Practice Exam 3 Key

Multiple choice:
1. F 2. F 3. T 4. T 5. T 6. T 7. F 8. F 9. T 10. T 11. T 12. F 13. F
Comments:

1. For this problem, it helps to write it out symbolically. The first statement is written as: $\forall a \in \mathbb{R},\ a \leq 0 \Longrightarrow f(a) \neq 0$. We can logically manipulate the negation of the first statement in this way:

$$\begin{aligned} & -(\forall a \in \mathbb{R}, a \leq 0 \Longrightarrow f(a) \neq 0) \\ \Longleftrightarrow \quad & \exists a \in \mathbb{R}, -(a \leq 0 \Longrightarrow f(a) \neq 0) \\ \Longleftrightarrow \quad & \exists a \in \mathbb{R}, a \leq 0 \text{ and } f(a) = 0 \end{aligned}$$

   Notice how we handled the negation of the conditional ($\Longrightarrow$) in the second line. At this point, it should be apparent that the final line is different than the second statement in the problem. Specifically, you can consider a function f satisfying $f(0) = 0$.

2. Again, it helps to write it out symbolically. The first statement can be written as: $\exists a \in \mathbb{R}$, $a \leq 0$ and $f(a) = 0$. We can logically manipulate the negation of the first statement in this way:

$$\begin{aligned} & -(\exists a \in \mathbb{R}, a \leq 0 \text{ and } f(a) = 0) \\ \Longleftrightarrow \quad & \forall a \in \mathbb{R}, -(a \leq 0 \text{ and } f(a) = 0) \\ \Longleftrightarrow \quad & \forall a \in \mathbb{R}, a > 0 \text{ or } f(a) \neq 0 \\ \Longleftrightarrow \quad & \forall a \in \mathbb{R}, f(a) = 0 \Longrightarrow a > 0 \end{aligned}$$

   This is similar to the second statement but not the same. If "negative" was replaced by "positive" in the second statement, the answer to this one would be "True".

3. These are contrapositives.

4. Make a truth table to check this.

5. See Theorem 6.3.4.

6. Compare this to Proposition 6.3.5.

7. We can take the constant function $f(x) = 0$.

8. The idea is that order matters in the definition of an *ordered pair*.

9. We can take $X = Y$.

10. [Advanced] This collection is too "big" to be a set, since there is no "set of all sets". Observe that it doesn't follow our methods for defining sets (the *conditional definition* and the *constructive definition*)

11. [Advanced] This problem is somewhat ambiguous. I consider it "True". The reason is that a function $f\colon X \to Y$ can be understood as an element of $\mathcal{P}(X \times Y)$ (the power set of the

Cartesian product $X \times Y$) satisfying a certain condition, namely that each $x \in X$ appears in exactly one ordered pair $(x, y) \in f$. So our collection can be written as

$$\{f \in \mathcal{P}(\mathbb{R}, \mathbb{R}) : f \text{ is a function, and } f(x) = f(-x) \text{ for all } x \in (0, \infty)\},$$

which does fit the conditional definition of a set.

12. [Advanced] I interpret the collection as

$$\{f \in \mathcal{P}(\mathbb{R}, \mathbb{R}) : f \text{ is a function, and } f(x) \neq f(x) \text{ for all } x \in (0, \infty)\},$$

which is a set containing no elements, hence the empty set $\emptyset$.

13. The order of the composition is backwards.

Proofs:
2. We give a proof by cases. There are five cases to consider:

Case 1. Suppose that $x < y < 0$. Then the multiplication law gives $0 < xy < x^2$ and $0 < y^2 < xy$. Applying the multiplication law a second time gives $x^3 < x^2y$, and $xy^2 < y^3$. Moreover, since $0 < xy$, the multiplication law also gives $x^2y < xy^2$. Combining these gives $x^3 < y^3$.

Case 2. Suppose that $x < y = 0$. Then $0 < x^2$ and $x^3 < 0$ by the multiplication law. But $y^3 = 0$, so $x^3 < y^3$.

Case 3. Suppose that $x < 0 < y$. Then the multiplication law gives $x^3 < 0$ and $0 < y^3$, so $x^3 < y^3$.

Case 4. Suppose that $x = 0 < y$. Then $0 < y^2$ and $0 < y^3$ by the multiplication law. But $x^3 = 0$, so $x^3 < y^3$.

Case 5. Suppose that $0 < x < y$. Then the multiplication law gives $0 < x^2 < xy$ and $0 < xy < y^2$. Applying the multiplication law a second time gives $0 < x^3 < x^2y$ and $0 < xy^2 < y^3$. Since $0 < xy$, the multiplication law also gives $x^2y < xy^2$. Combining these gives $x^3 < y^3$.

In all cases, we have shown that $x^3 < y^3$.

[Note: The idea here is that the multiplication law in the textbook has two cases depending on whether x (or y) is positive or negative. This suggests a proof using cases. The main difficulty is to make sure each case is covered. It's not necessary to write the phrase "multiplication law" as long as you apply it correctly. It's quite likely you can find a more efficient proof than the one here.]

3. [Note: I believe this is a technically flawed problem. As far as I can tell, solving this problem requires the fact that a number $n \in \mathbb{Z}$ is odd if and only if it can be written in the form $n = 2q + 1$ for some $q \in \mathbb{Z}$ (or at least the "only if" part of this statement). This is actually Proposition 11.3.4 on p. 142 of the book, which suggests that it should not be considered available right now. Basically, the only way to really solve the problem directly is to prove Proposition 11.3.4 along the way, which I think is more involved than our current level. (On the other hand, note that a statement like "if n^2 is odd, then n is odd" **is** perfectly provable with the current material.)]

We prove this by contrapositive. Assume that n is odd. By Proposition 11.3.4, n can be written in the form $n = 2q + 1$ for some $q \in \mathbb{Z}$. Then $n^2 = (2q + 1)^2 = 4q^2 + 4q + 1 = 2(2q^2 + 2q) + 1$.

Assuming Proposition 11.3.4, this implies that n^2 is odd. Alternatively, we can reason as in Proposition 2.2.4 in our book. Namely, if r is an integer, then either $r \leq 2q^2 + 2q$ or $r \geq 2q^2 + 2q + 1$. In the first case, $2r \leq 2(2q^2 + 2q) < n^2$. In the second case, $2r \geq 2(2q^2 + 2q) + 2 < n^2$. Thus $2r \neq n^2$ for all integers r. We conclude that n^2 is odd.

4. First, we check the case where $n = 1$. The equation reduces to $1/2 = 1/2$, which is true.
   Next, we do the inductive step. Assume that the equation is true for $n = k$. Then

$$\begin{aligned}\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} &= \frac{k(k+2)+1}{(k+1)(k+2)} \\ &= \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}.\end{aligned}$$

This completes the inductive step.