Sample Final Exam Answers (Summer 2018)

1. We prove the proposition by induction on n. In the base case $n = 1$, the given equation becomes $2^1 = 2^2 - 2$, which is true. For the inductive step, we assume that the proposition holds for $n = k$ for some $k \in \mathbb{N}$. We prove the case where $n = k + 1$. Applying the inductive hypothesis, we have

$$2^1 + 2^2 + \cdots + 2^k + 2^{k+1} = (2^{k+1} - 2) + 2^{k+1} = 2 \cdot 2^{k+1} - 2 = 2^{k+2} - 2.$$

This verifies the equation for $n = k + 1$ and completes the inductive step.

2. (a) f is not injective since $f(0, 1) = f(0, 2) = 0$.
(b) f is surjective since, for all $t \in \mathbb{R}$, $(t, 1) \in \mathbb{R} \times \mathbb{R}_{>0}$ $f(t, 1) = t$.

3. There are $6 \cdot 3 \cdot 5 = 90$ possible meals.

4. (a) False: take $r = 0$ (or any $r < 1$)
(b) False: let $r = 0$. Then $rx = 0 \in \mathbb{Q}$ for all $x \in \mathbb{R}$
(c) True: let $y = -x$

5. We start with the base case $n = 1$. Then we have $F_2^2 + F_1F_3 = 1 - 1 \cdot 2 = (-1)^1$. This verifies the base case.

Next, assume the proposition is true for $n = k$ for some $k \in \mathbb{N}$. Then

$$\begin{aligned}
F_{k+2}^2 - F_{k+1}F_{k+3} &= F_{k+2}^2 - F_{k+1}(F_{k+1} + F_{k+2}) \\
&= F_{k+2}^2 - F_{k+1}^2 + F_{k+1}F_{k+2} \\
&= F_{k+2}^2 - (-1)^k - F_kF_{k+2} + F_{k+1}F_{k+2} \text{ (by the inductive hypothesis)} \\
&= F_{k+2}^2 - (-1)^k - (F_{k+2} - F_{k+1})F_{k+2} - F_{k+1}F_{k+2} \\
&= F_{k+2}^2 - (-1)^k - F_{k+2}^2 + F_{k+1}F_{k+2} - F_{k+1}F_{k+2} \\
&= -(-1)^k = (-1)^{k+1}.
\end{aligned}$$

This completes the inductive step. [Note: This is a more difficult problem. The key is to play around with the rule for the Fibonacci sequence $F_{n+1} = F_n + F_{n-1}$ until you can manipulate the left-hand side into $(-1)^{k+1}$. There are probably several ways to do this.]

6. There are 9 students taking all three classes. Let a be the number of students taking analysis and algebra, b the number of students taking algebra and topology, c the number of students taking topology and analysis, and d the number of students taking all three. Then we have $a + c + d = 40 - 12 = 28$, $a + b + d = 40 - 13 = 27$, $b + c + d = 40 - 14 = 26$, and $a + b + c + d = 75 - 12 - 13 - 14 = 36$ (it might help to make a Venn diagram). Subtracting each of the first three equations from the last gives, respectively, $b = 8$, $c = 9$. $a = 10$. From this, the last equation gives $d = 9$.

7. Let X be a subset of $Y = \{1, 2, \ldots, 10\}$ having cardinality 6. Pair the elements in Y as $Y_1 = \{1, 10\}$, $Y_2 = \{2, 9\}$, $Y_3 = \{3, 8\}$, $Y_4 = \{4, 7\}$, $Y_5 = \{5, 6\}$. Since we are chosing six elements from the set $\{1, 2, \ldots, 10\}$, and there are five pairs $Y_1, \ldots, Y_5$, the Pigeonhole Principle implies that two elements must come from the same pair Y_j. Then the two numbers in Y_j are in our set and sum to 11.

8. See p. 142 in the textbook.

9. Let $(a, b), (c, d), (e, f) \in \mathbb{R}^2$. First, since $a^2 + b^2 = a^2 + b^2$, we have $(a, b) \sim (a, b)$. Thus $\sim$ is reflexive. Next, if $a^2 + b^2 = c^2 + d^2$, then $c^2 + d^2 = a^2 + b^2$. Thus $(a, b) \sim (c, d)$ implies $(c, d) \sim (a, b)$, so $\sim$ is symmetric. Finally, if $a^2 + b^2 = c^2 + d^2$ and $c^2 + d^2 = e^2 + f^2$, then $a^2 + b^2 = e^2 + f^2$. Thus, $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$ imply that $(a, b) \sim (e, f)$. This shows that $\sim$ is transitive.

The only element of $\mathbb{R}^2$ equivalent to $(0, 0)$ is $(0, 0)$ itself, since $a^2 + b^2 = 0$ implies $a = b = 0$.

10. Observe that $12 \equiv 50 \mod 19$. Thus the given equation is equivalent to $5x \equiv 50 \mod 19$. Since $\gcd(5, 19) = 1$, this equation is equivalent to $x \equiv 10 \mod 19$. So $x = 10 + 19q$ for some $q \in \mathbb{Z}$.