_A a$$

so that its action on $\epsilon \in \mathrm{Lie}(\mathcal{G}_Q)$ is

$$\int_\Sigma \langle \epsilon d_A a \rangle.$$

On the other hand, the vector field induced by ϵ is $d_A\epsilon$ so that

$$(\iota_\epsilon\omega)(a) = \int_\Sigma \langle d_A\epsilon \wedge a \rangle$$

which agrees with the previous expression after using Stokes theorem. This shows that the curvature is indeed a moment map. Since μ is $\mathcal{G}_Q$-equivariant the level set $\mu^{-1}(0)$ is $\mathcal{G}_Q$-invariant and we can form the symplectic quotient

$$\mathcal{M}_Q := \mathcal{A}_Q//\mathcal{G}_Q = \mu^{-1}(0)/\mathcal{G}_Q,$$

which is the moduli space of flat connections on Σ. It is a general fact that at smooth points the symplectic quotient inherits a symplectic structure from the original symplectic manifold. In this way the classical phase space $\mathcal{M}_Q$ has a natural symplectic structure; we will largely ignore the issue of singularities. We remark that while both $\mathcal{A}_Q$ and $\mathcal{G}_Q$ are infinite dimensional, the moduli space $\mathcal{M}_Q$ is in fact finite dimensional. We see this as follows. The flat connection A gives rise to a twisted de Rham complex

$$0 \to \Omega^0(\Sigma; \mathfrak{g}_Q) \xrightarrow{d_A} \Omega^1(\Sigma; \mathfrak{g}_Q) \xrightarrow{d_A} \Omega^2(\Sigma; \mathfrak{g}_Q) \to 0.$$

Denote the cohomology of this complex by $H^\bullet_A(\mathfrak{g}_Q)$. If $A_t = A + ta$, $a \in \Omega^1(\Sigma; \mathfrak{g}_Q)$ is a family of connections (A being flat) then F_{A+ta} will be flat to first order in t precisely when $d_A a = 0$. Also, an element $\epsilon \in \Omega^0(\Sigma; \mathfrak{g}_Q)$, thought of as an infinitesimal gauge transformation, acts on A by $A \mapsto A + d_A\epsilon$. Hence at a flat connection the Zariski tangent space is

$$T_{[A]}\mathcal{M}_Q \simeq H^1_A(\mathfrak{g}_Q).$$

The zeroth cohomology $H^0_A(\mathfrak{g}_Q)$ is simply the Lie algebra of the stabilizer subgroup of A in $\mathcal{G}_Q$, denoted by $\mathfrak{z}_A$. By Poincaré duality $H^0_A(\mathfrak{g}_Q) \simeq H^2_A(\mathfrak{g}_Q)$. Using the Index Theorem we find

$$\dim G \cdot \chi(\Sigma) = H^1_A(\mathfrak{g}_Q) - 2\dim \mathfrak{z}_A.$$

So, at smooth points, that is, connections A that have finite stabilizer, the moduli space is smooth of dimension $\dim G \cdot \chi(\Sigma)$.

We give here another construction of $\mathcal{M}_Q$ that will be useful in its generalization to marked surfaces. We have seen that for a given connection A on a principal G-bundle $P \to X$ the holonomy is a map

$$\Omega(X, x_0) \to G/G$$

where we write $\Omega(X, x_0)$ for smooth loops in X based at $x_0 \in X$. If A is flat it is easy to check that the holonomy descends to a homomorphism $\pi_1(X, x_0) \to G$. The group G acts on $\mathrm{Hom}(\pi_1(X, x_0), G)$ from the right by conjugation. We have the following result.

Proposition 3.2. *The holonomy map gives an bijection of sets*

$$\mathit{Hom}(\pi_1(X, x_0), G)/G \simeq \mathcal{M}_Q.$$

Sketch of proof: We have argued that a flat connection gives an element in $\mathrm{Hom}(\pi_1(X, x_0), G)$. One then needs to check how the holonomy changes under gauge transformations.

In the opposite direction, given a homomorphism $\rho : \pi_1(X, x_0) \to G$ we construct a principal G-bundle with connection having holonomy given by ρ. Let $\tilde{X}$ be the universal cover of X and $\tilde{x}_0$ a point in the fibre above x_0. Then $\pi_1(X, x_0)$ acts on $\tilde{X}$ by deck transformations. From this we get a principal G-bundle

$$Q_\rho = \tilde{X} \times_\rho G \to X.$$

To get a connection, consider the trival flat connection on $\tilde{X} \times G$. This descends to the quotient and one can check that its holonomy is given by ρ. □

In fact, it is in the guise of this result that many of the claims in this section were proven. This is partially because the space $\mathrm{Hom}(\pi_1(X, x_0), G)/G$ is finite dimensional whereas in the construction of $\mathcal{M}_Q$ using gauge theory necessarily requires the use of infinite dimensional spaces.

While we have succeeded in constructing a symplectic structure on $\mathcal{M}_Q$ canonically associated to Σ this is not sufficient to carry out the programme of geometric quantization. We would like to check that the symplectic from ω on $\mathcal{M}_Q$ is integral, so that it represents $\frac{i}{2\pi}$ times the curvature of a complex line bundle $\mathcal{L}_Q$ on $\mathcal{M}_Q$. Even more, we would like to take as our quantum Hilbert space the space of holomorphic sections of $\mathcal{L}_Q$. To even define this we need a complex structure on the moduli space and a holomorphic structure on the line bundle. Of course, the final result must be functorial, in that it does not depend on any additional choices made during the construction.

We begin by endowing $\mathcal{M}_Q$ with a complex structure, albeit not functorially. Pick a complex structure I on Σ, which induces a Hodge star $*$ on Σ. We then have an orthogonal decomposition[4]

$$\Omega^1(\Sigma;\mathfrak{g}_\mathbb{C}) = \Omega^{1,0}_A(\Sigma;\mathfrak{g}_\mathbb{C}) \oplus \Omega^{0,1}(\Sigma;\mathfrak{g}_\mathbb{C})$$

under which

$$T^{(1,0)}\mathcal{A}_Q = \Omega^{0,1}(\Sigma;\mathfrak{g}_\mathbb{C}).$$

The differential operator $d_A : \Omega^0(\Sigma;\mathfrak{g}_\mathbb{C}) \to \Omega^1(\Sigma;\mathfrak{g}_\mathbb{C})$ decomposes as $d_A = \partial_A + \overline{\partial}_A$. The connection $\overline{\partial}_A$ gives $\mathfrak{g}_Q \otimes \mathbb{C} \to \Sigma_I$ a holomorphic structure. In fact, by fixing a differential operator $\overline{\partial}_0$, say the standard Cauchy-Riemann operator, we may write any other such operator by adding an element of $T^{(1,0)}\mathcal{A}_Q$. That is, this vector space parametrizes holomorphic structures on $\mathfrak{g}_Q \otimes \mathbb{C} \to \Sigma$. The group of automorphisms $\mathrm{Aut}(\mathfrak{g}_Q \otimes \mathbb{C})$ acts on $T^{(1,0)}\mathcal{A}_Q$ and the quotient, denoted by $\mathcal{M}_I$, is the moduli space of (semi-stable) holomorphic structures on $\mathfrak{g}_P \otimes \mathbb{C}$. In our case, the automorphism group is $\mathcal{G}_\mathbb{C}$, the complexified gauge group and we are parameterizing holomorphic structures on the trivial $G_\mathbb{C}$-bundle on Σ_I. The quotient we take is the GIT quotient and gives $\mathcal{M}_I$ the structure of a projective variety. Its stable points form a Zariski open, dense subset and are the smooth points. It is a theorem of Narashimhan and Seshadri [13] that $\mathcal{M}_I$ and $\mathcal{M}_Q$ are homeomorphic. Using this we transfer the complex structure to $\mathcal{M}_Q$. In summary, if we endow Σ with a complex structure the moduli space of flat connections inherits a complex structure. Moreover, it can be shown that with this structure the symplectic form ω is Kähler.

We now turn to the question of whether the symplectic form ω on $\mathcal{M}_Q$ is integral. We again fix a complex structure I on Σ and so obtain an identification of $\mathcal{M}_Q$ with the moduli space of holomorphic structures on the trivial $G_\mathbb{C}$-bundle over Σ. As stated above, the choice of complex structure gives $\Omega^1(\Sigma;\mathfrak{g}) \simeq \Omega^{0,1}(\Sigma;\mathfrak{g}_\mathbb{C})$. Consider the determinant line bundle

$$\mathcal{L}_I \to \mathcal{A}_Q$$

whose fibre at $A \in \mathcal{A}_Q$ is $\Lambda^{max}(\ker \overline{\partial}_A)^* \otimes \Lambda^{max}(\ker \overline{\partial}_A^*)$. Clearly then $\mathcal{L}_I$ is $\mathcal{G}_\mathbb{C}$-equivariant and is holomorphic[5] in A. Hence $\mathcal{L}_I$ is a holomorphic line bundle. Note that a Hermitian connection on $\mathcal{L}_I$ is specified by a Hermitian metric. In our case, the Hermitian metric we use is the Quillen metric constructed as follows. Let $c > 0$ and $U_c \subset \mathcal{A}_Q$ the open set of connections such that $\Delta_A^+ = \overline{\partial}_A \overline{\partial}_A^*$ does not have c as an eigenvalue. Let H_c^+ be the span of the eigenspaces of Δ_A^+ with eigenvalue less than c; this forms a vector bundle of finite rank over U_c. Do the same construction with $\Delta_A^- = \overline{\partial}_A^* \overline{\partial}_A$. The $\overline{\partial}_A$ operator gives an isomorphism

$$\mathcal{L}_I|_{U_c} \simeq \Lambda^{max} H_c^+ \otimes \left(\Lambda^{max} H_c^-\right)^*.$$

[4]The actual decomposition is of forms with coefficients in $\mathfrak{g}_Q \otimes \mathbb{C} \simeq Q \times_G (\mathfrak{g} \otimes \mathbb{C})$. However, after making a choice of trivialization of Q this bundle is trivial with fibres $\mathfrak{g}_\mathbb{C} := \mathfrak{g} \otimes \mathbb{C}$. So as to avoid excessive notation, we simply write the coefficients as $\mathfrak{g}_\mathbb{C}$ instead of the more cumbersome $\mathfrak{g}_Q \otimes \mathbb{C}$.

[5]Had we chosen the perhaps more natural looking $T_A^{1,0}\mathcal{A}_Q \simeq \Omega^{1,0}(\Sigma, \mathfrak{g}_\mathbb{C})$ instead of $T_A^{1,0}\mathcal{A}_Q \simeq \Omega^{0,1}(\Sigma, \mathfrak{g}_\mathbb{C})$ then $\mathcal{L}_I$ would be anti-holomorphic.

Let $|\cdot|_c$ denote the L^2 metric on the right hand side and put

$$|\cdot|^2 = \left(\prod_{\lambda>c}\lambda\right)|\cdot|_c^2.$$

This is the Quillen metric on $\mathcal{L}_I$. It was shown by Quillen that $\frac{i}{2\pi}$ times the curvature of the induced connection is the symplectic form ω, showing that indeed ω is integral.

Combining the complex structure on $\mathcal{M}_Q$ and the integrality of the symplectic form, we see that giving Σ a complex structure makes $\mathcal{M}_Q$ into a Hodge manifold. Hence we have shown that to each Riemann surface (Σ, I) and level $k \in \mathbb{N}$ we can assign a complex vector space

$$\mathcal{H}_I^k = H^0(\mathcal{M}_Q, \mathcal{L}_I^{\otimes k}).$$

Since $\mathcal{M}_I \simeq \mathcal{M}_Q$ is projective $\mathcal{H}_I^k$ is finite dimensional. Even more is true: the dimension of $\mathcal{H}_I^k$ is independent of the choice of complex structure on Σ, as least for suitably large k. Indeed, using Hirzebruch-Riemann-Roch we can write the Euler characteristic for the Dolbeault complex with coefficients in $\mathcal{L}_I^{\otimes k}$ as an integral over characteristics class independent of the choice of complex structure. By Serre duality

$$H^p(\mathcal{M}_Q; \mathcal{L}^{\otimes k})^\vee \simeq H^{\dim \mathcal{M}_Q - p}(\mathcal{M}_Q; \mathcal{K}_Q \otimes (\mathcal{L}^\vee)^{\otimes k}).$$

For sufficiently large k the line bundle $\mathcal{K}_Q \otimes (\mathcal{L}^\vee)^{\otimes k}$ is negative and hence the right hand side will vanish by the Kodaira-Nakano Vanishing Theorem. This proves that $\dim_{\mathbb{C}} \mathcal{H}_I^k$ is independent of Kähler polarization. This is a good indication that our construction is in fact independent of the choice of complex structure. However, we need to clairify out understanding of the dependence of $\mathcal{H}_I^k$ on I. In particular, for two complex structures I, J on Σ, how, if at all, are $\mathcal{H}_I^k$ and $\mathcal{H}_J^k$ related?

It turns out that, by elliptic regularity, the Hilbert spaces $\mathcal{H}_I^k$ glue together to give a complex vector bundle over Teichmüller space, $\mathcal{H}^k \to \mathcal{T}_\Sigma$. To be able to canonically identify any two of the fibres is the the statement that $\mathcal{H}^k$ carries a flat connection; the identification is then given by parallel transport. Since constant phases of a wave function are not observable we can actually relax our condition and require only a projectively flat connection, i.e. a connection with constant scalar curvature. It then follows that the connection gives an isomorphism between the projectification of the fibres, $\mathbb{P}\mathcal{H}_I^k \simeq \mathbb{P}\mathcal{H}_J^k$. Indeed, such a connection exists, as was constructed by Hitchin [10].

It is an interesting and difficult problem to compute the dimension of the vector spaces constructed above. In the case that $G = U(1)$ we have

$$\dim_{\mathbb{C}} H^0(\mathcal{M}_Q^{U(1)}, \mathcal{L}_Q^{\otimes k}) = k^g.$$

For $G = SU(2)$, using the Verlinde formula, we have

$$\dim_{\mathbb{C}} H^0(\mathcal{M}_Q^{SU(2)}, \mathcal{L}_Q^{\otimes k}) = \left(\frac{k+2}{2}\right)^{1-g} \sum_{j=1}^{k+1} \sin^{2(g-1)}\left(\frac{j\pi}{k+2}\right).$$

More complicated formulae exist for compact Lie groups whose complexifications are of type A, B, C, D or G.

We have thus described a solution to the geometric quantization problem of the moduli space of flat connections. It is in the constructed vector spaces $\mathcal{H}_\Sigma^k$ that path integrals on 3-manifolds with boundary Σ take values. For a thorough discussion see [3]. However, the constructed Hilbert spaces cannot be used in case in which Wilson loops are present. For that wee need to study surfaces with marked points, corresponding to intersections with the Wilson loops, to which we now turn.

3.1 The Case of Marked Surfaces

Let Σ again be a compact oriented 2-manifold without boundary. Denote its genus by g. Let $p_1, \ldots, p_n$ be distinct points on Σ. Colour the points by irreducible representations R_i of the gauge group G. We would like to functorialy assign a complex vector space $\mathcal{H}_{\Sigma, p_i, R_i}$ to this marked surface.

To begin, we need to decide what classical system we are actually quantizing. By a classical system we mean roughly a symplectic manifold. As discussed above if there are no marked points we know how to proceed. What we need to understand is how to the coloured points, i.e. points together with a representation. Representations, being linear, should correspond to quantum objects, ideally obtained by quantizing some corresponding classical object. Luckily, there is a manner in which we can make this idea precise in the case at hand, Kirillov's orbit method.

Let $\xi \in \mathfrak{g}^*$ and consider the orbit $\mathcal{O}_\xi$ of ξ under the coadjoint action of G. For generic ξ the stabilizer of the G action will be a maximal torus $T \subset G$. Hence $\mathcal{O}_\xi \simeq G/T$. Recall that we have a diffeomorphism

$$G/T \simeq G_{\mathbb{C}}/B$$

where $B \subset G_{\mathbb{C}}$ is the Borel subgroup corresponding to T. The right hand side is a complex projective manifold and we can use the diffeomorphism to endow G/T with a complex structure. Also by the identification $\mathcal{O}_\xi \simeq G/T$ we have an isomorphsism

$$T_\xi \mathcal{O}_\xi \simeq \mathfrak{g}/\mathfrak{t},$$

and a similar isomorphism at all other tangent spaces of the coadjoint orbit. Define a skew-symmetric bilinear form on $T_\xi \mathcal{O}_\xi$ by

$$\omega_\xi(x, y) = (\xi, [x, y]) \quad x, y \in \mathfrak{g}/\mathfrak{t}.$$

It is easily checked that this is well defined. Extend ω to a differential 2-form on $\mathcal{O}_\xi$ so that it is G-invariant. The Jacobi identity implies ω is closed and by construction it is non-degenerate. Hence $(\mathcal{O}_\xi, \omega)$ is a symplectic manifold. It can be checked that with the induced complex structure on $\mathcal{O}_\xi$ the form ω is in fact Kähler. In order to quantize $(\mathcal{O}_\xi, \omega)$ we need ω to be integral. In general, this will not be so. However, if λ is a dominant weight and the coadjoint orbit intersects the positive Weyl chamber at $\frac{\lambda}{2\pi i}$ the form ω will be integral. Furthermore, by Borel-Weil-Bott the associated holomorphic line bundle on $\mathcal{O}_\xi$ will have as its space of holomorphic sections the irreducible representation of G of dominant weight λ. In summary, given any irreducible representation of R we can always find a classical phase space $(G/T, \omega_R)$ whose quantization is R.

We now use the above discussion to give a gauge theoretic description of the classical phase space for a marked surface. Let $Q \to \Sigma$ be the trivial G-bundle. Let $\mathcal{A}_Q^{sing}$ be the space of singular connections on Q. Consider the map

$$\mu_{R_i} : \mathcal{A}_Q^{sing} \to \mathrm{Lie}(\mathcal{G}_Q)^*$$

that sends $A \in \mathcal{A}_Q^{sing}$ to $\sum_i T_i \delta(p_i - x)$ where $T_i \in \mathfrak{g}$ is in the (co)adjoint orbit determined by R_i. Let $\mathcal{A}_{Q,R_i}$ be the pre-image under μ_{R_i} of a product of coadjoint orbits in $\mathrm{Lie}(\mathcal{G}_Q)^*$ corresponding to the R_i. Then the quotient of $\mathcal{A}_{Q,R_i}/\mathcal{G}_Q$ being a symplectic quotient is again symplectic (at smooth points) and corresponds to the connections on Q flat away from the marked points and with holonomies at the marked points contained in the corresponding conjugacy classes of G after exponentiation. We denote this moduli space by $\mathcal{M}_{Q,R_i}$.

Again using the holonomy we can obtain a construction of $\mathcal{M}_{Q,R_i}$ in terms of group representations. Denote by c_i a small oriented loop around the marked point p_i. Then

$$\pi_1(\Sigma \setminus \{p_1, \ldots, p_n\}) = \langle A_1, \ldots, A_g, B_1, \ldots, B_g, c_1, \ldots, c_n \mid \prod_{i=1}^{g} [A_i, B_i] \prod_{j=1}^{n} c_i \rangle.$$

By our discussion above, to each irreducible representation R_i of G we have a coadjoint orbit in $\mathfrak{g}^*$ to which we can associate a conjugacy class C_i in G. Then we have the following analogue of the set bijection obtained in the unmarked case.

Proposition 3.3. *The holonomy map gives a bijection of sets*

$$\mathcal{M}_{Q,R_i} \simeq \{\rho \in Hom(\pi_1(\Sigma \backslash \{p_1, \dots, p_n\}), G) \mid \rho(c_i) \in C_i\} / G.$$

Proceeding as in the unmarked case, we need to endow $\mathcal{M}_{Q,R_i}$ with a complex structure. This is accomplished by the following theorem of Mehta and Seshadri [12] which we state for $G = SU(N)$.

Theorem 3.4. *The moduli space of of semi-stable parabolic vector bundles of rank N, paradegree 0 and paradeterminant $\mathcal{O}_\Sigma$ for $\{p_i, R_i\}$ is homeomorphic to $\mathcal{M}_{Q,R_i}$.*

The construction of moduli space of semi-stable parabolic vector bundles is as a GIT quotient and makes it a projective algebraic variety with regular points the stable bundles. It can be checked that at regular points the symplectic from on $\mathcal{M}_{Q,R_i}$ becomes a Kähler form. Again the determinant bundle over $\mathcal{M}_{Q,R_i}$ has $\frac{i}{2\pi}$ times the curvature equal to the symplectic form. Taking the space of holomorphic sections of this bundle gives the quantum state space; this is the space of conformal blocks. The method of Hitchin can be modified to prove that the quantum state spaces corresponding to different choices of complex structures on Σ are projectively isomorphic as desired.

3.2 Relation Between the Hamiltonian and Lagrangian Approaches

Now that we have described the basics of both the Hamiltonian and Lagrangian approaches to quantum Chern-Simons theory we briefly explain how the two are related [14]. Consider the situation in which X is a compact, oriented three manifold with boundary Σ. We saw that it is no longer the case that the path integral

$$Z(X) = \int_{\mathcal{A}} \mathcal{D}A \, e^{2\pi i k S_X(A)}$$

is invariant under gauge transformations. Indeed, we should think of $Z(X)$ as a section of the Chern-Simons line $\mathcal{L}_Q \to \mathcal{A}_Q$, Q being the restriction of a principal G-bundle on X to Σ, given by

$$Z(X)(\eta) = \int_{\{A \in \mathcal{A}_P \mid A|_\Sigma = \eta\}} \mathcal{D}A \, e^{2\pi i k S_X(A)}.$$

This section will be holomorphic once complex structures are fixed. Now say X is a compact, oriented three manifold without boundary. Cut X along an oriented two manifold Σ so that

$$X = X_1 \cup_\Sigma X_2.$$

Note that $\partial X_1 = \Sigma = -\partial X_2$. Then we have holomorphic sections $Z(X_1) \in H^0(\mathcal{A}_\Sigma, \mathcal{L}_\Sigma^{\otimes k})$ and $Z(X_2) \in H^0(\mathcal{A}_\Sigma, (\mathcal{L}_\Sigma^\vee)^{\otimes k})$. As the canonical pairing $(Z(X_2), Z(X_2))$ is $Z(X)$ we see

$$Z(X) = \int_{\mathcal{A}_Q} \mathcal{D}\eta \int_{\left\{A_1 \in \mathcal{A}_{P_1} \mid A_1|_\Sigma = \eta\right\}} \mathcal{D}A_1 \, e^{2\pi i k S_{X_1}(A_1)} \cdot \int_{\left\{A_2 \in \mathcal{A}_{P_2} \mid A_2|_\Sigma = \eta\right\}} \mathcal{D}A_2 \, e^{-2\pi i k S_{X_1}(A_2)}.$$

3.3 Chern-Simons Lines

We have seen above how to construct from a compact oriented surface Σ without boundary a symplectic manifold, which we called $\mathcal{M}_Q$; this is the moduli space of flat connections on the (necessarily trivializable) principal G-bundle $Q \to \Sigma$. In this section with give a direct construction of the symplectic structure and prove that it satisfies the integrality criterion of geometric quantization. Direct here means that all the structure derives from the Chern-Simons lagrangian. That such a construction should be possible is the general philosophy that a local lagrangian field theory should give rise to an integral symplectic structure on its classical phase space. We follow [8]. For a different approach see [14].

We consider a compact oriented 3-manifold X with boundary Σ. We have seen that the Chern-Simons action changes under gauge transformations according to[6]

$$S_X(\phi \cdot s, A) = S_X(s, A) + \int_\Sigma s^* \langle \mathrm{Ad}_{g_\phi^{-1}} A \wedge \theta_\phi \rangle + \int_X -\frac{1}{6} \langle \theta_\phi \wedge [\theta_\phi \wedge \theta_\phi] \rangle.$$

Motivated by this, put

$$c_\Sigma(A, g) = exp\left(2\pi i \int_\Sigma \langle \mathrm{Ad}_{g^{-1}} A \wedge \theta_\phi \rangle + 2\pi i \int_X -\frac{1}{6} \langle \theta_g \wedge [\theta_g \wedge \theta_g] \rangle\right)$$

where $A \in \Omega^1(\Sigma; \mathfrak{g})$ and $g : \Sigma \to G$. Note that this depends only on data on Σ. It is easy to check that c_Σ is a cocycle and so determines a complex line bundle $\mathcal{L}_Q \to \mathcal{A}_Q$, which we call the Chern-Simons line. By definition, given a section $s : X \to P$ the Chern-Simons actions defines a section of unit norm

$$e^{2\pi i S_X(A)} \in L_Q|_{\partial A}$$

where ∂A denotes the connection restricted to Σ. The main result of this section is the following theorem.

Theorem 3.5. *The Chern-Simons action determines a unitary connection on $\mathcal{L}_Q \to \mathcal{A}_Q$ with curvature*

$$\frac{i}{2\pi} \omega(a, b) = -2 \int_\Sigma \langle a \wedge b \rangle \quad \forall a, b \in T_A \mathcal{A}_Q.$$

Moreover, the curvature makes $\mathcal{A}_Q$ into a symplectic affine space. The action of the gauge group $\mathcal{G}_Q$ lifts to $\mathcal{L}_Q$ making it into a $\mathcal{G}_Q$-equivariant line bundle and preserves the metric and connection. The moment map is $\mu(A) = 2F_A$ viewed as an element of $Lie(\mathcal{G}_Q)^\vee$.

Sketch of proof: For full details see [8]. Let A_t be a one-parameter family of connections in $\mathcal{A}_Q$. This determines a connection A on the bundle $Q \times [0, 1] \to \Sigma \times [0, 1]$. We have $\partial A = A_0 \coprod A_1$. Hence

$$e^{2\pi i S_{\Sigma \times [0,1]}(A)} \in \mathcal{L}_{A_0} \otimes \mathcal{L}_{A_1}^\vee \simeq \mathrm{Isom}(\mathcal{L}_{A_0}, \mathcal{L}_{A_1}).$$

Now fix a section $s : \Sigma \to Q$ and hence a trivialization of $\mathcal{L}_Q \to \mathcal{A}_Q$. A calculation shows

$$S_{\Sigma \times [0,1]} s, (A) = -\int_0^1 dt \int_\Sigma s^* \langle A_t \wedge A_t' \rangle$$

where $'$ denotes the derivative with respect to t. Put

$$(\theta_s)_\eta(a) = 2\pi i \int_\Sigma s^* \langle \eta \wedge a \rangle \quad \eta \in \mathcal{A}_Q, \ a \in T_\eta \mathcal{A}_Q.$$

[6]We omit the choice of level in this section.

One can check that θ_s transforms so that it defines a connection on $\mathcal{L}_Q$. Then by definition, the Chern-Simons action is the parallel transport of θ. An easy calculation shows that the curvature of the connection is as claimed. The non-degeneracy of the curvature follows from the non-degeneracy of the bilinear form on $\mathfrak{g}$. Hence $\mathcal{A}_Q$ is symplectic and, being the curvature of a line bundle, its symplectic form satisfies the integrality criterion.

We now compute the moment map of the $\mathcal{G}_Q$ action. Consider more generally the case in which $\pi : P \to M$ is a G-equivariant hermitian line bundle carrying a G-invariant connection ∇ with curvature F_∇, the symplectic form on M. Then a moment map for the G-action is

$$\pi^*\mu(\xi) = \frac{i}{2\pi}\nabla(\xi) \quad \xi \in \mathfrak{g}$$

where on the right hand side we identify $\xi \in \mathfrak{g}$ with the horizontal lift of the vector field on M it determines. Indeed,

$$\pi^* d\mu(\xi) = d\nabla(\xi) = \pi^* F_\nabla(\xi).$$

We use this method to determine the moment map in the case at hand. Working in a trivialization determined by the section $s : \Sigma \to Q$ the infinitesimal parallel transport determined by a gauge transformation $\epsilon \in \Omega^0(\Sigma; \mathfrak{g}_Q)$ is

$$-2\pi i \int_\Sigma \langle \eta \wedge d_\eta \epsilon \rangle.$$

Also, the gauge transformation acting on the fibre is

$$2\pi i \int_\Sigma \langle \eta \wedge d\epsilon \rangle.$$

Taking the difference of the two expressions, which is precisely the vertical component of the infinitesimal gauge transformation, is

$$2 \int_\Sigma \langle F_\eta \epsilon \rangle,$$

which completes our sketch of the proof.

□

Corollary 3.6. *The hermitian line bundle $\mathcal{L}_Q \to \mathcal{A}_Q$ descends (at smooth points) to a hermitian line bundle on $\mathcal{M}_Q$. The induced connection has $\frac{i}{2\pi}$ times its curvature equal to an integral symplectic form on $\mathcal{M}_Q$.*

Thus we have constructed, canonically from the Chern-Simons action, a classical phase space. However, we cannot complete the quantization of $\mathcal{M}_Q$ as there is no complex structure present. Indeed, the polarization should be viewed as some additional geometric data needed to define the quantization.

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