MAT 322: THE LIST

We will be covering essentially the entire book *Analysis on manifolds* by James Munkres, with the exception of a small handful of sections. This book is fairly thick. Here, we will reduce the book down to **terms** whose definitions should become automatic, and roughly **50 theorems** whose proofs you are expected to master. Along with the definition of the terms listed, you should be familiar with the basic properties and minor theorems related to them. You should also know relevant examples, for example showing that the hypothesis of a theorem cannot be weakened, and be able to do relevant computations such as those in the homework exercises. (These aren't listed here.) Some of the proofs are fairly lengthy. On a test, you would likely only be asked to prove one of the main subclaims needed to prove the theorem.

Section 1

- vector, vector space
- span, linear combination, linear independence, basis, dimension, standard basis
- inner product, inner product space, dot product
- norm, Euclidean norm, sup norm, triangle inequality
- matrix, entry, size, row index, column index
- matrix multiplication, product, identity matrix
- linear transformation, linear isomorphism, row matrix, column matrix, transpose
- column space, column rank, row space, row rank, rank
- Gauss–Jordan reduction, elementary row operations, echelon form, reduced echelon form

Section 2

- elementary matrix, left inverse, right inverse, inverse, invertible
- singular, non-singular
- determinant function
- minor, cofactor, Cramer's rule

Theorem 1. A matrix A of size n by m is invertible if and only if $n = m = \operatorname{rank} A$.

Theorem 2. The determinant function is uniquely characterized by its three axioms.

Section 3

- metric, metric space, subspace, Euclidean metric, sup metric
- ε-neighborhood, open, closed, neighborhood, limit point, closure
- open ball, open cube, topological property
- continuous at a point, continuous, isolated point, limit, interior, exterior, boundary, rectangle, open/closed rectangle/cube, cube

Section 4

- covering, open covering, compact, bounded, uniform continuity, connected, convex
- Extreme value theorem, intermediate value theorem

Theorem 3. A subspace of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

Theorem 4. Let X be a compact subspace of $\mathbb{R}^n$, and let $U \subset \mathbb{R}^n$ be an open set containing X. There exists $\varepsilon > 0$ such that the ε-neighborhood of X is contained in U.

Theorem 5. Let X be a compact metric space and $f\colon X \to \mathbb{R}^n$ a continuous function. Then f is uniformly continuous.

Section 5

- derivative, differentiable, partial derivative, directional derivative, Jacobian
- parametrized curve, velocity vector, scalar field, gradient

Section 6

- Mean value theorem, continuously differentiable, of class C^1
- second-order partial derivatives, of class C^r, of clas C^∞

Theorem 6. Let A be an open set in $\mathbb{R}^n$. Let $f\colon A \to \mathbb{R}^m$ be a function such that the partial derivatives $D_j f_i$ of the component functions of f exist at each point in A and are continuous in A. Then f is differentiable at each point of A.

Theorem 7. Let $A \subset \mathbb{R}^m$ be open and let $f\colon A \to \mathbb{R}$ be a function of class C^2. Then for each $a \in A$ and $j, k \in \{1, \ldots, m\}$,

$$D_k D_j f(a) = D_j D_k f(a).$$

Section 7

- chain rule
- mean value theorem

Theorem 8. Let $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^n$. Let $f\colon A \to \mathbb{R}^n$ and $g\colon B \to \mathbb{R}^p$ be functions with $f(A) \subset B$. Suppose $f(a) = b$. If f is differentiable at a, and if g is differentiable at b, then the composite function $g \circ f$ is differentiable at a. Furthermore,

$$D(g \circ f)(a) = Dg(b) \cdot Df(a).$$

Section 8

Theorem 9. Let A be open in $\mathbb{R}^n$ and let $f\colon A \to \mathbb{R}^n$ be of class C^1. If $Df(a)$ is non-singular, then there exists $\varepsilon > 0$, $\alpha > 0$ such that

$$|f(x_0) - f(x_1)| \geq \alpha |x_0 - x_1|$$

for all x_0, x_1 in the open cube $C(a; \varepsilon)$.

Theorem 10. Let A be open in $\mathbb{R}^n$ and let $f\colon A \to \mathbb{R}^n$ be of class C^r. Let $B = f(A)$. If f is one-to-one on A and if $Df(x)$ is non-singular for some $x \in A$, then the set B is open in $\mathbb{R}^n$ and the inverse function $g\colon B \to A$ is of class C^r.

Section 9

Theorem 11. Let A be open in $\mathbb{R}^{k+n}$ and let $f\colon A \to \mathbb{R}^n$ be of class C^r. Write f in the form $f(x, y)$ for $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^n$. Suppose that (a, b) is a point of A such that $f(a, b) = 0$ and

$$\det \frac{\partial f}{\partial y}(a, b) \neq 0.$$

Then there is a neighborhood B of a in $\mathbb{R}^k$ and a unique continuous function $g\colon B \to \mathbb{R}^n$ such that $g(a) = b$ and $f(x, g(x)) = 0$ for all $x \in B$. The function g is of class C^r.

Section 10

- component interval, width, volume, length, partition, subrectangle, lower sum, upper sum, common refinement
- lower integral, upper integral, integrable, integral, Riemann integral

Theorem 12. Let Q be a rectangle and let $f\colon Q \to \mathbb{R}$ be a bounded function. Then

$$\underline{\int_Q} f \leq \overline{\int_Q} f.$$

Equality holds if and only if for all $\varepsilon > 0$ there exists a corresponding partition P of Q for which $U(f,P) - L(f,P) < \varepsilon$.

Section 11

- measure zero

Theorem 13. Let Q be a rectangle in $\mathbb{R}^n$ and let $f\colon Q \to \mathbb{R}$ be a bounded function. Let D be the set of points of Q at which f fails to be continuous. Then $\int_Q f$ exists if and only if D has measure zero in $\mathbb{R}^n$.

Section 12

- fundamental theorem of calculus, antiderivative

Theorem 14. Let $Q = A \times B$, where A is a rectangle in $\mathbb{R}^k$ and B is a rectangle in $\mathbb{R}^n$. Let $f\colon Q \to \mathbb{R}$ be a bounded function; write f in the form $f(x,y)$ for $x \in A$ and $y \in B$. If f is integrable over Q, then $\underline{\int_{y\in B}} f(x,y)$ and $\overline{\int_{y\in B}} f(x,y)$ are integrable over A as functions of x and

$$\int_Q f = \int_{x\in A} \underline{\int_{y\in B}} f(x,y) = \int_{x\in A} \overline{\int_{y\in B}} f(x,y).$$

Section 13

- integral over a bounded set, linearity/comparison/monotonicity/additivity of the integral

Section 14

- rectifiable set, volume, (Jordan) content, simple region

Theorem 15. If S is a simple region in $\mathbb{R}^n$, then S is compact and rectifiable.

Section 16

- partition of unity

Theorem 16. Let $\mathcal{A}$ be a collection of open sets and A their union. There exists a partition of unity of A subordinate to the cover $\mathcal{A}$.

Section 17

- change of variables, diffeomorphism

Theorem 17. Let $g\colon A \to B$ be a diffeomorphism of open sets in $\mathbb{R}^n$. Let $f\colon B \to \mathbb{R}$ be a continuous function. Then f is integrable over B if and only if the function $(f \circ g)|\det Dg|$ is integrable over A. In this case,

$$\int_B f = \int_A (f \circ g)|\det Dg|.$$

Section 18

- primitive diffeomorphism

Theorem 18. Let $A \subset \mathbb{R}^n$ be open and let $g \colon A \to \mathbb{R}^n$ be a function of class C^1. If the subset E of A has measure zero in $\mathbb{R}^n$, then the set $g(E)$ also has measure zero in $\mathbb{R}^n$.

Theorem 19. Let $g \colon A \to B$ be a diffeomorphism of open sets in $\mathbb{R}^n$, where $n \geq 2$. Given $a \in A$, there is a neighborhood U_0 of a contained in A and a sequence of diffeomorphisms of open sets in $\mathbb{R}^n$

$$U_0 \xrightarrow{h_1} U_1 \xrightarrow{h_2} U_2 \longrightarrow \cdots \xrightarrow{h_k} U_k$$

such that the composite $h_k \circ \cdots \circ h_2 \circ h_1$ equals $g|U_0$ and such that each h_i is a primitive diffeomorphism.

Section 20

- parallelopiped, n-frame, right-handed, orientation, orientation-preserving
- orthogonal set, orthonormal set, orthogonal matrix, orthogonal transformation, Euclidean isometry

Theorem 20. Let A be an $n \times n$ matrix. Let $h \colon \mathbb{R}^n \to \mathbb{R}^n$ be the linear transformation $h(x) = A \cdot x$. Let S be a rectifiable set in $\mathbb{R}^n$, and let $T = h(S)$. Then $v(T) = |\det A| \cdot v(S)$.

Theorem 21. Let $h \colon \mathbb{R}^n \to \mathbb{R}^n$. Then h is an isometry if and only if h has the form

$$h(x) = A \cdot x + p,$$

where A is an orthogonal matrix.