MAT 322: MIDTERM 2 LIST

The test will consist of eight short answer/computational problems (8×3 points) and one proof (16 points). The short answer problems will focus on definitions and statements of theorems, computations, relevant examples, homework problems, and the relationship between the various concepts. See below for the pool of specific topics. The sample exams should give a good idea of what to expect. The proof problem will be one of the three listed here, each of which contains one piece of the proof of the change of variables theorem.

As a general rule, you can use facts that appear earlier in the book freely as needed. While any correct proof is of course acceptable, certainly the proofs as presented in the book will be adequate. The theorems listed here may be slightly different than what is in the book. Of course, you only need to prove what is asked by the problem.

Given the format of this test, no notes or other resources will be allowed.

Theorems:

1. Let $\mathcal{A}$ be a collection of open sets that cover $\mathbb{R}^n$. There exists a sequence $\phi_1, \phi_2, \ldots$ of smooth functions $\phi_i \colon \mathbb{R}^n \to \mathbb{R}$ (of class C^∞) such that the following are satisfied (let S_i denote the support of ϕ_i):
   (1) $\phi_i(x) \geq 0$ for all $x \in \mathbb{R}^n$ for each $i \in \mathbb{N}$.
   (2) Each set S_i is compact and contained in an element of $\mathcal{A}$.
   (3) Each point of A has a neighborhood that intersects finitely many of the sets S_i.
   (4) $\sum_{i=1}^\infty \phi_i(x) = 1$ for all $x \in \mathbb{R}^n$.
   Notes:
   - You may assume Lemma 16.1: for any rectangle $Q \subset \mathbb{R}^n$, there is a function $\phi \colon \mathbb{R}^n \to \mathbb{R}$ of class C^∞ such that $\phi(x) > 0$ for all $x \in \operatorname{Int} Q$ and $\phi(x) = 0$ otherwise.
   - Do not assume Lemma 16.2.
2. Let $A, B \subset \mathbb{R}^n$ be open sets and $g \colon A \to B$ be a diffeomorphism, where $n \geq 2$. For all $a \in A$, there is a neighborhood U_0 of a contained in A and a sequence of diffeomorphisms
$$U_0 \xrightarrow{h_1} U_1 \xrightarrow{h_2} U_2 \longrightarrow \cdots \xrightarrow{h_k} U_k,$$
   where each $U_j \subset \mathbb{R}^n$ is open, such that the composite $h_k \circ \cdots \circ h_2 \circ h_1$ equals $g|U_0$ and such that each h_i is a primitive diffeomorphism.
3. **(CoV)** Let $g \colon A \to B$ be a diffeomorphism, where $A, B \subset \mathbb{R}^n$ are open sets. Then for every continuous function $f \colon B \to \mathbb{R}$ that is integrable over B, the function $(f \circ g)|\det Dg|$ is integrable over A and $\int_B f = \int_A (f \circ g)|\det Dg|$.
   Prove the inductive step of (Cov). Namely, assume that (CoV) holds in dimension $n-1$. Prove that (CoV) holds for each primitive diffeomorphism $f \colon U \to V$, where U, V are open sets in $\mathbb{R}^n$.

Short answer topic pool:

- smooth bump functions (Lemma 16.1), partition of unity
- change of variables, diffeomorphism, change of variables theorem
- polar coordinates, other computational examples
- diffeomorphisms preserve sets of measure zero

- primitive diffeomorphism
- parallelopiped, geometric meaning of determinant
- orientation, orientation-preserving maps
- orthogonal matrix, orthogonal group, classification of isometries of Euclidean space
- volume of a parallelopiped
- parametrized manifold, integral over a parametrized manifold (with respect to volume), volume of a parametrized manifold
- k-manifold, k-manifold without boundary, coordinate chart/patch
- examples and non-examples of manifolds
- interior and boundary of a manifold
- level sets/superlevel sets of functions of class C^r are manifolds (Theorem 24.4)
- integrating a scalar function over a manifold
- multilinear function, k-tensor
- elementary k-tensor, representation of tensors as linear combinations of elementary k-tensors, manipulation of tensors
- tensor product $\otimes$, its algebraic properties
- dual transformation T^*
- permutation, symmetric group, elementary permutation
- inversion, sign of a permutation
- alternating tensor, elementary alternating tensor
- wedge product $\wedge$, its algebraic properties, the map A
- tangent vector, velocity vector, tangent space, tangent bundle
- tensor field, differential form