MAT 322 Analysis in Several Dimensions
Midterm II Spring 2015

Name:____

I.D.:____

No calculators, computers, tablets or smartphones can be used on this test. Answer each question in the space provided and on the reverse side of the sheets. Write carefully and show your work: no credit will be given for unjustified answers.

1	2	3	4	5	EC	SUM
20 pts	20 pts	20 pts	20 pts	20 pts	20 pts	120 pts

1. (a) (5 points) Find the area of the region in $\mathbb{R}^2$ bounded by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

(b) (15 points) Let A be the portion of the first quadrant in $\mathbb{R}^2$ lying between the hyperbolas $xy = 1$ and $xy = 2$ and the two straight lines $y = x$ and $y = 4x$. Evaluate

$$\int_A x^2 y^3 dxdy.$$

Hint: Set $x = u/v$ and $y = uv$.

2. (a) (7 points) Let $\mathcal{P} = \mathcal{P}(\mathbf{a}, \mathbf{b})$ be a parallelogram in $\mathbb{R}^n$ spanned by the vectors

$$\mathbf{a} = a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n \quad \text{and} \quad \mathbf{b} = b_1\mathbf{e}_1 + \cdots + b_n\mathbf{e}_n,$$

where $\mathbf{e}_1, \ldots, \mathbf{e}_n$ is the standard basis in $\mathbb{R}^n$. Find $V(\mathcal{P})$.

(b) (13 points) Let A be open in $\mathbb{R}^2$ and $f \in C^1(A, \mathbb{R})$. Let Y be the graph of f in $\mathbb{R}^3$ — a parameterized manifold in $\mathbb{R}^3$ with $\alpha(x, y) = (x, y, f(x, y))$ for $(x, y) \in A$. Express $v(Y)$ as an integral.

3. (a) (6 points) Let $\alpha : \mathbb{R} \to \mathbb{R}^2$ be the map $\alpha(x) = (x, x^3)$. Show that $M = \alpha(\mathbb{R})$ is a 1-manifold in $\mathbb{R}^2$ covered by a single coordinate patch α.

(b) (7 points) Let $\alpha : \mathbb{R}^2 \to \mathbb{R}^3$ be the map

$$\alpha(x, y) = (x(x^2 + y^2), y(x^2 + y^2), x^2 + y^2).$$

Is $M = \alpha(\mathbb{R}^2)$ a 2-manifold in $\mathbb{R}^3$ with a coordinate patch α?

(c) (7 points) Repeat part (b) for the map

$$\alpha(x, y) = (x, y, x^2 + y^2).$$

4. (a) (5 points) Let $\mathcal{O}$ be open in $\mathbb{R}^n$ and let $f \in C^r(\mathcal{O}, \mathbb{R})$. Denote by M the set of points in $\mathbb{R}^n$ for which $f(\mathbf{x}) = 0$ and by N — the set of points for which $f(\mathbf{x}) \geq 0$. Assuming that M is non-empty, state a condition on f which guarantees that N is an n-manifold in $\mathbb{R}^n$ and $M = \partial N$.

(b) (10 points) Let $B^n(a) = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| \leq a\}$ be the closed ball of radius a in $\mathbb{R}^n$ and $S^{n-1}(a) = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| = a\}$ — the $(n-1)$-sphere in $\mathbb{R}^n$. Prove that the ball $B^n(a)$ is an n-manifold in $\mathbb{R}^n$ and $\partial B^n(a) = S^{n-1}(a)$.
Hint: Use part (a).

(c) (5 points) Show that $S^{n-1}_+(a) = S^{n-1}(a) \cap \mathbb{H}^n$ is an $(n-1)$-manifold in $\mathbb{R}^n$ and find its boundary.

5. (a) (5 points) Determine which of the following are 2-tensors on $\mathbb{R}^3$ and express those who are in terms of elementary 2-tensors $\phi_{i,j} = \phi_i \otimes \phi_j$ corresponding to the standard basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ on $\mathbb{R}^3$:

$$f(\mathbf{x}, \mathbf{y}) = x_1 x_2 + x_2 y_1 + x_3 y_2,$$
$$g(\mathbf{x}, \mathbf{y}) = x_1 y_1 + x_2 y_2 + 1,$$
$$h(\mathbf{x}, \mathbf{y}) = x_1 y_1 + x_1 y_3 + 10 x_2 y_3 - 2 x_3 y_1 + x_3 y_3.$$

(b) (5 points) Let f and g be the following tensors on $\mathbb{R}^3$: $f(\mathbf{x}, \mathbf{y}, \mathbf{z}) = x_1 y_2 z_3$ and $g = \phi_{1,1} - \phi_{1,3} + \phi_{2,3}$. Express $f \otimes g$ as a linear combination of elementary tensors and as a multilinear function $(f \otimes g)(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{u}, \mathbf{v})$.

(c) (5 points) Determine which of the following are alternating 2-tensors on $\mathbb{R}^3$ and express those who are in terms of elementary alternating tensors :

$$f(\mathbf{x}, \mathbf{y}) = x_1 y_3 - x_1 y_2 + x_2 y_1,$$
$$g(\mathbf{x}, \mathbf{y}) = x_1 y_3 - x_3 y_1,$$
$$h(\mathbf{x}, \mathbf{y}) = x_1^2 y_2^2 - x_2^2 y_1^2.$$

(d) (5 points) Let

$$f = \phi_1 + \phi_2 + \phi_3 \quad \text{and} \quad g = \phi_1 \wedge \phi_2 + \phi_1 \wedge \phi_3 + \phi_2 \wedge \phi_3$$

be alternating tensors on $\mathbb{R}^3$. Find $f \wedge g$ as a linear combination of elementary alternating tensors and as a multilinear function $(f \wedge g)(\mathbf{x}, \mathbf{y}, \mathbf{z})$.

Extra Credit (a) (5 points) A k-tensor T on a vector space V is called *symmetric*, if $T^\sigma = T$ for all permutations $\sigma \in S_k$. Prove that a 2-tensor T is symmetric if and only if $\operatorname{Alt} T = 0$.

(b) (5 points) Prove that any 2-tensor F on V can be uniquely written as

$$F = S + A,$$

where S is a symmetric 2-tensor, and A is an alternating 2-tensor.

(c) (10 points) Let A be open in $\mathbb{R}^n$ and $f \in C^1(A, \mathbb{R})$. Let Y be the graph of f in $\mathbb{R}^{n+1}$ — a parameterized manifold in $\mathbb{R}^{n+1}$ with $\alpha(\mathbf{x}) = (\mathbf{x}, f(\mathbf{x}))$, $\mathbf{x} \in A$. Express $v(Y)$ as an integral. (This is extension of Problem 2 (b) to general n).