

# Combinatorial Lie bialgebras of curves on surfaces

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**Abstract** Goldman [7] and Turaev [11] found a Lie bialgebra structure on the vector space generated by non-trivial free homotopy classes of curves on a surface. When the surface has non-empty boundary, this vector space has a basis of cyclic reduced words in the generators of the fundamental group and their inverses. We give a combinatorial algorithm to compute this Lie bialgebra on this vector space of cyclic words. Using this presentation, we prove a variant of Goldman's result relating the bracket to disjointness of curve representatives when one of the classes is simple. We exhibit some examples we found by programming the algorithm which answer negatively Turaev's question about the characterization of simple curves in terms of the cobracket. Further computations suggest an alternative characterization of simple curves in terms of the bracket of a curve and its inverse. Turaev's question is still open in genus zero.

## 1 Introduction

A Lie bialgebra structure on vector space  $W$  consists in two linear operations, a bracket from  $W \otimes W$  to  $W$  and a cobracket, from  $W$  to  $W \otimes W$ , satisfying certain identities (see Appendix A). Goldman [7] and Turaev [11] found, in stages, a Lie bialgebra structure on the vector space generated by all non-trivial free homotopy classes of curves on an orientable surface. The desire to understand better the beautiful structure of Goldman and Turaev and to answer some of the questions posed by them motivated this work. The Lie algebra of Goldman, as well as the Goldman-Turaev Lie bialgebra, can be generalized via "String topology" to manifolds of all dimensions, see [2] and [3].

Here,

- we give explicit presentations of the Goldman-Turaev Lie bialgebra of curves on a surface with boundary, (there is one for each surface symbol, see Section 2).

These presentations define purely combinatorial Lie bialgebra structures on the vector space of reduced cyclic words on certain alphabets and, therefore, give algorithms to compute algebraically the bracket and cobracket. These algorithms can be programmed

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and so we did, finding examples which answer certain questions about the Goldman-Turaev Lie bialgebra we describe now.

Goldman [7] showed that if the bracket of the two free homotopy classes is zero, and one of them has a simple representative, then these classes have disjoint representatives. His proof uses Kerckhoff's convexity property of Teichmuller space [8]. We extend this result by showing that

- the number of terms of the bracket of two classes, one of them simple and non-homologous to zero, equals the minimum number of intersection points of these classes (Theorem 5.3).

**Theorem 5.3** *Let  $\mathcal{V}$  and  $\mathcal{W}$  be cyclic reduced words and such that  $\mathcal{V}$  has a simple representative which is non-homologous to zero. Then there exists two representatives  $\alpha$  and  $\beta$  of  $\mathcal{V}$  and  $\mathcal{W}$  respectively such that the bracket of  $\mathcal{V}$  and  $\mathcal{W}$  computed using the intersection points of  $\alpha$  and  $\beta$  does not have cancellation. In other words, the number of terms (counted with multiplicity) of  $\langle \mathcal{V}, \mathcal{W} \rangle$  equals the minimal number of intersection points of representatives of  $\mathcal{V}$  and  $\mathcal{W}$ .*

On the last page of [7], Goldman asked whether it would be possible to replace Kerckhoff convexity by a topological argument,

- the proof here of the variant of Goldman's result is essentially topological.

The hypothesis that one of the classes is simple cannot be omitted. By running our program, we found out that

- there exist pairs of distinct classes, which are not multiples of simple curves, have bracket zero and do not have disjoint representatives (Example 5.5).

Goldman recently found these examples independently (email communication), see also the last page of [7].

In [11], Turaev formulated a statement dual to that of Goldman's, that is, if the cobracket of a class is zero, then the class is a multiple of a simple curve and asked whether this is true. Again, by the aid of the computer, we found that

- in every surface of negative Euler characteristic and positive genus there exists classes with cobracket zero which are not multiples of simple curves (Example 5.7, Figure 1).
- Turaev's conjectural characterization of simple curves is still possible for genus zero surfaces.

$E_{a_1}$

$E_{a_2}$

Figure 1: A representative of  $c(a_1a_1a_2a_2)$  in the punctured torus

- a possible replacement for Turaev's condition in surfaces of all genus is that multiples of simple curves are characterized by the vanishing of the bracket of a class with its inverse. Moreover, the output of our program suggests a stronger statement: the number of terms of the bracket of a primitive class with its inverse is twice the minimal number of self-intersection points of the class.

In fact we have quantitative results about the last two questions.

**Theorem 5.9** (1) *On the sphere with three punctures all the cyclic words with at most sixteen letters, except the multiples of the three peripheral curves, have non-zero cobracket.*

(2) *On the torus with two punctures all the cyclic words  $\alpha$  with at most fifteen letters have the property that twice the minimal number of self-intersection points equals the number of terms of the bracket  $[\alpha, \bar{\alpha}]$  in the natural basis.*

Some examples we have computed suggest that even a more general result may hold:

**Question.** *Let  $n$  and  $m$  be two different non-zero integers and let  $\mathcal{V}$  be a primitive reduced cyclic word. Is the number of terms of the bracket of  $\mathcal{V}^n$  with  $\mathcal{V}^m$  equal to  $2|m \cdot n|$  multiplied by the minimal number of self-intersection points of  $\mathcal{V}$ ?*

Now we wonder about the possible implications of the computer program for counterexamples in three-manifold theory using complicated collections of disjoint simple curves to generate Heegard decompositions.

A new version of the programs to compute bracket and cobracket, and to compute intersection numbers of curves on orientable surfaces with boundary can be found in the author's homepage: <http://www.math.sunysb.edu/~moira/>

The rest of the paper is organized as follows. In Section 2, we define  $\mathbb{V}$ , the vector space of reduced cyclic words on certain alphabets, fix a surface symbol, and associate to each reduced cyclic word  $\mathcal{V}$ , a certain subset  $\mathbf{LP}_1(\mathcal{V})$  of pairs of subwords which will play a key role later. Analogously, to each pair of reduced cyclic words  $\mathcal{V}$  and  $\mathcal{W}$

we associate a subset  $\mathbf{LP}_2(\mathcal{V}, \mathcal{W})$  of pairs of the form (subword of  $\mathcal{V}$ , subword of  $\mathcal{W}$ ). Using  $\mathbf{LP}_1(\mathcal{V})$  we define a linear map from  $\mathbb{V}$  to  $\mathbb{V} \otimes \mathbb{V}$  and using  $\mathbf{LP}_2(\mathcal{V}, \mathcal{W})$  we define a linear map from  $\mathbb{V} \otimes \mathbb{V}$  to  $\mathbb{V}$ .

Free homotopy classes of curves on a surface with boundary, the latter described by a surface symbol in an appropriate alphabet, are in one-to-one correspondence with cyclic reduced words in that alphabet, that is, with the basis of  $\mathbb{V}$ . Since the Goldman-Turaev Lie bialgebra is defined using intersection points of representing geometric curves, in Section 3 we study the relation between cyclic words and intersection points of certain representatives. More precisely, we show that each primitive cyclic word  $\mathcal{V}$  in  $\mathbb{V}$  has a representative such that its self-intersection points are in one-to-one correspondence with  $\mathbf{LP}_1(\mathcal{V})$  quotiented by an involution (Theorem 3.9). Furthermore, for each pair of cyclic words  $\mathcal{V}, \mathcal{W}$  there exists a pair of representatives such that the intersection points of  $\mathcal{V}$  and  $\mathcal{W}$  are in one-to-one correspondence with elements of  $\mathbf{LP}_2(\mathcal{V}, \mathcal{W})$  (Theorem 3.13).

Using the correspondence between free homotopy classes and cyclic reduced words, the Goldman-Turaev Lie bialgebra operations become defined on  $\mathbb{V}$ . In Section 4, using the representatives of Section 3, we prove that two linear maps we define in Section 2 are the Goldman-Turaev bracket and cobracket (Proposition 4.1).

In Section 5 we give a topological proof of the variant of the result of Goldman about simple curves and disjointness (Corollary 5.4) and a generalization of this variant (Theorem 5.3), and we exhibit some examples that shows that a dual statement to that of Goldman asked by Turaev does not hold (Example 5.7). We conclude the section by stating open problems related to Turaev's characterization and its possible replacement.

In Appendix A a definition of involutive Lie bialgebra is given and in Appendix B, we describe the Goldman-Turaev Lie bialgebra of curves and we prove that it is involutive (Proposition B.1).

In the final stages of the combinatorial treatment we benefited from the basic papers [1] of Birman and Series, and [4] of Cohen and Lustig which helped us to understand the relation between our combinatorics and hyperbolic geometric. Joint work with Jane Gilman about a concrete homology intersection matrix for a surface with a symmetric adapted basis [5] also gave impulse and new ideas for our efforts. This work also benefited from discussions with Bill Goldman, Feng Luo, Dennis Sullivan, and Vladimir Turaev, and from a visit to Renaissance Technology.

## 2 The vector space of cyclic words

### 2.1 Cyclic words and linked pairs

For each non-negative integer  $n$ , the  $n$ -*alphabet* or, briefly, the *alphabet* is the set of  $2n$  symbols  $\mathbb{A}_n = \{a_1, a_2, \dots, a_n, \bar{a}_1, \bar{a}_2, \dots, \bar{a}_n\}$ . We shall consider linear words, denoted with capital roman characters, and cyclic words, denoted by capital caligraphy-

cal characters, both in the letters of  $\mathbb{A}_n$ . The reader should think of cyclic words as symbols placed at the vertices of the  $n$ -th root of unity in  $\mathbb{C}$  up to circular symmetry,  $n = 1, 2, 3, \dots$ , see Figure 2.

If  $x_0x_1 \dots x_{m-1}$  is a linear word, then, by definition,  $\overline{x_0x_1 \dots x_{m-1}} = \overline{x_{m-1}x_{m-2} \dots x_0}$  and for each letter  $x$ ,  $\overline{\overline{x}} = x$ . A linear word  $x_0x_1 \dots x_{m-1}$  is *freely reduced* if  $x_i \neq \overline{x_{i+1}}$  for each  $i \in \{0, 1, \dots, m-1\}$ . A linear word  $\mathbf{W}_1$  is a *linear representative* of a cyclic word  $\mathcal{W}$  if  $\mathbf{W}_1$  can be obtained from  $\mathcal{W}$  by making a cut between two consecutive letters of  $\mathcal{W}$ . In such a case, we write  $\mathcal{W} = \mathbf{c}(\mathbf{W}_1)$ . If  $\mathcal{W}$  is a cyclic word,  $\mathbf{W}_1$  is a linear representative of  $\mathcal{W}$ , and  $n$  is a positive integer, we define  $\mathcal{W}^n$  as  $\mathbf{c}(\mathbf{W}_1^n)$ ,  $\overline{\mathcal{W}}$  as  $\mathbf{c}(\overline{\mathbf{W}_1})$ , and  $\mathcal{W}^{-n}$  as  $\overline{\mathcal{W}^n}$ . (These are the basic operations on conjugacy classes in groups and are well defined by these prescriptions). A cyclic word is *reduced* if it is non-empty and all its linear representatives are freely reduced, i.e., if the arrangement of symbols or ring is reduced. A reduced cyclic word is *primitive* if it cannot be written as  $\mathcal{W}^r$  for some  $r \geq 2$  and some reduced cyclic word  $\mathcal{W}$ . The *length* of a linear (resp. cyclic) word  $\mathbf{W}$  (resp.  $\mathcal{W}$ ) is the number of letters counted with multiplicity that it contains and it is denoted by  $l(\mathbf{W})$  (resp.  $l(\mathcal{W})$ ). By a *subword of a cyclic word*  $\mathcal{W}$  we mean a linear subword of one of the linear representatives of  $\mathcal{W}$ .

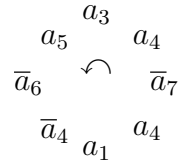


Figure 2: A ring in the letters of  $\mathbb{A}_7$

Let  $\mathcal{O}$  be a reduced cyclic word such that every letter of  $\mathbb{A}_n$  appears exactly once. From now on, we shall work with a fixed word  $\mathcal{O}$  and all our constructions will depend on this choice (nevertheless, see Remark 2.17). Such an  $\mathcal{O}$  is called the *surface symbol*.

To each cyclic word  $\mathcal{W}$ , we associate a number,  $o(\mathcal{W}) \in \{-1, 0, 1\}$  as follows. If  $\mathcal{W}$  is reduced and there exists an injective orientation preserving (resp. orientation reversing) map of oriented rings, from the letters of  $\mathcal{W}$  to the letters of  $\mathcal{O}$  then  $o(\mathcal{W}) = 1$  (resp.  $o(\mathcal{W}) = -1$ ). In all other cases (that is, if  $\mathcal{W}$  is not reduced or if there is no such orientation preserving or reversing map )  $o(\mathcal{W}) = 0$ .

**Definition 2.1.** Let  $\mathbf{P}, \mathbf{Q}$  be two linear words. The ordered pair  $(\mathbf{P}, \mathbf{Q})$  is  $\mathcal{O}$ -*linked* or briefly, *linked*, if  $\mathbf{P}$  and  $\mathbf{Q}$  are reduced words of length at least two and one of the following conditions holds.

- (1)  $\mathbf{P} = p_1p_2$ ,  $\mathbf{Q} = q_1q_2$  and  $o(\mathbf{c}(\overline{p_1q_1p_2q_2})) \neq 0$ .
- (2)  $\mathbf{P} = p_1Yp_2$ ,  $\mathbf{Q} = q_1Yq_2$ ,  $p_1 \neq q_1$ ,  $p_2 \neq q_2$  and  $Y$  is a linear word of length at least one, and if one writes  $Y$  as  $x_1Xx_2$  (where  $X$  is an empty word if  $Y$  has length two and  $x_1$  coincides with  $x_2$  if the length of  $Y$  is one), then  $o(\mathbf{c}(\overline{p_1q_1x_1})) = o(\mathbf{c}(p_2q_2\overline{x_2}))$ .
- (3)  $\mathbf{P} = p_1Yp_2$ ,  $\mathbf{Q} = q_1\overline{Y}q_2$ ,  $p_1 \neq \overline{q_2}$ ,  $p_2 \neq \overline{q_1}$  and  $Y$  is a linear word of length at least one, and if one writes  $Y$  as  $x_1Xx_2$  (where  $X$  may be an empty word and  $x_1$  may be

equal to  $x_2$ ), then  $o(c(q_2\bar{p}_1x_1)) = o(c(\bar{q}_1p_2\bar{x}_2))$ .

□

Linked pairs capture the following idea: Two strands on a surface come close, stay together for some time and then separate. If one strand enters the strip from above and exits below and the other vice versa we must have an intersection. This is measured by linked pairs (see Figure 6)

The above definition surfaced studying the structure of the intersection and self-intersection points of curves in an orientable surface with boundary. It may seem obscure at the first reading but the reader will find the full motivation in Section 3. In particular, we will show how Figure 6 illustrates this definition.

If  $\mathcal{W}$  is a reduced cyclic word, denote by  $\mathbf{LP}_1(\mathcal{W})$  the set of linked pairs  $(P, Q)$ , where  $P$  and  $Q$  are occurrences of linear subwords of  $\mathcal{W}$ . For understanding the definition better right now we suggest the reader do the calculations of Example 2.2.

**Example 2.2.** Set  $\mathcal{O} = c(a_1a_2\bar{a}_1\bar{a}_2a_3a_4\bar{a}_3\bar{a}_4)$  and consider  $\mathcal{W} = c(a_1a_2\bar{a}_3a_1a_1a_3\bar{a}_2a_1)$ . There are linked pairs in  $\mathbf{LP}_1(\mathcal{W})$  of all types.

- (a) the pairs;  $c(a_3a_1a_3a_1a_2a_2)$  (1)  $(a_2\bar{a}_3, \bar{a}_3a_1)$ ,  $(\bar{a}_3a_1, a_2\bar{a}_3)$   $(a_2\bar{a}_3, a_1a_3)$ ,  $(a_1a_3, a_2\bar{a}_3)$ ,  $(\bar{a}_3a_1, a_3\bar{a}_2)$ ,  $(a_3\bar{a}_2, \bar{a}_3a_1)$ ,  $(a_1a_3, a_3\bar{a}_2)$ ,  $(a_3\bar{a}_2, a_1a_3)$  satisfy Definition 2.1(1);
- (b)  $(\bar{a}_3a_1a_1, a_1a_1a_3)$ ,  $(a_1a_1a_3, \bar{a}_3a_1a_1)$ ,  $(\bar{a}_2a_1a_1, a_1a_1a_2)$ ,  $(a_1a_1a_2, \bar{a}_2a_1a_1)$  satisfy Definition 2.1(2);
- (c)  $(a_1a_2\bar{a}_3a_1, a_1a_3\bar{a}_2a_1)$ ,  $(a_1a_3\bar{a}_2a_1, a_1a_2\bar{a}_3a_1)$  satisfy Definition 2.1(3)

□

**Remark 2.3.** Since a linear subword of a cyclic word is determined by its ordered pair of endpoints, the set of occurrences of linear subwords in a reduced cyclic word  $\mathcal{W}$  contains  $l(\mathcal{W})^2$  elements. Therefore, the set of linked pairs of  $\mathcal{W}$ ,  $\mathbf{LP}_1(\mathcal{W})$  is finite and contains at most  $l(\mathcal{W})^4$  elements. A more careful study (Proposition 2.9) will show that  $l(\mathcal{W}).(l(\mathcal{W}) - 1)$  is an upper bound for the cardinality of  $\mathbf{LP}_1(\mathcal{W})$ . □

## 2.2 Definition of the cobracket

We denote by  $\mathbb{V}$  the vector space generated by non-empty reduced cyclic words in the letters of  $\mathbb{A}_n$ . Our next objective consists in defining a Lie cobracket on  $\mathbb{V}$ , which is a linear map  $\delta: \mathbb{V} \rightarrow \mathbb{V} \otimes \mathbb{V}$  satisfying certain identities (see Appendix A). Let us first motivate the definition: Observe that making two cuts between two different pairs of consecutive letters of a cyclic word, one gets two linear words. By gluing together the ends of each linear word, and reducing if necessary, one obtains a pair of reduced or empty cyclic words. On the other hand, as we shall soon see, every linked pair of subwords of a cyclic word  $\mathcal{W}$  determines two pairs of consecutive letters where one can make two cuts. Therefore, every linked pair of subwords of a cyclic word determines a pair of reduced or empty cyclic words. We will also see that these two cyclic words are

non-empty (Proposition 2.5) and that the linked pair determines an ordering of this pair of cyclic words.

Here is the precise definition of the procedure of the above paragraph. To each ordered pair  $(P, Q) \in \mathbf{LP}_1(\mathcal{W})$  we associate two cyclic words  $\delta_1(P, Q) = c(W_1)$  and  $\delta_2(P, Q) = c(W_2)$  by the following.

- (i) Assume that (1) or (2) of Definition 2.1 hold. Make two cuts on  $\mathcal{W}$ , one immediately before  $p_2$  and the other immediately before  $q_2$ . We obtain two linear words,  $W_1$  and  $W_2$ , the former, starting at  $p_2$ , and the latter, starting at  $q_2$ .
- (ii) If condition (3) holds, let  $W_1$  be the linear subword of  $\mathcal{W}$  starting at  $p_2$  and ending at  $q_1$ , and let  $W_2$  be the linear subword of  $\mathcal{W}$  starting at  $q_2$  and ending at  $p_1$ .

**Lemma 2.4.** *Let  $\mathcal{W}$  be a cyclic reduced word. For each  $(P, Q) \in \mathbf{LP}_1(\mathcal{W})$ , the linear words  $W_1$  and  $W_2$  of the above definition are disjoint in  $\mathcal{W}$ . Moreover,  $W_1$  and  $W_2$  are non-empty and one can write  $\mathcal{W} = c(W_1W_2)$  in the case (i) above and  $\mathcal{W} = c(YW_1\bar{Y}W_2)$  in the case (ii), where  $Y$  is as in Definition 2.1(3).*

*Proof.* The proof of the result in the case (i) is straightforward. Let us study the case (ii). We claim that  $Y$  and  $\bar{Y}$  cannot overlap. Indeed, assume, for instance that there exist a pair of linear words  $Y_1$  and  $Y_2$  such that  $Y = Y_1Y_2$  and  $\bar{Y}$  starts with  $Y_2$ . Notice that, by definition,  $\bar{Y} = \bar{Y}_2\bar{Y}_1$ . Therefore,  $Y_2 = \bar{Y}_2$ . Since  $Y_2$  is reduced this cannot happen.

Finally, if one of the words, for instance,  $W_1$  is empty, then since  $Y$  is non-empty,  $\mathcal{W} = c(Y\bar{Y}W_2)$  is not reduced, which contradicts our hypothesis. ■

**Proposition 2.5.** *Let  $\mathcal{W}$  be a reduced cyclic word and let  $(P, Q) \in \mathbf{LP}_1(\mathcal{W})$  be a linked pair. Then  $\delta_1(P, Q)$  and  $\delta_2(P, Q)$  are reduced cyclic words. Moreover,  $\delta_1(P, Q)$  and  $\delta_2(P, Q)$  are non-empty.*

*Proof.* Observe that, by Lemma 2.4, the linear words  $W_1$  and  $W_2$  are non-empty. Hence in order to prove that the cyclic words  $\delta_1(P, Q)$  and  $\delta_2(P, Q)$  are non-empty, it is enough to prove that  $c(W_1)$  and  $c(W_2)$  are reduced.

Assume that  $(P, Q)$  satisfies Definition 2.1(1). Now,  $\delta_1(P, Q)$  is not reduced if and only if  $p_2 = \bar{q}_1$  and  $\delta_2(P, Q)$  is not reduced if and only if  $p_1 = \bar{q}_2$ . Since  $o(c(\bar{p}_1\bar{q}_1p_2q_2)) \neq 0$ ,  $p_2 \neq \bar{q}_1$  and  $p_1 \neq \bar{q}_2$ .

If  $(P, Q)$  satisfies Definition 2.1(2) then  $\delta_1(P, Q)$  (resp.  $\delta_2(P, Q)$ ) is not reduced if and only if  $\bar{x}_2 = p_2$  (resp.  $\bar{x}_2 = q_2$ ). Since  $\mathcal{W}$  is reduced, none of those equations can be satisfied.

Finally, if  $(P, Q)$  satisfies Definition 2.1(3) then  $\delta_1(P, Q)$  is not reduced if and only if  $p_2 = \bar{q}_1$  and  $\delta_2(P, Q)$  is not reduced if and only if  $p_1 = \bar{q}_2$ . On the other hand, by Definition 2.1(3),  $p_2 \neq \bar{q}_1$  and  $p_1 \neq \bar{q}_2$ . Hence the proof is complete. ■

**Example 2.6.** Let  $\mathcal{O}$  and  $\mathcal{V}$  be as in Example 2.2. Hence for instance,

$$\begin{aligned}\delta_1(a_2\bar{a}_3, \bar{a}_3a_1) &= \mathbf{c}(\bar{a}_3) \text{ and } \delta_2(a_2\bar{a}_3, \bar{a}_3a_1) = \mathbf{c}(a_1a_1a_3\bar{a}_2a_1a_1a_2), \\ \delta_1(a_1a_2\bar{a}_3a_1, a_1a_3\bar{a}_2a_1) &= \mathbf{c}(a_1a_1) \text{ and } \delta_2(a_1a_2\bar{a}_3a_1, a_1a_3\bar{a}_2a_1) = \mathbf{c}(a_1a_1).\end{aligned}$$

□

By the following equation, one associates a sign to each linked pair  $(P, Q)$ .

$$\text{sign}(P, Q) = \begin{cases} \mathbf{o}(\mathbf{c}(\bar{p}_1\bar{q}_1p_2q_2)) & \text{if } (P, Q) \text{ satisfies Definition 2.1(1).} \\ \mathbf{o}(\mathbf{c}(\bar{p}_1\bar{q}_1x_1)) & \text{if } (P, Q) \text{ satisfies Definition 2.1(2).} \\ \mathbf{o}(\mathbf{c}(q_2\bar{p}_1x_1)) & \text{if } (P, Q) \text{ satisfies in Definition 2.1(3).} \end{cases}$$

**Lemma 2.7.** (a) For every linked pair  $(P, Q)$ ,  $\text{sign}(P, Q) = 1$  or  $\text{sign}(P, Q) = -1$ .

(b) If  $(P, Q)$  is a linked pair then  $(Q, P)$  is also a linked pair. Moreover,  $\text{sign}(P, Q) = -\text{sign}(Q, P)$ .

*Proof.* Let us prove (a) and leave the proof of (b) to the reader. If  $(P, Q)$  satisfies Definition 2.1(1), the result holds by definition. Otherwise, observe that if  $Z$  is a reduced cyclic word of three letters then  $\mathbf{o}(Z) = 1$  or  $\mathbf{o}(Z) = -1$ , which implies the result. ■

Now, using the above sign and the definitions of  $\delta_1$  and  $\delta_2$ , we define  $\delta: \mathbb{V} \longrightarrow \mathbb{V} \otimes \mathbb{V}$  as the linear map such that for every reduced cyclic word  $\mathcal{W}$ ,

$$\delta(\mathcal{W}) = \sum_{(P, Q) \in \mathbf{LP}_1(\mathcal{W})} \text{sign}(P, Q) \delta_1(P, Q) \otimes \delta_2(P, Q).$$

By definition, the set  $\mathbf{LP}_1(\mathcal{W})$  is finite, hence, the above sum is finite.

## 2.3 Definition of the bracket

Let  $\mathcal{V}$  and  $\mathcal{W}$  be two cyclic words and choose a pair of consecutive letters in each word. Performing a cut between each of these pair of letters, one obtains two linear words,  $V_1$  and  $W_1$ . The linear word  $V_1W_1$  determines a cyclic word,  $\mathbf{c}(V_1W_1)$  (possibly not reduced). In other words, a pair of cuts on a pair of cyclic words determines a third cyclic word. Now, we want to define an object analogous to  $\mathbf{LP}_1(\mathcal{W})$  not for a single cyclic word  $\mathcal{W}$  but for a pair of cyclic words  $\mathcal{V}$  and  $\mathcal{W}$ . This object will be denoted by  $\mathbf{LP}_2(\mathcal{V}, \mathcal{W})$  and should contain linked pairs  $(P, Q)$  such that  $P$  is a subword of  $\mathcal{V}$  and  $Q$  is a subword of  $\mathcal{W}$  (and, as we shall see, it must also contain other linked pairs). Since the formal definition could seem unnatural, we have chosen to give first a motivation for it. The first naive definition is (at least, so it was for the author of this paper) let the set of linked pairs of a pair of cyclic words  $\mathcal{V}$  and  $\mathcal{W}$  be the set of linked pairs  $(P, Q)$  such that  $P$  is a subword of  $\mathcal{V}$  and  $Q$  is a subword of  $\mathcal{W}$ . The following example illustrates the limitations of this naive definition.

**Example 2.8.** Let us consider the alphabet  $\mathbb{A}_2 = \{a_1, a_2, \bar{a}_1, \bar{a}_2\}$  and define  $\mathcal{O} = a_1 a_2 \bar{a}_1 \bar{a}_2$ . Set  $\mathcal{V} = a_1 a_1 a_2$ ,  $\mathcal{W} = a_1 a_1 a_2 a_1 a_1 a_2 a_1$ ,  $\mathbf{P} = a_2 a_1 a_1 a_2 a_1 a_1 a_2 a_1 a_1 a_2$  and  $\mathbf{Q} = a_1 a_1 a_1 a_2 a_1 a_1 a_2 a_1 a_1 a_1$ . Then  $(\mathbf{P}, \mathbf{Q})$  is an  $\mathcal{O}$ -linked pair. On the other hand,  $\mathbf{P}$  is a subword of  $\mathcal{V}^4$  but not a subword of  $\mathcal{V}$ ,  $\mathcal{V}^2$  or  $\mathcal{V}^3$ . Analogously,  $\mathbf{Q}$  is a subword of  $\mathcal{W}^2$  but not a subword of  $\mathcal{W}$ .  $\square$

Because linked pairs correspond to intersection points we will have to consider subwords of all powers of pairs of cyclic words as the set of linked pairs of a two cyclic words, (see below). Unlike the case of the set of linked pairs of a single word  $\mathbf{LP}_1(\mathcal{W})$ , the definition of  $\mathbf{LP}_2(\mathcal{V}, \mathcal{W})$  does not obviously imply that this set is finite. The finiteness of  $\mathbf{LP}_2(\mathcal{V}, \mathcal{W})$  is true and the size will be estimated in Proposition 2.9. Another issue concerning linked pairs of subwords of all powers of a cyclic word is whether the length of the subwords is bounded. It turns out that there exists a bound on the length, depending on the length of the pair of original cyclic words (Proposition 2.13). Notice that Proposition 2.13 also implies that the cardinality of the set of linked pairs of two words,  $\mathbf{LP}_2(\mathcal{V}, \mathcal{W})$  is bounded, but unlike Proposition 2.9 it does not give a sharp upper bound for the number of elements in  $\mathbf{LP}_2(\mathcal{V}, \mathcal{W})$ .

For each pair of cyclic reduced words,  $\mathcal{V}$  and  $\mathcal{W}$ , the *set of linked pairs of  $\mathcal{V}$  and  $\mathcal{W}$* , denoted by  $\mathbf{LP}_2(\mathcal{V}, \mathcal{W})$ , is defined to be the set of all pairs  $(\mathbf{P}, \mathbf{Q})$  for which there exists positive integers  $j$  and  $k$  such that  $\mathbf{P}$  is an occurrence of a subword of  $\mathcal{V}^j$ ,  $\mathbf{Q}$  is an occurrence of a subword of  $\mathcal{W}^k$ , where  $l(\mathcal{V}^{j-1}) < l(\mathbf{P}) \leq l(\mathcal{V}^j)$  and  $l(\mathcal{W}^{k-1}) < l(\mathbf{Q}) \leq l(\mathcal{W}^k)$ . (Here, we set  $l(\mathcal{V}^0) = l(\mathcal{W}^0) = 0$ .)

**Proposition 2.9.** *Let  $\mathcal{V}$  and  $\mathcal{W}$  be reduced cyclic words. Then*

- (a) *There are at most  $l(\mathcal{V}) \cdot l(\mathcal{W})$  elements in  $\mathbf{LP}_2(\mathcal{V}, \mathcal{W})$ , the set of linked pairs of  $\mathcal{V}$  and  $\mathcal{W}$ .*
- (b) *The set of linked pairs of one word,  $\mathbf{LP}_1(\mathcal{W})$  contains at most  $l(\mathcal{W}) \cdot (l(\mathcal{W}) - 1)$  elements.*

*Proof.* We first prove (a). Let  $n_1, n_2, n_3$  be the number of linked pairs satisfying (1), (2) and (3) respectively of Definition 2.1 for a pair of cyclic reduced words  $\mathcal{V}$  and  $\mathcal{W}$ .

Let  $C$  be the set of pairs of the form  $(x_1 x_2, z_1 z_2)$  such that  $x_1 x_2$  is an occurrence of a subword of  $\mathcal{V}$ , and  $z_1 z_2$  is an occurrence of a subword of  $\mathcal{W}$ . Notice that the cardinality of  $C$  is  $l(\mathcal{V}) \cdot l(\mathcal{W})$ .

Let  $C_1$  be the set of pairs  $(x_1 x_2, z_1 z_2) \in C$  such that  $x_1 \neq z_1$  and  $x_2 \neq z_2$ . Clearly, the set of linked pairs satisfying Definition 2.1(1) is contained in  $C_1$  and so any upper bound of the cardinality of  $C_1$  is larger than  $n_1$ .

We claim that each linked pair satisfying Definition 2.1(2) determines two different elements in  $C \setminus C_1$ , and that for every positive integer  $k$ ,  $k$  different linked pairs satisfying Definition 2.1(2) determine  $2k$  such elements. By this claim, since there are  $n_2$  pairs satisfying Definition 2.1(2), there are  $2n_2$  pairs in  $C \setminus C_1$ . Then, the cardinality of  $C_1$  is at most  $l(\mathcal{V}) \cdot l(\mathcal{W}) - 2n_2$  and so  $n_1 \leq l(\mathcal{V}) \cdot l(\mathcal{W}) - 2n_2$ .

In order to prove the claim, consider a linked pair satisfying (2), let  $Y$  be the “middle” linear word of this pair as in Definition 2.1(2) and let  $y$  be the first letter of  $Y$ . Let  $(P_1, Q_1), (P_2, Q_2)$  be the elements of  $C$ , such that  $P_1$  and  $Q_1$  start with the first letter of  $Y$  and  $P_2$  and  $Q_2$  end with the first letter of  $Y$ . Since all the linear words  $Y$  are different and, by definition,  $(P_1, Q_1)$  and  $(P_2, Q_2)$  are not in  $C_1$ , the claim is proved

Let  $C_2$  be the set of pairs  $(x_1x_2, z_1z_2) \in C$  such that  $\bar{x}_1 \neq z_2$  and  $\bar{x}_2 \neq z_1$ . The set of linked pairs satisfying (1) is contained in  $C_2$ . By an argument similar to the one we used to determine the cardinality of  $C_1$ , one can show that the cardinality of  $C_2$  is  $l(\mathcal{V}) \cdot l(\mathcal{W}) - 2n_3$ . Hence  $n_1 \leq l(\mathcal{V}) \cdot l(\mathcal{W}) - 2n_3$ .

Adding both inequalities, one gets  $2n_1 \leq 2l(\mathcal{V}) \cdot l(\mathcal{W}) - 2n_2 - 2n_3$ . Hence  $n_1 + n_2 + n_3 \leq l(\mathcal{V}) \cdot l(\mathcal{W})$ , and the proof of (a) is complete.

In order to prove (b) we proceed analogously as we did in (a), but defining  $C$  as the set of pairs of linear two-letter subwords  $(P, Q)$  of  $\mathcal{V}$  such that  $P$  and  $Q$  do not start at the same letter, i.e.,  $P$  and  $Q$  are different occurrences of subwords of  $\mathcal{V}$ . Hence  $C$  has  $l(\mathcal{W}) \cdot (l(\mathcal{W}) - 1)$  elements and all the pairs satisfying Definition 2.1(1) are in  $C$ . Now, we can complete the proof as above. ■

**Example 2.10.** The following examples show that the bounds of Proposition 2.9 are sharp. Let  $\mathcal{O} = a_1a_2\bar{a}_1\bar{a}_2a_3a_4\bar{a}_3\bar{a}_4$ . Then  $\mathbf{LP}_1(c(a_1\bar{a}_3a_2))$  contains exactly six elements, which are

$$\{(a_1\bar{a}_3, \bar{a}_3a_2), (a_1\bar{a}_3, a_2a_1), (\bar{a}_3a_2, a_1\bar{a}_3), (\bar{a}_3a_2, a_2a_1), (a_2a_1, a_1\bar{a}_3), (a_2a_1, \bar{a}_3a_2)\}.$$

Also  $\mathbf{LP}_2(c(a_1\bar{a}_3), c(a_2\bar{a}_4))$  contains four elements which are,

$$\{(a_1\bar{a}_3, a_2\bar{a}_4), (a_1\bar{a}_3, \bar{a}_4a_2), (\bar{a}_3a_1, a_2\bar{a}_4), (\bar{a}_3a_1, \bar{a}_4a_2)\}$$

□

The next result about linear words is well known and it will be used in the proofs of the Lemmas 2.12 and 3.7.

**Lemma 2.11.** *If  $P = x_0x_1 \dots x_{m-1}$  is a linear word and for some  $i \in \{1, 2, \dots, m-1\}$ ,  $P = x_i x_{i+1} \dots x_{m-1} x_0 x_1 \dots x_{i-1}$  then there exists a linear word  $Q$  and an integer  $r$  such that  $r \geq 2$ ,  $P = Q^r$  and  $l(Q)$  divides  $i$ .*

The next lemma will be used in the proof of Proposition 2.13.

**Lemma 2.12.** *Let  $\mathcal{V}, \mathcal{W}$  be cyclic words which are not powers of the same cyclic word and let  $P$  be a linear word. Let  $k, l$  be a pair of positive integers such that  $P$  is a subword of  $\mathcal{V}^k$  and either  $P$  or  $\bar{P}$  is a subword of  $\mathcal{W}^l$ . Moreover, assume that  $(k-1)l(\mathcal{V}) < l(P)$  and  $(l-1)l(\mathcal{W}) < l(P)$ . Then  $l(P) < l(\mathcal{V}) + l(\mathcal{W})$ .*

*Proof.* Let us first consider the case that  $P$  is a subword of both,  $\mathcal{V}^k$  and  $\mathcal{W}^l$ . Assume that  $l(P) \geq l(\mathcal{V}) + l(\mathcal{W})$ . Then we can write  $P = p_0 p_1 p_2 \dots p_{m-1}$  where  $m \geq l(\mathcal{V}) + l(\mathcal{W})$ . Since  $P$  is a subword of  $\mathcal{W}^l$ ,

$$p_0 p_1 \dots p_{l(\mathcal{V})-1} = p_{l(\mathcal{W})} p_{l(\mathcal{W})+1} \dots p_{l(\mathcal{W})+l(\mathcal{V})-1}.$$

Since  $P$  is a subword of  $\mathcal{V}^k$ ,

$$p_{l(\mathcal{W})} p_{l(\mathcal{W})+1} \dots p_{l(\mathcal{W})+l(\mathcal{V})-1} = p_r p_{r+1} \dots p_{r+l(\mathcal{V})-1} = p_r p_{r+1} \dots p_{l(\mathcal{V})-1} p_0 p_1 \dots p_{r-1},$$

where  $r$  is the remainder of dividing  $l(\mathcal{W})$  by  $l(\mathcal{V})$ . Since  $\mathcal{V}$  is not a power of  $\mathcal{W}$ ,  $r > 0$ . By Lemma 2.11, there exists a linear word  $X$  and a positive integer  $d$  such that  $l(X)$  divides  $r$  and  $p_0 p_1 \dots p_{l(\mathcal{V})-1} = X^d$ . So  $\mathcal{V} = c(X)^d$ . Since  $l(X)$  divides  $r$  and  $l(\mathcal{V})$ ,  $l(X)$  divides  $l(\mathcal{W})$ . Thus,  $\mathcal{W}$  is also a power of  $c(X)$ , contradicting our hypothesis. Therefore,  $l(P) < l(\mathcal{V}) + l(\mathcal{W})$ .

To prove the result in the other case, one observes that  $\bar{P}$  is a subword of a cyclic word  $\mathcal{W}$  if and only if  $P$  is a subword of  $\bar{\mathcal{W}}$ . Then replacing  $\mathcal{W}$  by  $\bar{\mathcal{W}}$ , the result follows.  $\blacksquare$

Up to certain special cases left to the reader, the following proposition follows from Lemma 2.12.

**Proposition 2.13.** *Let  $\mathcal{V}$  and  $\mathcal{W}$  be two reduced cyclic words. Then  $\mathbf{LP}_2(\mathcal{V}, \mathcal{W})$  is the set of all linked pairs  $(P, Q)$  such that  $P$  is an occurrence of a subword of  $\mathcal{V}^j$ ,  $Q$  is an occurrence of a subword of  $\mathcal{W}^k$ ,  $l(\mathcal{V}^{j-1}) < l(P) \leq l(\mathcal{V}^j)$  so that  $l(\mathcal{W}^{k-1}) < l(Q) \leq l(\mathcal{W}^k)$  where  $j, k$  are positive integers such that  $j < 2 + \frac{l(\mathcal{W})}{l(\mathcal{V})}$  and  $k < 2 + \frac{l(\mathcal{V})}{l(\mathcal{W})}$ .*

Now, we will define the bracket. Firstly, we associate to each linked pair  $(P, Q) \in \mathbf{LP}_2(\mathcal{W}, \mathcal{Z})$  a cyclic word  $\gamma(P, Q) = c(W_1 Z_1)$ , where  $W_1$  and  $Z_1$  are linear words defined as follows.

- (i) If conditions (1) or (2) of Definition 2.1 holds for the linked pair  $(P, Q)$ ,  $W_1$  is the representative of  $\mathcal{W}$  obtained by cutting  $\mathcal{W}$  immediately before  $p_2$  and  $Z_1$  is the representative of  $\mathcal{Z}$  obtained by cutting  $\mathcal{Z}$  immediately before  $q_2$ .
- (ii) If condition (3) of Definition 2.1 holds for the pair  $(P, Q)$ . Then  $W_1$  is the linear subword of  $\mathcal{W}$  that starts right after the end of  $Y$  and ends right before the first letter of  $Y$ , and  $Z_1$  is the subword of  $\mathcal{Z}$  that starts right after the last letter of  $\bar{Y}$  and ends right before the beginning of  $\bar{Y}$ . (Observe that  $Y$  may not be a subword of  $\mathcal{W}$  nor of  $\mathcal{Z}$ , but we can always find the first and last letters of  $Y$  in  $\mathcal{W}$  and  $\mathcal{Z}$ )

**Example 2.14.** Set  $\mathcal{O} = c(a_1 a_2 \bar{a}_1 \bar{a}_2 a_3 a_4 \bar{a}_3 \bar{a}_4)$ . Then

$$\mathbf{LP}_2(c(a_1 a_2 a_2 a_3), c(\bar{a}_2 \bar{a}_2)) = \{(a_1 a_2 a_2 a_3, \bar{a}_2 \bar{a}_2 \bar{a}_2 \bar{a}_2), (a_1 a_2 a_2 a_3, \bar{a}_2 \bar{a}_2 \bar{a}_2 \bar{a}_2)\},$$

and both pairs  $(a_1 a_2 a_2 a_3, \bar{a}_2 \bar{a}_2 \bar{a}_2 \bar{a}_2)$ ,  $(a_1 a_2 a_2 a_3, \bar{a}_2 \bar{a}_2 \bar{a}_2 \bar{a}_2)$  satisfy (3) of Definition 2.1. Notice that  $Z_1$  in this case is the empty word.  $\square$

The next result is analogous to Proposition 2.5.

**Proposition 2.15.** *For each pair of reduced cyclic words  $\mathcal{W}$  and  $\mathcal{Z}$ , and for each linked pair  $(P, Q) \in \mathbf{LP}_2(\mathcal{W}, \mathcal{Z})$ ,  $\gamma(P, Q)$  is a cyclically reduced word. In particular,  $\gamma(P, Q)$  is non-empty.*

*Proof.* This proof follows the same ideas of the proof of Proposition 2.5, except that in the case (ii), the words  $W_1$  or  $Z_1$  may be empty. Hence it is necessary to see that they cannot both be empty. Observe that if  $W_1$  and  $Z_1$  are empty, then  $Y = W_2^n$ ,  $\bar{Y} = Z_2^m$ , for some positive integers  $n, m$ , and some linear representative  $W_2$  of  $\mathcal{W}$  and  $Z_2$  of  $\mathcal{Z}$ . But in this case, if we write  $P = p_1 Y p_2$ ,  $Q = q_1 \bar{Y} q_2$ , as in Definition 2.1, we have that  $p_1 = \bar{q}_2$ ,  $p_2 = \bar{q}_1$ . So  $(P, Q)$  does not satisfy (3) of Definition 2.1. ■

We define the bracket  $[\cdot, \cdot]: \mathbb{V} \otimes \mathbb{V} \longrightarrow \mathbb{V}$  as the linear map such that for each pair of cyclic words,  $\mathcal{W}$  and  $\mathcal{Z}$ ,

$$[\mathcal{W}, \mathcal{Z}] = \sum_{(P, Q) \in \mathbf{LP}_2(\mathcal{W}, \mathcal{Z})} \text{sign}(P, Q) \gamma(P, Q).$$

By Proposition 2.9,  $\mathbf{LP}_2(\mathcal{V}, \mathcal{W})$  is finite and so the bracket is well defined.

**Example 2.16.** If  $\mathcal{O} = c(a_1 a_2 \bar{a}_1 \bar{a}_2 a_3 a_4 \bar{a}_3 \bar{a}_4)$  then  $[c(a_1 a_2 a_2 a_3), c(\bar{a}_2 \bar{a}_2)] = -2c(a_3 a_1)$ . (see Example 2.14) □

In Section 4 we prove that for each  $\mathcal{O}$ ,  $(\mathbb{V}, \gamma, \delta)$  is an involutive Lie bialgebra (Theorem 4.2).

**Remark 2.17.** Even though there is a vast number of  $\mathcal{O}$ 's for each  $n$  (exactly,  $(n-1)!$ ), and each  $\mathcal{O}$  determines a Lie bialgebra, there are at most  $n/2 + 1$  Lie bialgebras up to isomorphism, (since a surface is determined by its genus and the number of boundary components.) □

### 3 Intersection points and linked pairs

The goal of this section is to prove that there are bijective correspondences between “unordered” linked pairs of a cyclic reduced word and self-intersection points of representatives with minimal self-intersection and between linked pairs of a pair of words and intersection points of a pair of representatives with minimal intersection. Our representatives are similar to the ones constructed by Reinhart [9] but improves them in the sense that intersection and self-intersections of our representatives are minimal and are organized by the combinatorics of linked pairs.

We start by gathering together some well known results.

**Lemma 3.1.** *Let  $\Sigma$  be an oriented surface with boundary. There is a bijective correspondence between any two of the following sets.*

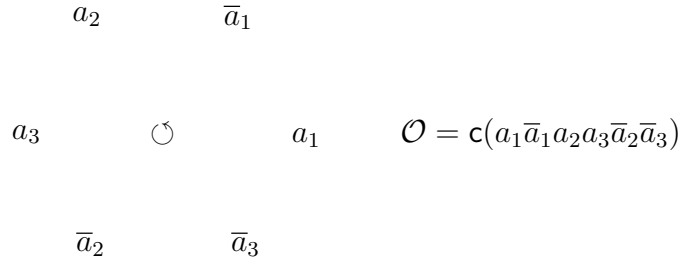


Figure 3: The  $4n$ -gon  $P_{\mathcal{O}}$

- (1) Conjugacy classes of non-trivial elements of the fundamental group of  $\Sigma$ .
- (2) Non-trivial free homotopy classes of maps from the circle to  $\Sigma$ .
- (3) Non-empty cyclically reduced cyclic words in  $\mathbb{A}_n = \{a_1, a_2, \dots, a_n, \bar{a}_1, \bar{a}_2, \dots, \bar{a}_n\}$ , where  $1 - n$  is the Euler characteristic of  $\Sigma$ .

Lemma 3.1 allows us to identify non-empty cyclically reduced cyclic words and non-trivial free homotopy classes, and we will often make use of this identification.

### 3.1 Arc representatives

As in Section 2, throughout this section we fix a surface symbol, that is, a cyclic word  $\mathcal{O} = c(o_1 o_2 \dots o_{2n})$  such that every letter of  $\mathbb{A}_n$  appears exactly once. Denote by  $P_{\mathcal{O}}$  the  $4n$ -gon with edges labeled counterclockwise in the following way: One chooses an edge as first and labels it with  $o_1$ , the second has no label, the third is labeled with  $o_2$ , the fourth with no label and so on as is shown in the example of Figure 3.

For each  $i \in \{1, 2, \dots, n\}$ , one identifies the edge  $a_i$  with the edge  $\bar{a}_i$  without creating Moebius bands. In this way, one gets a surface  $\Sigma_{\mathcal{O}}$  with non-empty boundary and Euler characteristic  $1 - n$ . Furthermore, every surface with non-empty boundary can be obtained from such a  $4n$ -gon (one has to take  $\mathcal{O}$  as the empty word in order to get the disk). Denote by  $\pi: P_{\mathcal{O}} \rightarrow \Sigma_{\mathcal{O}}$  the projection map and denote by  $E_{x_i}$  the projection of the edge labeled with  $x_i$ . Observe that  $E_{x_i} = E_{\bar{x}_i}$ .

A loop in  $\Sigma_{\mathcal{O}}$  is a piecewise smooth map from the circle to  $\Sigma_{\mathcal{O}}$ . We will often identify a loop with its image.

Choose a point  $c$  in the interior of  $P_{\mathcal{O}}$ . For each  $i \in \{1, 2, \dots, n\}$ , let  $q_i$  and  $s_i$  be a pair of points such that  $\pi(q_i) = \pi(s_i)$ , and that  $q_i$  (resp.  $s_i$ ) is in the interior of the edge labeled with  $a_i$  (resp.  $\bar{a}_i$ ). Denote by  $a_i$  the based homotopy class of the loop that starts at  $\pi(c)$ , runs along the projection of the segment from  $c$  to  $q_i$  and then along the projection of the segment from  $s_i$  to  $c$ . Hence  $\{a_1, a_2, \dots, a_n\}$  is a set of generators of the fundamental group of  $\Sigma_{\mathcal{O}}$ . If  $\mathcal{W}$  is a reduced cyclic word in the letters of  $\mathbb{A}_n$ , a loop  $\alpha$  in  $\Sigma_{\mathcal{O}}$  is a representative of  $\mathcal{W}$  if  $\alpha$  is freely homotopic to a curve  $\beta$  passing through  $c$  such that the (based) homotopy class of  $\beta$  written in the generators  $\{a_1, a_2, \dots, a_n\}$

is a linear representative of  $\mathcal{W}$  (see Lemma 3.1).

A one dimensional manifold (possibly with boundary)  $\beta$  (resp. a pair of loops  $\alpha, \beta$ ) on a surface  $\Sigma$  has a *bigon* if there exists a pair  $u$  and  $v$  of subarcs of  $\beta$  (resp. such that  $u$  is a subarc of  $\alpha$  and  $v$  is a subarc of  $\beta$ ), such that endpoints of  $u$  equal the endpoints of  $v$  and the loop formed by running first along  $u$  and then along  $v$  is null-homotopic in  $\Sigma$ .

From now on, when considering a cyclic word  $\mathcal{W} = \mathbf{c}(x_0x_1x_2\dots x_{m-1})$ , for each  $i \in \mathbb{Z}$ ,  $x_i$  denotes the letter of  $\mathcal{W}$  with subindex  $i$  modulo  $m$ . This convention also applies to other objects with subindex.

A *segment* is a map from a closed interval into the surface. If  $A_0, A_1, A_2, \dots, A_{s-1}$  is a sequence of oriented segments of  $P_{\mathcal{O}}$  such that for each  $i \in \{0, 1, \dots, s-2\}$ , the projection of the final point of  $A_i$  is equal to the projection of the initial point of  $A_{i+1}$ , then  $\pi(A_0)\pi(A_1)\dots\pi(A_{s-1})$  is an arc in  $\Sigma_{\mathcal{O}}$ . We will write  $\pi(A_iA_{i+1}\dots A_{i+j})$  instead of  $\pi(A_i)\pi(A_{i+1})\dots\pi(A_{i+j})$ . Clearly, if the endpoint of  $\pi(A_{s-1})$  equals the initial point of  $\pi(A_0)$  then  $\pi(A_0A_1\dots A_{s-1})$  is a loop.

Let  $\mathcal{W} = \mathbf{c}(x_0x_1x_2\dots x_{m-1})$  be a reduced cyclic word and let  $\alpha$  be a representative of  $\mathcal{W}$ . We say that  $\alpha$  is a *segmented representative of  $\mathcal{W}$*  if there exist a sequence of oriented segments  $A_0, A_1, A_2, \dots, A_{m-1}$  in  $P_{\mathcal{O}}$  such that  $\alpha = \pi(A_0A_1\dots A_{m-1})$ , and

- (1) For each  $i \in \{0, 1, 2, \dots, m-1\}$ ,  $A_i$  is an oriented arc starting at a point in the interior of the edge labeled with  $\bar{x}_i$ , and ending at a point in the interior of the edge labeled with  $x_{i+1}$ .
- (2) For each  $i, j \in \{0, 1, 2, \dots, m-1\}$ ,  $A_i$  intersects  $A_j$  in at most one point.
- (3) The endpoints of the arcs  $A_0, A_1, \dots, A_{m-1}$ , are all different.
- (4) There are no triple intersections between the arcs  $A_0, A_1, \dots, A_{m-1}$ .
- (5) For each  $i \in \{0, 1, 2, \dots, m-1\}$ , the interior of  $A_i$  is contained in the interior of  $P_{\mathcal{O}}$ .

The following lemma characterizes the segmented representatives of a reduced cyclic word with bigons.

**Lemma 3.2.** *Let  $\mathcal{W} = \mathbf{c}(x_0x_1x_2\dots x_{m-1})$  be a reduced cyclic word and let  $\alpha = \pi(A_0A_1\dots A_{m-1})$  be a segmented representative of  $\mathcal{W}$ . Then  $\alpha$  has a bigon if and only if there exist  $i, j \in \{0, 1, \dots, m-1\}$ ,  $j \neq i$ , and  $k \geq 0$  such that one of the following holds*

- (1)  $\pi(A_i) \cap \pi(A_j) \neq \emptyset$ ,  $\pi(A_{i+k}) \cap \pi(A_{j+k}) \neq \emptyset$  and for each  $h \in \{1, 2, \dots, k-1\}$ ,  $\pi(A_{i+h}) \cap \pi(A_{j+h}) = \emptyset$ .
- (2)  $\pi(A_i) \cap \pi(A_j) \neq \emptyset$ ,  $\pi(A_{i+k}) \cap \pi(A_{j-k}) \neq \emptyset$  and for each  $h \in \{1, 2, \dots, k-1\}$ ,  $\pi(A_{i+h}) \cap \pi(A_{j-h}) = \emptyset$ .

*Proof.* Clearly, if (1) or (2) hold then  $\alpha$  has a bigon. We prove now the reverse implication. Let  $U, V$  be the subarcs that bound a bigon, and let  $p, q$  be the endpoints of

$U$  and  $V$ . Assume that  $U$  goes from  $p$  to  $q$  and  $V$  goes from  $q$  to  $p$ . The proof in the other case is similar.

Let  $i, j \in \{0, 1, \dots, m-1\}$ ,  $s, l \geq 0$ , be such that  $p \in \pi(A_i) \cap \pi(A_j)$ ,  $q \in \pi(A_{i+s}) \cap \pi(A_{j-l})$ , and  $s$  and  $l$  are minimal with such a property. Let  $P = x_{i+1}x_{i+2}\dots x_{i+s}$  and  $Q = x_{j-l}x_{j-l+1}\dots x_{j-1}$ . Since the union of  $U$  and  $V$  is a null-homotopic loop, and  $\mathcal{W}$  is reduced,  $P = \overline{Q}$ . Hence  $s = l$  and taking  $h = s$ , the integers  $i, j$  and  $h$  satisfy (2). ■

**Proposition 3.3.** *Let  $\mathcal{W} = c(x_0x_1x_2\dots x_{m-1})$  be a primitive reduced cyclic word. Then there exist a segmented representative  $\alpha = \pi(A_0A_1\dots A_{m-1})$  of  $\mathcal{W}$  such that  $\alpha$  does not have bigons.*

*Proof.* Consider  $U_i$  and  $V_i$  thin tubular neighborhoods of the edges  $a_i$  and  $\bar{a}_i$  in  $P_{\mathcal{O}}$  respectively. The projection of  $U_i$  and  $V_i$  determine two sides of  $E_i$ : the  $a_i$ -side, containing  $\pi(U_i)$  and the  $\bar{a}_i$ -side, containing  $\pi(V_i)$ .

Now, to each loop  $\varphi$  such that each small arc of  $\varphi$  intersects  $\cup_{1 \leq i \leq n} E_{a_i}$  transversely, we associate a cyclic word  $\mathcal{V}_\varphi$ , which describes the way that  $\varphi$  crosses  $\cup_{1 \leq i \leq n} E_{a_i}$ : Choose a point  $p$  in  $\varphi$  and not in  $\cup_{1 \leq i \leq n} E_{a_i}$ . Let  $y_0y_1\dots y_{k-1}$  be an ordered sequence of letters of  $\mathbb{A}_n$  such that, starting at  $p$ , the first edge crossed by  $\varphi$  is  $E_{y_0}$ , from the  $y_0$ -side to the  $\bar{y}_0$ -side. The second edge crossed by  $\varphi$  is  $E_{y_1}$  from the  $y_1$ -side to the  $\bar{y}_1$ -side and, in general, for each  $i \in \{1, \dots, k\}$ , the  $i$ -th edge crossed by  $\varphi$  is  $E_{y_{i-1}}$  from the  $y_{i-1}$ -th side to the  $\bar{y}_{i-1}$ -side. Finally, take  $\mathcal{V}_\varphi = c(y_0y_1\dots y_{k-1})$ .

Endow  $\Sigma_{\mathcal{O}}$  with a hyperbolic metric with geodesic boundary such that the arcs  $\cup_{1 \leq i \leq n} E_{a_i}$  are also geodesic. (Such a metric exists because we can assume that  $P_{\mathcal{O}}$  is a hyperbolic polygon  $P_{\mathcal{O}}$  with geodesic edges and right angles.) Let  $\beta$  be a geodesic representative of  $\mathcal{W}$ . Then,  $\beta \cup (\cup_{1 \leq i \leq n} E_{a_i})$  does not have bigons, because a bigon cannot be bounded by geodesic segments.

Let us prove that  $\mathcal{V}_\beta$  is reduced and therefore,  $\mathcal{V}_\beta = c(x_0x_1\dots x_{m-1})$ . Indeed, suppose that  $\mathcal{V}_\beta = c(y_0y_1\dots y_{k-1})$  is not reduced. Hence  $y_{i+1} = \bar{y}_i$ , for some  $i \in \{0, 1, 2, \dots, k-1\}$ . Then there exists a subarc  $B$  of  $\beta$ , a subarc  $S$  of  $E_{y_i}$  such that  $S$  and  $B$  bound a disk, which is absurd, being a bigon bounded by geodesics.

Now, we claim that there exists an homotopy  $\beta_t$ ,  $t \in [0, 1]$  such that:

- (i)  $\beta_0 = \beta$ .
- (ii) For each  $t \in [0, 1]$ ,  $\beta_t \cup (\cup_{1 \leq i \leq n} E_{a_i})$  does not have bigons.
- (iii) All the self-intersection points of  $\beta_1 \cup (\cup_{1 \leq i \leq n} E_{a_i})$  are double.
- (iv) All the intersection points of pairs of arcs of  $\beta_t \cup (\cup_{1 \leq i \leq n} E_{a_i})$  are transverse.

Indeed, if all the self-intersection points of  $\beta \cup (\cup_{1 \leq i \leq n} E_{a_i})$  are double, the claim holds trivially. Otherwise, we remove self-intersection points which are not double in the following way: Let  $p$  be one such point. Consider a small disk  $D$  centered at  $p$  such that each connected component of  $D \cap (\beta \cup (\cup_{1 \leq i \leq n} E_{a_i}))$  passes through  $p$ . First, assume that one of these connected components is included in  $\cup_{1 \leq i \leq n} E_{a_i}$ . In this case, deform  $\beta$  as in Figure 4(a) and then as in Figure 4(b). Observe that in the second step, if  $h$  is the number of connected components of  $\beta \cap D$ , then after the homotopy

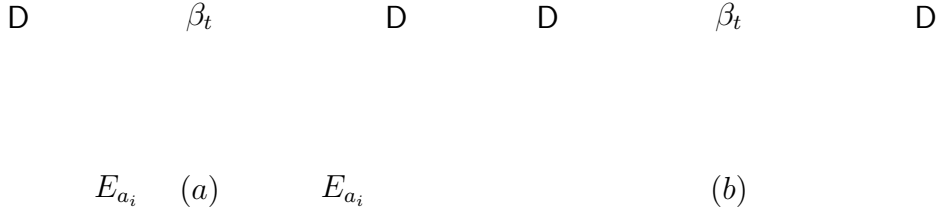


Figure 4: Proof of Proposition 3.3

there are exactly  $\frac{h(h-1)}{2}$  transverse double points in  $D$ . We can assume that in both homotopies,  $\beta \setminus D$  is fixed. Now, assume that none of the connected components of  $D \cap (\beta \cup (\cup_{1 \leq i \leq n} E_{a_i}))$  is included in  $\cup_{1 \leq i \leq n} E_{a_i}$ . Deform  $\beta$  as we did in the second step of the previous case. Hence we have an homotopy  $\beta_t$ ,  $t \in [0, v]$  for some  $v < 1$ , such that  $\beta_0 = \beta$  and the number of self-intersection points of  $\beta_t \cup (\cup_{1 \leq i \leq n} E_{a_i})$  which are not double, is strictly smaller than the number of self-intersection points of  $\beta \cup (\cup_{1 \leq i \leq n} E_{a_i})$  which are not double. We can assume that all the intersection points of pairs of arcs of  $\beta_t \cup (\cup_{1 \leq i \leq n} E_{a_i})$  are transverse for each  $t \in [0, v]$ . Let us check now that for all  $u \in [0, v]$ ,  $\beta_u \cup (\cup_{1 \leq i \leq n} E_{a_i})$  does not have bigons. Indeed, if for some  $u \in [0, v]$ ,  $\beta_u \cup (\cup_{1 \leq i \leq n} E_{a_i})$  has a bigon  $B$ , we can follow  $B$  in  $\beta_t \cup (\cup_{1 \leq i \leq n} E_{a_i})$ . Since all intersection points of pair of arcs are transverse, there exists a bigon in  $\beta_t \cup (\cup_{1 \leq i \leq n} E_{a_i})$  for every  $t \in [0, v]$ . In particular, there exists a bigon in  $\beta \cup (\cup_{1 \leq i \leq n} E_{a_i})$ , a contradiction. (Note that the dotted regions of Figure 4(b) are possible traces of bigons). Now we can extend the homotopy, removing at each step non-double self-intersection points. Thus the claim follows.

By (ii),  $\mathcal{V}_{\beta_1}$  is reduced. Thus, as above, one can prove that  $\mathcal{V}_{\beta_1} = \mathbf{c}(x_0 x_1 \dots x_{m-1})$ .

Set  $\alpha = \beta_1$ . There exists  $m$  oriented subarcs of  $\alpha$ ,  $B_0, B_1, \dots, B_{m-1}$  such that  $\alpha = B_0 B_1 \dots B_{m-1}$  and for each  $i$ , the interior of  $B_i$  does not intersect  $\cup_{1 \leq i \leq n} E_{a_i}$  and  $B_i$  starts at  $E_{x_i}$ , runs on the  $\bar{x}_i$ -side and ends at  $E_{x_{i+1}}$ , on the  $x_{i+1}$ -side. For each  $i \in \{0, 1, \dots, m-1\}$  there exists a unique arc  $A_i$  joining two edges of  $P_{\mathcal{O}}$  such that  $\pi(A_i) = B_i$ . Thus,  $\alpha = \pi(A_0 A_1 \dots A_{m-1})$  satisfies the required properties of a segmented representative without bigons. ■

We will now show that there exists a segmented representative of a power of a primitive cyclic reduced word  $\mathcal{W}^r$ , for which there exists  $r-1$  self-intersection points which appear because the curve wraps around itself  $r$  times. Moreover, we see that we can identify in which pair of segments of the segmented representative one can find these  $r-1$  points and that the endpoints of every bigon are among these  $r-1$  points.

**Proposition 3.4.** *Let  $r \geq 1$  and let  $\mathcal{W} = \mathbf{c}(y_0 y_1 \dots y_{m-1})^r$  be a reduced cyclic word such that  $\mathbf{c}(y_0 y_1 \dots y_{m-1})$  is primitive. Then there exists a segmented representative of  $\mathcal{W}$ ,  $\alpha = \pi(A_0 A_1 \dots A_{mr-1})$  such that for every pair  $i, j \in \{0, 1, \dots, mr-1\}$  the following are equivalent*

- (1)  $A_i \cap A_j \neq \emptyset$  and  $i \equiv j \pmod{m}$ .

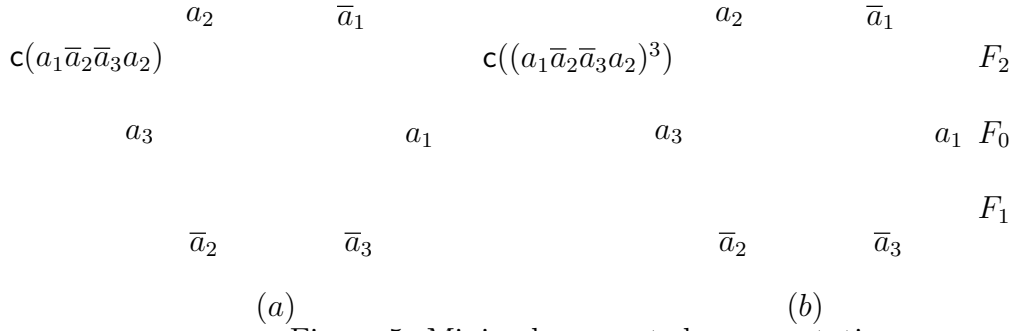


Figure 5: Minimal segmented representatives

(2)  $i = j$  or  $i = mr - 1$  or  $j = mr - 1$ .

Moreover, all the bigons of  $\alpha$  have endpoints in the intersection of  $A_{mr-1}$  with  $A_{mj-1}$  for some  $j \in \{1, 2, \dots, r-1\}$ .

*Proof.* By Proposition 3.3 there exists a segmented representative  $\gamma$  of  $\mathbf{c}(y_0 y_1 \dots y_{m-1})$  which does not have bigons (see Figure 5(a)). Assume that  $\gamma = \pi(C_0 C_1 \dots C_{m-1})$ . Let  $A \subset \Sigma_{\mathcal{O}}$  be an annulus, having  $\gamma$  as one of the boundary components. Subdivide this annulus in  $r-1$  parallel annuli. Let  $\gamma_1, \gamma_2, \dots, \gamma_r$  be the boundary components of these annuli. We can assume going along the arc  $E_{y_0}$  in one of the two possible directions, right after the initial arc of  $\gamma_1$  (that is, the arc “parallel” to  $C_0$ ), one finds the initial arc of  $\gamma_2$ , then the initial arc of  $\gamma_2$  and so on. Each  $\gamma_i$  is homotopic to  $\gamma$ , transversal to  $\cup_{1 \leq i \leq n} E_{a_i}$  and such that for each  $i \in \{1, 2, \dots, r\}$   $\gamma_i = \pi(D_{i,0} D_{i,1} \dots D_{i,m-1})$ , where  $D_{i,j}$  is an arc in  $P_{\mathcal{O}}$  from the edge  $\bar{x}_i$  to the edge  $x_{i+1}$ . Moreover, all the endpoints of the edges  $D_{i,s}$  are different for every  $i \in \{0, 1, \dots, r-1\}$  and  $s \in \{0, 1, \dots, m-1\}$  and if  $i \neq j$ ,  $D_{i,h} \cap D_{j,h} = \emptyset$ .

For each  $j \in \{1, 2, \dots, r-1\}$ , let  $F_j$  be an arc in  $P_{\mathcal{O}}$  joining the initial point of  $D_{j,m-1}$  with the last point of  $D_{j+1,m-1}$ . We can assume that for every  $j, k \in \{1, 2, \dots, r-1\}$ , if  $j \neq k$  then  $F_j \cap F_k = \emptyset$ . Let  $F_r$  be an arc in  $P_{\mathcal{O}}$  joining the initial point of  $D_{r,m-1}$  with the last point of  $D_{0,m-1}$ . We may also assume that each of the arcs  $F_i$  intersects each of the arcs  $D_{h,l}$  in at most one point.

For each  $i \in \{0, 1, \dots, mr-1\}$ , write  $i = mh + s$  where  $0 \leq s < m$ . If  $s \neq m-1$ , set  $A_i = D_{h+1,s}$  and if  $r = m-1$ ,  $A_i = F_{h+1}$ . By definition,  $\alpha$  is a loop. Moreover,  $\alpha$  is a segmented representative.

Assume that  $\alpha = \pi(A_0 A_1 \dots A_{r_{m-1}})$  has a bigon. Let  $i, j$  and  $k$  be as in Lemma 3.2. Write  $i = mh_i + s_i$ ,  $j = mh_j + s_j$ , with  $0 \leq s_i, s_j \leq m-1$ . We claim that  $i, i+k, j$  and  $j+k$  are congruent to  $m-1$  modulo  $m$ .

Indeed, suppose that Lemma 3.2(1) holds. (If Lemma 3.2(2) holds the proof can be completed with analogous arguments). Since  $A_i \cap A_j \neq \emptyset$ , then either  $s_i = s_j = m-1$  or  $s_i \neq s_j$ . If  $s_i \neq s_j$ ,  $C_{s_i} \cap C_{s_j} \neq \emptyset$  and  $C_{s_i+k} \cap C_{s_j+k} \neq \emptyset$ . Also,  $C_{s_i+h} \cap C_{s_j+h} = \emptyset$  if  $h \in \{1, 2, \dots, k-1\}$ . Then  $\gamma$  has a bigon, a contradiction. So  $s_i = s_j = m-1$ . Since  $A_{i+k} \cap A_{j+k} \neq \emptyset$  and  $i+k$  and  $j+k$  are congruent modulo  $m$ ,  $i+k$  and  $j+k$  must be

congruent to  $m - 1$  modulo  $m$ .

By definition, if  $A_{sm-1}$  intersects  $A_{tm-1}$ , then  $s = r$  or  $t = r$  and the proof is complete. ■

### 3.2 Self-intersection points and linked pairs

We start to study by an example the relation of pairs of subwords of a cyclic reduced word and self-intersection points of a segmented representative.

**Example 3.5.** Let  $\mathcal{O}$  and  $\mathcal{W}$  be as in Example 2.2. Since  $a_2\bar{a}_3$  and  $\bar{a}_3a_1$  are subwords of  $\mathcal{W}$ , any segmented representative of  $\mathcal{W}$  contains the projection of two transversal segments,  $B_1$  from  $\bar{a}_2$  to  $\bar{a}_3$  and  $B_2$ , from  $a_3$  to  $a_1$  (see Figure 6(a)). One might guess that the occurrence of  $a_2\bar{a}_3$  and  $\bar{a}_3a_1$  as subwords of a cyclic word will imply a self-intersection point in every representative of  $\mathcal{W}$ . We will prove that a generalization of this holds, that is, certain pairs of subwords of two letters imply self-intersection points of the representatives.

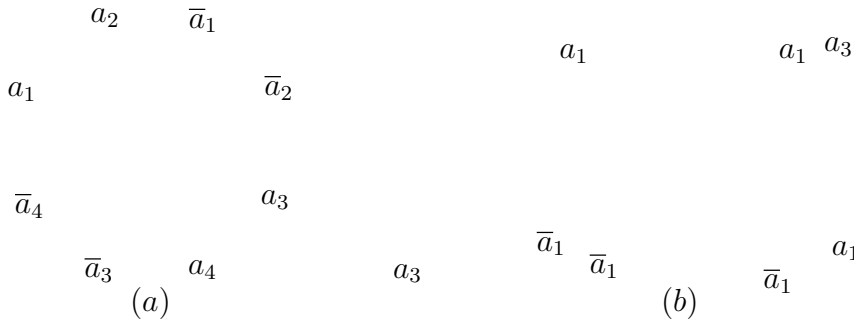


Figure 6: *Example 3.5*

Now, consider the pair of subwords of  $\mathcal{W}$ ,  $a_1a_1$  and  $\bar{a}_3a_1$ , see Figure 6(b). Since both the segments corresponding to this pair of subwords land in the edge  $a_1$ , the occurrence of these two subwords does not provide enough information to deduce the existence of a self-intersection point. In order to understand better this configuration of segments, we prolong the subwords starting with  $a_1a_1$  and  $\bar{a}_3a_1$  until they have different letters at the beginning and at the end. Then we study how the arcs corresponding to these subwords intersect. So for instance, in our example we get  $\bar{a}_3a_1a_1$  and  $a_1a_1a_3$ , and we see a self-intersection point (Figure 6(b)). We will see that certain pairs of subwords of  $\mathcal{W}$  imply self-intersection points of representatives. □

Let  $\alpha = \pi(A_0A_1 \dots, A_{m-1})$  be a segmented representative of a reduced cyclic word  $\mathcal{W} = \mathbf{c}(x_0x_1 \dots x_{m-1})$ . By an *arc* of  $\alpha$  we mean a finite subsequence of segments of the infinite sequence  $\prod_{i \in \mathbb{N}} \pi(A_1)\pi(A_2) \dots \pi(A_m)$ . The *underlying subword of the arc*  $\pi(A_iA_{i+1} \dots A_{i+j})$  is the subword  $x_i x_{i+1} \dots x_{i+j+1}$  of  $\mathcal{W}^\infty$ .

**Definition 3.6.** Let  $U, V$  be a pair of arcs of segmented representatives of  $\alpha$  and  $\beta$  respectively, such that the underlying words of  $U$  and  $V$  are  $x_i x_{i+1} \dots x_{i+j+1}$ , and  $y_k y_{k+1} \dots y_{k+j+1}$ , respectively. Moreover, assume that exactly one of the following holds (see Figure 7).

- (1)  $x_i \neq y_j, x_{i+j+1} \neq y_{k+j+1}$  and  $x_{i+1} x_{i+2} \dots x_{i+j} = y_{k+1} y_{k+2} \dots y_{k+j}$ .
- (2)  $x_i \neq \bar{y}_{k+j+1}, x_{i+j+1} \neq \bar{y}_k$  and  $x_{i+1} x_{i+2} \dots x_{i+j} = \bar{y}_{k+j} \bar{y}_{k+j-1} \dots \bar{y}_{k+1}$ .

If  $\alpha \neq \beta$  (resp.  $\alpha = \beta$ ) we say that  $\{U, V\}$  is a *semiparallel pair of arcs of  $\alpha$  and  $\beta$*  (resp. of  $\alpha$ ) parallel in case (1) and antiparallel in case (2).  $\square$

Let  $\mathcal{V}$  be a primitive reduced cyclic word of length  $m$ , let  $r \geq 1$  let  $\mathcal{W} = \mathcal{V}^r$ . We call a segmented representative of  $\mathcal{W}$  as in Proposition 3.3 if  $r = 1$  and as in Proposition 3.4 if  $r \geq 1$  *minimal*. Let  $r > 1$  and  $\alpha$  be a minimal segmented representative of  $\mathcal{W}$ . We denote by  $P_\alpha$  the set of intersection points of  $\pi(A_{r m-1})$  with  $\pi(A_{k m-1})$ , for  $k \in \{1, 2, \dots, r-1\}$  and by  $I_\alpha$  the set of self-intersection points of  $\alpha$  not in  $P_\alpha$ . Hence the set of self-intersection points of a minimal segmented representative  $\alpha$  is the disjoint union of  $P_\alpha$  and  $I_\alpha$ . When  $r = 1$ ,  $P_\alpha$  is empty by definition and  $I_\alpha$  is the set of self-intersection points of  $\alpha$ .

**Lemma 3.7.** *Let  $\mathcal{W} = c(x_0 x_1 \dots x_{m-1})$  be a cyclically reduced cyclic word and let  $\alpha$  be a minimal segmented representative of  $\mathcal{W}$ . Let  $p$  be a self-intersection point of  $\alpha$ , such that  $p \in I_\alpha$ . Then there exists a unique semiparallel pair of arcs of  $\alpha$ ,  $\{U, V\}$ , such that  $p \in U \cap V$ . Moreover, if  $U = \pi(A_i A_{i+1} \dots A_{i+j})$  and  $V = \pi(A_k A_{k+1} \dots A_{k+j})$  then  $0 \leq j \leq l(\mathcal{W}) - 1$  and there exists a unique  $u \in \{0, 1, \dots, j\}$  such that  $p \in \pi(A_{i+u})$  and  $p \in \pi(A_{k+u})$  in parallel case and  $p \in \pi(A_{k+j-u})$  in the antiparallel case.*

*Proof.* Suppose that  $p \in \pi(A_r) \cap \pi(A_s)$ . Since  $\mathcal{W}$  is cyclically reduced exactly one of the following holds.

- (i)  $\{\bar{x}_r, x_{r+1}\} \cap \{\bar{x}_s, x_{s+1}\} = \emptyset$
- (ii)  $\{\bar{x}_r, x_{r+1}\} \cap \{\bar{x}_s, x_{s+1}\} \neq \emptyset$ , and  $\bar{x}_r = \bar{x}_s$  or  $x_{r+1} = x_{s+1}$ .
- (iii)  $\{\bar{x}_r, x_{r+1}\} \cap \{\bar{x}_s, x_{s+1}\} \neq \emptyset$ , and  $\bar{x}_r = x_{s+1}$  or  $x_{r+1} = \bar{x}_s$ .

If (i) holds, setting  $\{U, V\} = \{\pi(A_r), \pi(A_s)\}$ , taking  $i = r, k = s$  and  $j = 0$ , the result holds.

In the cases (ii) and (iii) one keeps adding segments before and after  $A_r$  and  $A_s$ , until finding arcs landing in different edges at the beginning and at the end. More precisely, assume that (ii) holds. By Lemma 2.11 and since  $p \notin P_\alpha$ , there exist integers  $t, l \geq -1$ , such that  $1 \leq t + l < l(\mathcal{W}) - 1$  and  $t + l$  is maximum with the property that

$$x_{r-t} x_{r-t+1} \dots x_{r+l} = x_{s-t} x_{s-t+1} \dots x_{s+l}.$$

Here, we set  $\{U, V\} = \{\pi(A_{r-t-1} A_{r-t} \dots A_{r+l}), \pi(A_{s-t-1} A_{s-t} \dots A_{s+l})\}$ . Clearly, Definition 3.6(1) holds. Thus by taking  $i = r - t - 1 \pmod{m}, k = s - t - 1 \pmod{m}$ , and  $u = t + 1, j = t + l + 1$ , the proof is complete for this case.

Finally, assume that (iii) holds. Let  $t, l$  be non-negative integers such that  $t + l \geq 1$  and  $t + l$  is the maximum positive integer such that

$$x_{r-t+1}x_{r-t+2}\dots x_r x_{r+1}\dots x_{r+l} = \bar{x}_{s+t}\bar{x}_{s+t-1}\dots \bar{x}_{s+1}\bar{x}_s\dots \bar{x}_{s-l+1}.$$

Here, we define  $U = \pi(A_{r-t}A_{r-t+1}\dots A_{r+l})$  and  $V = \pi(A_{s-l}A_{s-l+1}\dots A_{s+t})$ . Hence  $\{U, V\}$  satisfies Definition 3.6(2), and so taking  $i = r - t - 1 \pmod{m}$ ,  $k = s - l \pmod{m}$ , and  $u = t$  and  $j = t + l + 1$  the proof of the lemma is complete.  $\blacksquare$

We associate to linear word  $x_i x_{i+1} \dots x_{i+j}$  a sequence of copies of  $P_{\mathcal{O}}$  glued in a certain way: Consider  $j - 1$  copies  $P_{\mathcal{O}}^1, P_{\mathcal{O}}^2, \dots, P_{\mathcal{O}}^{j-1}$ , of the polygon  $P_{\mathcal{O}}$ . Set  $\mathbf{S} = (P_{\mathcal{O}}^1 \cup P_{\mathcal{O}}^2 \cup \dots \cup P_{\mathcal{O}}^{j-1}) / \sim$ , where  $\sim$  is the equivalence relation generated by the pairs  $(y, z)$  for which there exists  $h \in \{1, 2, \dots, j - 1\}$  such that  $y$  is in the edge of  $P_{\mathcal{O}}^{i+h-1}$  labeled with  $x_{i+h}$ ,  $z \in P_{\mathcal{O}}^{i+h}$ , and  $z$  is in the edge of  $P_{\mathcal{O}}^{i+h}$  labeled with  $\bar{x}_{i+h}$  and  $\pi(y) = \pi(z)$ . Such an  $\mathbf{S}$  is a *strip of the word*  $x_i x_{i+1} \dots x_{i+j}$  (see Figure 7). Observe that for each  $h \in \{1, 2, \dots, j\}$ ,  $P_{\mathcal{O}}^h$  is embedded in  $\mathbf{S}$ . Thus we can think of  $P_{\mathcal{O}}^h$  as a subset of  $\mathbf{S}$ . The map  $P_{\mathcal{O}}^1 \cup P_{\mathcal{O}}^2 \cup \dots \cup P_{\mathcal{O}}^j \rightarrow \Sigma_{\mathcal{O}}$  which restricted to each of the copies of  $P_{\mathcal{O}}$  is the projection  $\pi$ , induces a map,  $\Pi: \mathbf{S} \rightarrow \Sigma_{\mathcal{O}}$ .

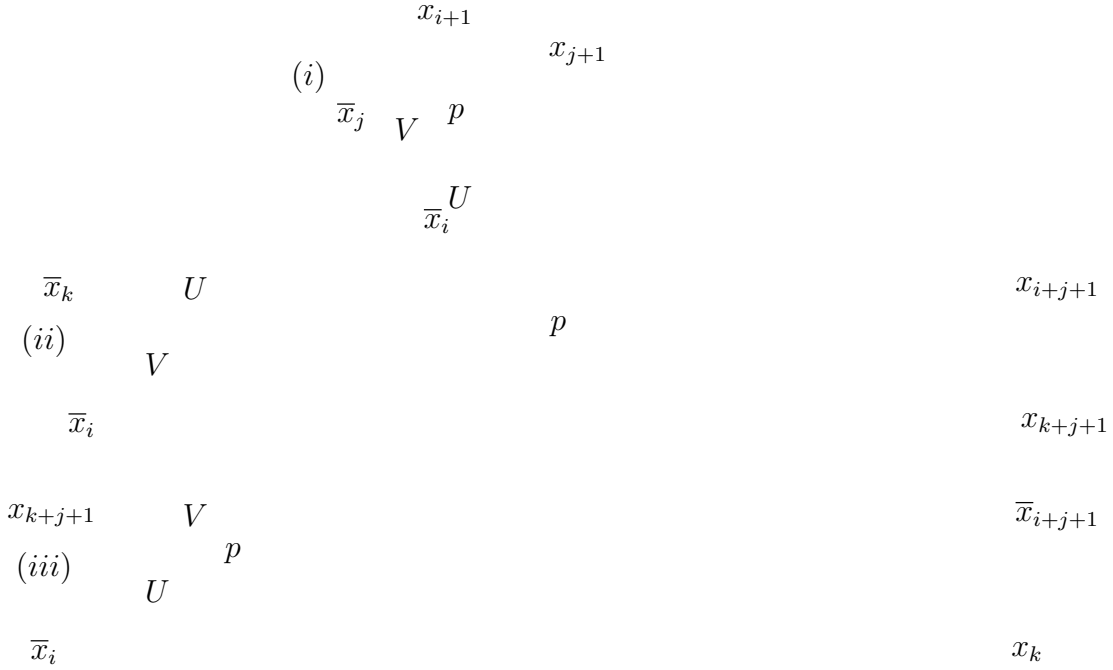


Figure 7: Strips and semiparallel pairs.

Let  $C(\{U, V\})$  denote the subset of points of  $\mathfrak{l}_\alpha$  which are associated to  $\{U, V\}$  by Lemma 3.7.

Denote the image of a map  $\beta$  by  $\text{Im}(\beta)$ . By Lemma 3.7 we have:

**Lemma 3.8.** *Let  $U = \pi(A_i A_{i+1} \dots A_{i+j})$ ,  $V = \pi(A_k A_{k+1} \dots A_{k+j})$  be a semiparallel pair of arcs of a segmented representative  $\alpha$  and let  $\mathbf{S}$  be the strip of the underlying word of  $U$ . Then there exist two continuous maps,  $\mu, \nu: [0, 1] \rightarrow \mathbf{S}$  such that*

- (1)  $\Pi(\text{Im}(\mu)) = U$  and  $\Pi(\text{Im}(\nu)) = V$ . Moreover, for each  $h \in \{0, 1, 2, \dots, j\}$ ,  $\Pi(\text{Im}(\mu) \cap P_{\mathcal{O}}^h) = \pi(A_{i+h})$  and if the pair of arcs  $\{U, V\}$  are parallel (resp. antiparallel) then  $\Pi(\text{Im}(\nu) \cap P_{\mathcal{O}}^h) = \pi(A_{k+h})$  (resp.  $\Pi(\text{Im}(\nu) \cap P_{\mathcal{O}}^h) = \pi(A_{k+j-h})$ ).
- (2)  $\Pi(\text{Im}(\mu) \cap \text{Im}(\nu)) = C(\{U, V\})$ . Furthermore, the restriction of  $\Pi$  to  $\text{Im}(\mu) \cap \text{Im}(\nu)$  is a bijection between  $\text{Im}(\mu) \cap \text{Im}(\nu)$  and  $C(\{U, V\})$ .

**Theorem 3.9.** *Let  $\mathcal{W} = \mathbf{c}(x_0 x_1 \dots x_{m-1})$  be a cyclically reduced cyclic word in the letters of  $\mathbb{A}_n$  and let  $\alpha$  be a minimal segmented representative of  $\mathcal{W}$ . Then self-intersection points of  $\alpha$  in  $\mathbf{l}_\alpha$  correspond bijectively to the set of pairs of linked pairs of  $\mathcal{W}$  of the form  $\{(P, Q), (Q, P)\}$ .*

*Proof.* We claim that different points in  $\mathbf{l}_\alpha$  cannot be assigned to the same pair of arcs by Lemma 3.7. Indeed let  $U = \pi(A_i A_{i+1} \dots A_{i+j})$ ,  $V = \pi(A_k A_{k+1} \dots A_{k+j})$  be a semiparallel pair of arcs and assume that  $\{U, V\}$  are assigned to more than one point. Take two of these points  $p, q$ . By Lemma 3.7 we can assume that there exist  $s, t \in \{0, 1, \dots, j\}$  such that  $s < t$ ,  $p \in \pi(A_{i+s})$ ,  $q \in \pi(A_{i+t})$  and there are no points assigned to  $\{U, V\}$  in the arc  $\pi(A_{i+s+1} \dots A_{i+t-1})$ . Let  $\mu, \nu$  and  $\mathbf{S}$  be as in Lemma 3.8. Let  $P, Q \in \mathbf{S}$  be such that  $\Pi(P) = p$  and  $\Pi(Q) = q$ . The subarc of  $\text{Im}(\mu)$  from  $P$  to  $Q$  and the subarc of  $\text{Im}(\nu)$  from  $Q$  to  $P$  bound a disk. Therefore the subarc of  $U$  from  $p$  to  $q$  and the subarc of  $V$  from  $q$  to  $p$  bound the image of a disk. Since the bigons of  $\alpha$  have endpoints in  $\mathbf{P}_\alpha$ ,  $p, q \notin \mathbf{l}_\alpha$ , a contradiction. So the claim holds.

Now, let  $p \in \mathbf{l}_\alpha$ , let  $\{U, V\}$  be the pair of arcs assigned to  $p$  by Lemma 3.7 and let  $\mathbf{P}$  and  $\mathbf{Q}$  be the underlying words of  $U$  and  $V$  respectively. Let  $\mu, \nu$  and  $\mathbf{S}$  be as in Lemma 3.8. By the above claim  $p$  is the only point in  $\text{Im}(\mu) \cap \text{Im}(\nu)$ . Therefore  $(\mathbf{P}, \mathbf{Q})$  is a linked pair (see Figure 7).

Conversely, if  $(\mathbf{P}, \mathbf{Q})$  is a linked pair of occurrences of subwords of  $\mathcal{W}$ ,  $U$  and  $V$  are the arcs with underlying words  $\mathbf{P}, \mathbf{Q}$  respectively, then  $U, V$  is a semiparallel pair of arcs. Hence we can apply Lemma 3.8. By the definition of linked pair, one has  $\text{Im}(\mu) \cap \text{Im}(\nu)$  contains a single point  $P$ , (see Figure 7). The pair of arcs assigned to  $P$  by Lemma 3.7 is  $\{U, V\}$ . Hence the proof of the theorem is complete. ■

**Remark 3.10.** By construction, a minimal segmented representative of a primitive cyclic reduced word  $\mathcal{W}$  has the minimal number of self-intersection points in its free homotopy class. Hence by Theorem 3.9, the minimal number of self-intersection points of pairs of representatives  $\mathcal{W}$  equals half of the cardinality of the set of linked pairs of  $\mathcal{W}$ ,  $\mathbf{LP}_1(\mathcal{W})$ . This is an equivalent form of a result found in stages by Birman and Series [1], and Cohen and Lustig in [4], (see also [10]). □

### 3.3 Study of intersection points of a pair of segmented representatives

A pair of representatives  $\alpha, \beta$  of a pair of reduced cyclic words  $\mathcal{V}$  and  $\mathcal{W}$  such that all intersections are transverse double points has *minimal intersection* if every pair of representatives of  $\mathcal{V}$  and  $\mathcal{W}$ , such that their intersection consists in finitely many transverse double points, has at least as many intersection points as  $\alpha$  with  $\beta$ .

The next result is analogous to Proposition 3.4.

**Proposition 3.11.** *For each pair of reduced cyclic words  $\mathcal{V}$  and  $\mathcal{W}$  there exists a pair of segmented representatives  $\alpha$  and  $\beta$  with the following properties: The union of  $\alpha$  and  $\beta$  does not have bigons, the endpoints of the arcs of  $\alpha$  do not intersect the endpoints of the arcs of  $\beta$  and there are no triple intersection points between  $\alpha$  and  $\beta$ . Furthermore, if  $\mathcal{V}$  and  $\mathcal{W}$  are not powers (positive or negative) of the same cyclic word then  $\alpha$  and  $\beta$  have minimal intersection.*

*Finally, if  $\mathcal{V}$  and  $\mathcal{W}$  are (positive or negative) powers of a primitive cyclic word of length  $k$ ,  $\alpha = \pi(A_0 A_1 \dots A_m)$  and  $\beta = \pi(B_0 B_1 \dots B_l)$ , and if  $k$  divides  $i - j$ , for some  $i \in \{0, 1, \dots, m\}$ , and  $j \in \{0, 1, \dots, l\}$  then  $\pi(A_i) \cap \pi(B_j) = \emptyset$ .*

*Proof.* First assume that  $\mathcal{V} = \mathcal{V}_1^r$ ,  $\mathcal{W} = \mathcal{W}_1^s$  where  $\mathcal{V}_1$  and  $\mathcal{W}_1$  are two different primitive cyclic words,  $\mathcal{V}_1 \neq \overline{\mathcal{W}_1}$  and  $r, s \geq 1$ .

Construct two representatives of  $\mathcal{V}_1$  and  $\mathcal{W}_1$   $\alpha_1$  and  $\beta_1$ , as in Proposition 3.3, considering two geodesic represented by  $\mathcal{V}_1$  and  $\mathcal{W}_1$ , respectively. Observe that  $\alpha_1 \cup \beta_1$  does not have bigons. If there are triple intersection points of  $\cup_{1 \leq i \leq n} E_{a_i} \cup \alpha_1 \cup \beta_1$ , deform  $\alpha \cup \beta$  as in Proposition 3.3 to  $\alpha_1 \cup \beta_1$  where  $\alpha_1$  and  $\beta_1$  are a pair of curves such that  $\alpha_1 \cup \beta_1$  does not have bigons, nor triple intersection points. Now, construct  $\alpha$  and  $\beta$  using  $\alpha_2$  and  $\beta_2$  respectively as in the proof of Proposition 3.4. Observe that if  $\alpha \cup \beta$  has a bigon then  $\alpha_2 \cup \beta_2$  has a bigon.

The minimal number of intersection points of  $\alpha$  and  $\beta$ , equals the number of intersection points of  $\alpha_1^r$  and  $\beta_1^s$  (where the  $p$ -th power of a representative means run along the representative  $p$  times). Using this, it is not hard to see that  $\alpha$  and  $\beta$  have minimal intersection.

Now, assume that  $\mathcal{V}$  and  $\mathcal{W}$  are powers (positive or negative) of a word  $\mathcal{X}$ . We can assume that  $\mathcal{X}$  is primitive. Let  $\gamma$  be a segmented representative of  $\mathcal{X}$ . Consider an annulus  $N$  in  $\Sigma_{\mathcal{O}}$  around  $\gamma$ , so that both boundary components of  $N$  are also segmented representatives of  $\mathcal{X}$ . Then  $N \setminus \gamma$  is the disjoint union of two annuli,  $N_1$  and  $N_2$ . Let  $\alpha$  and  $\beta$  be arcs representatives of  $\mathcal{V}$  and  $\mathcal{W}$  such that  $\alpha \subset N_1$  and  $\beta \subset N_2$ , and every arc of  $\alpha$  intersects every arc of  $\beta$  in at most one point. Notice that if  $\alpha$  and  $\beta$  have a bigon, then  $\gamma$  has a bigon. So the proof is complete.  $\blacksquare$

Let  $\mathcal{V}$  and  $\mathcal{W}$  be a pair of reduced cyclic words. A pair of representatives  $\alpha$  and  $\beta$  as in Proposition 3.11 will be called *good representatives of  $\mathcal{V}$  and  $\mathcal{W}$*

The proof of the next result is analogous to the proof of Lemma 3.7 but one needs to consider the following two cases separately: either the two curves are powers (positive or negative) of the same primitive word or they are not. In the former case, one also needs to apply Proposition 2.13.

**Lemma 3.12.** *Let  $\mathcal{V} = \mathbf{c}(y_0y_1 \dots y_{h-1})$  and  $\mathcal{W} = \mathbf{c}(x_0x_1 \dots x_{m-1})$  be cyclically reduced cyclic words and let  $\alpha, \beta$  be good representatives of  $\mathcal{V}$  and  $\mathcal{W}$ . Let  $p \in \alpha \cap \beta$ . Then there exists a unique semiparallel pair of arcs of  $\alpha$  and  $\beta$ ,  $\{U, V\}$ , such that  $p \in U \cap V$ . Moreover, if  $U = \pi(A_iA_{i+1} \dots A_{i+j})$  and  $V = \pi(B_kB_{k+1} \dots B_{k+j})$  then there exists a unique  $u \in \{0, 1, \dots, j\}$  such that  $p \in \pi(A_{i+u})$  and  $p \in \pi(B_{j+u})$  if  $U$  and  $V$  are parallel and  $p \in \pi(B_{k+j-u})$ , if  $U$  and  $V$  are antiparallel.*

The next result is the equivalent of Theorem 3.9 for pairs of cyclic words, and the proof uses analogous ideas.

**Theorem 3.13.** *Let  $\mathcal{V}$  and  $\mathcal{W}$  be two primitive reduced cyclic words. Let  $\alpha$  and  $\beta$  be a good pair of representatives of  $\mathcal{V}$  and  $\mathcal{W}$  (as in Proposition 3.11). Then there is a one to one correspondence between intersection points of  $\alpha$  and  $\beta$  and  $\mathbf{LP}_2(\mathcal{V}, \mathcal{W})$ , the set of linked pairs of  $\mathcal{V}$  and  $\mathcal{W}$ .*

The next remark is analogous to Remark 3.10.

**Remark 3.14.** Let  $\mathcal{V}$  and  $\mathcal{W}$  be primitive reduced cyclic words. The minimal arc representatives of  $\mathcal{V}$  and  $\mathcal{W}$  have minimal intersection, and then, the minimal number of intersection points of representatives of  $\mathcal{V}$  and  $\mathcal{W}$  equals the cardinality of the set of linked pairs of  $\mathcal{V}$  and  $\mathcal{W}$ ,  $\mathbf{LP}_2(\mathcal{V}, \mathcal{W})$ . An equivalent form of this result was obtained by Cohen and Lustig, [4].  $\square$

## 4 The isomorphism

Let  $\langle, \rangle$  and  $\Delta$  be the bracket and cobracket of the Lie bialgebra structure defined geometrically in Appendix A on the vector space generated by non-trivial free homotopy classes of curves on an oriented smooth surface (Appendix B). By Lemma 3.1,  $\langle, \rangle$  and  $\Delta$  are defined on  $\mathbb{V}$ , the vector space generated by all cyclically reduced cyclic words on the alphabet  $\mathbb{A}_n$ . In other words,  $(\mathbb{V}, \langle, \rangle, \Delta)$  is a Lie bialgebra. On the other hand, we have defined in Section 2 two linear maps  $\mathbb{V} \otimes \mathbb{V} \xrightarrow{[\cdot, \cdot]} \mathbb{V}$  and  $\mathbb{V} \xrightarrow{\delta} \mathbb{V} \otimes \mathbb{V}$ . We will show that the geometrically defined maps  $\langle, \rangle$  and  $\Delta$  coincide with the combinatorially defined maps  $[\cdot, \cdot]$  and  $\delta$ .

**Proposition 4.1.** *For each  $\mathcal{V}, \mathcal{W} \in \mathbb{V}$ ,  $\delta(\mathcal{V}) = \Delta(\mathcal{V})$  and  $[\mathcal{V}, \mathcal{W}] = \langle \mathcal{V}, \mathcal{W} \rangle$ .*

*Proof.* We prove that  $\delta(\mathcal{V}) = \Delta(\mathcal{V})$ . Write  $\mathcal{V} = \mathcal{W}^r$  where  $\mathcal{W}$  is a primitive reduced cyclic word. Let  $\alpha$  be a segmented representative of  $\mathcal{V}$  and let  $S$  be the set of self-intersection points of  $\alpha$ . Recall that  $S$  is the disjoint union of two subsets,  $l_\alpha$ , and  $P_\alpha$

where  $P_\alpha$  is the set of intersection points of  $A_{rm}$  with  $A_{km}$ , for  $k \in \{1, 2, \dots, r-1\}$ . By Theorem 3.9,  $I_\alpha$  is in correspondence with the pairs of linked pairs of  $\mathcal{W}$  of the form  $\{(P, Q), (Q, P)\}$ . For each  $q \in S$ , if  $\mathcal{V}_1^q$  and  $\mathcal{V}_2^q$  are as in Appendix B then  $\Delta(\mathcal{V}) = \sum_{q \in S} \mathcal{V}_1^q \otimes \mathcal{V}_2^q - \mathcal{V}_2^q \otimes \mathcal{V}_1^q$ . By definition of  $P_\alpha$ ,  $\sum_{q \in P_\alpha} \mathcal{V}_1^q \otimes \mathcal{V}_2^q - \mathcal{V}_2^q \otimes \mathcal{V}_1^q = \sum_{i=1}^{r-1} \mathcal{W}^i \otimes \mathcal{W}^{r-i} - \mathcal{W}^{r-i} \otimes \mathcal{W}^i = 0$ .

Consider  $p \in I_\alpha$  and let  $(P, Q) \in \mathbf{LP}_1(\mathcal{V})$  be the linked pair assigned to  $p$  by Theorem 3.9. If  $\text{sign}(P, Q) = 1$  then  $\mathcal{W}_1^p = \delta_1(P, Q)$  and  $\mathcal{W}_2^p = \delta_2(P, Q)$  and if  $\text{sign}(P, Q) = -1$  then  $\mathcal{W}_1^p = \delta_2(P, Q)$  and  $\mathcal{W}_2^p = \delta_1(P, Q)$ . Hence  $\Delta(\mathcal{V}) = \sum_{q \in S_\alpha} \mathcal{W}_1^q \otimes \mathcal{W}_2^q - \mathcal{W}_2^q \otimes \mathcal{W}_1^q = \delta(\mathcal{V})$ .

The proof of  $[\mathcal{V}, \mathcal{W}] = \langle \mathcal{V}, \mathcal{W} \rangle$  is analogous. ■

By Propositions 4.1 and B.1, we have the following.

**Theorem 4.2.**  $(\mathbb{V}, [, ], \delta)$  is an involutive Lie bialgebra.

## 5 Applications

A free homotopy class is *simple* if it has a simple representative, i.e., a non-selfintersecting representative. As simple closed curves, simple free homotopy classes satisfy statements that do not hold for the rest of the free homotopy classes. The extension of Goldman's result, Theorem 5.3, is one of these. On the other hand, by definition, if a free homotopy class is simple then its cobracket is zero. Using the algorithm we describe in this paper, we developed a program in  $C^{++}$  that, given the surface symbol  $\mathcal{O}$ , computes the bracket and the cobracket of reduced cyclic words. By running this program we found classes with cobracket zero which are not powers of a simple class, some of which we list in Examples 5.7 and 5.8. By means of the same programs, we found examples of pairs of non-simple classes with bracket zero, and no disjoint representatives (Example 5.5).

### 5.1 Topological proof of Goldman's result

The next lemma will be used in the proof of Theorem 5.3.

**Lemma 5.1.** *Let  $\mathcal{V}$  be a cyclically reduced cyclic word and let  $x \in \mathbb{A}_n$ . If  $(P_1, Q_1), (P_2, Q_2) \in \mathbf{LP}_2(\mathcal{V}, c(x))$  are linked pairs and  $\gamma(P_1, Q_1) = \gamma(P_2, Q_2)$  then  $(P_1, Q_1) = (P_2, Q_2)$  and so  $\text{sign}(P_1, Q_1) = \text{sign}(P_2, Q_2)$ .*

*Proof.* Linked pairs of  $\mathcal{V}$  and  $c(x)$  have the form  $(u_1 x^k u_2, x^{k+2})$  and  $(u_1 \bar{x}^k u_2, x^{k+2})$  where  $u_1 x^k u_2$  and  $u_1 \bar{x}^k u_2$  respectively are subwords of  $\mathcal{V}$ ,  $u_1 \neq x$ ,  $u_2 \neq x$  and  $k \geq 0$  (in the first case, Definition 2.1(1) or (2) holds and in the second case, Definition 2.1(3) holds).

Observe that if  $(u_1 \bar{x}^k u_2, x^{k+2})$  (resp.  $(u_1 x^k u_2, x^{k+2})$ ) is a linked pair then

$$l(\gamma(u_1 \bar{x}^k u_2, x^{k+2})) = l(\mathcal{V}) + 1 \quad (\text{resp. } l(\gamma(u_1 x^k u_2, x^{k+2})) = l(\mathcal{V}) - 1).$$

Hence if  $(P_1, Q_1), (P_2, Q_2) \in \mathbf{LP}_2(\mathcal{V}, c(x))$  are linked pairs such that  $\gamma(P_1, Q_1) = \gamma(P_2, Q_2)$  then either  $P_1 = y_1 x^k y_2$  and  $P_2 = z_1 x^j z_2$  or  $P_1 = y_1 \bar{x}^k y_2$  and  $P_2 = z_1 \bar{x}^j z_2$ .

Assume that  $P_1 = y_1 \bar{x}^k y_2$ ,  $P_2 = z_1 \bar{x}^j z_2$ ,  $\gamma(P_1, x^{k+2}) = \gamma(P_2, x^{j+2})$ , where  $k, j \geq 1$ . There exist (possibly empty) linear words  $X, Y$  such that  $\mathcal{W} = c(P_1 X P_2 Y)$ , where, if  $X$  (resp.  $Y$ ) is empty then the last letter of  $P_1$  (resp.  $P_2$ ) can coincide with the first letter of  $P_2$  (resp.  $P_1$ ). Hence  $\gamma(P_1, x^{k+2}) = c(y_1 x^{k-1} y_2 X P_2 Y)$  and  $\gamma(P_2, x^{j+2}) = c(P_1 X z_1 x^{j-1} z_2 Y)$ .

If  $\mathcal{W}$  is a cyclic word, denote by  $\mathcal{C}_{\mathcal{W}}$  the set of subwords of  $\mathcal{W}$  of the form  $u_1 \bar{x}^h u_2$  with  $u_1, u_2 \neq x$  and  $h \geq 0$ . Thus,

$$\mathcal{C}_{\mathcal{W}} \setminus \{P_1, P_2\} = \mathcal{C}_{\gamma(P_1, x^{k+2})} \setminus \{y_1 x^{k-1} y_2, P_2\} = \mathcal{C}_{\gamma(P_2, x^{j+2})} \setminus \{P_1, z_1 x^{j-1} z_2\}.$$

Since  $\gamma(P_1, x^{k+2}) = \gamma(P_2, x^{j+2})$ , we have that  $\mathcal{C}_{\gamma(P_1, x^{k+2})} = \mathcal{C}_{\gamma(P_2, x^{j+2})}$ . Hence

$$\{y_1 x^{k-1} y_2, P_2\} = \{P_1, z_1 x^{j-1} z_2\}$$

and then, finally,  $P_1 = P_2$ , as desired.

In order to prove the above result for the other case, one considers  $\mathcal{C}_{\mathcal{W}}$ , the set of subwords of  $\mathcal{W}$  of the form  $u_1 x^h u_2$  with  $u_1, u_2 \neq x$  and  $h \geq 0$  and proceeds as above. ■

**Lemma 5.2.** *Let  $\Sigma$  be an oriented surface with boundary and let  $\lambda$  be a simple closed curve non-homologous to zero. Then there exists an alphabet  $\mathbb{A}_n$ , a surface word  $\mathcal{O}$  for that alphabet and a homeomorphism  $\rho: \Sigma \rightarrow \Sigma_{\mathcal{O}}$  such that the cyclic word for the free homotopy class of  $\rho(\lambda)$  has one letter.*

*Proof.* One chooses a non-selfintersecting arc  $a_1$  transversal to  $\lambda$ , with endpoints on boundary components and intersecting  $\lambda$  exactly once. Then one cuts  $\Sigma$  open along  $\lambda$  and  $a_1$ . Now, one studies the separating and non-separating case.

In the separating case, this procedure yields two surfaces. Then one continues cutting open these two surfaces along non-selfintersecting arcs  $a_2, a_3, \dots, a_n$  with endpoints in the boundary components of  $\Sigma$ , until obtaining two disks.

In the non-separating case one cuts the surface open along non-selfintersecting arcs  $a_2, a_3, \dots, a_{n-1}$  with endpoints in the boundary components of  $\Sigma$ , until obtaining a disk. After gluing this disk along  $\lambda$ , one obtains a cylinder. Then one chooses an arc  $a_n$  with endpoints in the boundary components of  $\Sigma$  such that cutting the cylinder open along  $a_n$  yields a disk.

Denote by  $P_{\mathcal{O}}$  the disk obtained by cutting  $\Sigma$  along  $a_1, a_2, \dots, a_n$ . Then taking the identity map as the homeomorphism  $\rho$  completes the proof. ■

We state the next theorem using the correspondence between free homotopy classes and reduced cyclic words given by Lemma 3.1.

**Theorem 5.3.** *Let  $\mathcal{V}$  and  $\mathcal{W}$  be cyclic reduced words and such that  $\mathcal{V}$  has a simple representative which is non-homologous to zero. Then there exists two representatives  $\alpha$  and  $\beta$  of  $\mathcal{V}$  and  $\mathcal{W}$  respectively such that the bracket of  $\mathcal{V}$  and  $\mathcal{W}$  computed using the intersection points of  $\alpha$  and  $\beta$  does not have cancellation. In other words, the number of terms (counted with multiplicity) of  $\langle \mathcal{V}, \mathcal{W} \rangle$  equals the minimal number of intersection points of representatives of  $\mathcal{V}$  and  $\mathcal{W}$ .*

*Proof.* We can assume that  $\mathcal{V}$  and  $\mathcal{W}$  are not powers of the same word because if  $\mathcal{W}$  is a power of  $\mathcal{V}$ , then  $\mathcal{V}$  and  $\mathcal{W}$  have disjoint representatives.

By Proposition 3.11,  $\mathcal{V}$  and  $\mathcal{W}$  have minimal intersection. Moreover, the number of terms of the brackets is the minimal number of intersection points. Therefore, by Proposition 4.1 if the bracket of two words in a certain alphabet has cancellation, then the bracket of the corresponding words in another alphabet also has cancellation.

Thus by Lemma 5.2 we can assume that  $\mathcal{V}$  has only one letter. Then the result follows from Lemma 5.1. ■

The next is a particular case of Theorem 5.3.

**Corollary 5.4.** *Let  $\mathcal{V}$  and  $\mathcal{W}$  be two free homotopy classes of curves on a surface with boundary and assume that  $\mathcal{V}$  has a simple representative which is non-homologous to zero. Then the bracket of  $\mathcal{V}$  and  $\mathcal{W}$  is zero if and only if  $\mathcal{V}$  and  $\mathcal{W}$  have disjoint representatives.*

The next example shows that the hypothesis that one of the classes is simple cannot be omitted from Theorem 5.3.

**Example 5.5.** Let  $\mathcal{O} = a_1\bar{a}_1a_2\bar{a}_2$  and  $\mathcal{V} = a_1\bar{a}_2\bar{a}_2$ ,  $\mathcal{W} = a_1\bar{a}_2$ . Then  $[\mathcal{V}, \mathcal{W}] = 0$  but, by Theorem 3.13 and Remark 3.14 every pair of geometric representatives of  $\mathcal{V}$  and  $\mathcal{W}$  intersects at least in two points. □

**Example 5.6.** Let  $\mathcal{O} = a_1a_2\bar{a}_1\bar{a}_2a_3a_4\bar{a}_3\bar{a}_4$ . The following is a list of pairs of cyclic words  $(\mathcal{V}, \mathcal{W})$  such that  $[\mathcal{V}, \mathcal{W}] = 0$  but each pair of representatives of  $\mathcal{V}$  and  $\mathcal{W}$  is not disjoint, in the parenthesis there is written the minimal number of intersection points of  $\mathcal{V}$  and  $\mathcal{W}$ . (We thank Vladimir Chernov for pointing out a typo in our previous version).

- (i)  $\mathcal{V} = c(a_1\bar{a}_2\bar{a}_4a_1\bar{a}_2)$  and  $\mathcal{W} = c(a_1\bar{a}_2\bar{a}_4\bar{a}_4)$ , (2)
  - (ii)  $\mathcal{V} = c(a_1a_3a_3a_3\bar{a}_2)$  and  $\mathcal{W} = c(a_1a_3a_3\bar{a}_2)$ , (4)
- 

## 5.2 A counterexample

**Example 5.7.** Let  $\mathcal{O} = a_1a_2\bar{a}_1\bar{a}_2$ . So the surface  $S_{\mathcal{O}}$  associated to  $\mathcal{O}$  is a punctured torus. Let  $\mathcal{W} = c(a_1a_1a_2a_2)$ . Then  $\delta(\mathcal{W}) = 0$ . On the other hand, by Theorem 3.9

every geometric representative of  $\mathcal{W}$  has at least one self-intersection point. More generally, for every pair of integers  $i, j$ , if  $\mathcal{W} = \mathbf{c}(a_1^i a_2^j)$  then every representative of  $\mathcal{W}$  has at least  $(i - 1)(j - 1)$  self-intersection points and  $\delta(\mathcal{W}) = 0$ .  $\square$

$E_{a_1}$   
 $E_{a_2}$

Figure 8: A representative of  $\mathbf{c}(a_1 a_1 a_2 a_2)$  in the punctured torus

**Example 5.8.** Let  $\mathcal{O} = a_1 a_2 \bar{a}_1 \bar{a}_2 a_3 a_4 \bar{a}_3 \bar{a}_4$ . Then  $S_{\mathcal{O}}$  is a punctured surface of genus two. Here is a sample of cyclic words with cobracket zero, with the minimal number of self-intersection points of the representatives written in parenthesis:  $\mathbf{c}(a_3 a_4 \bar{a}_3 a_4), (1)$ ;  $\mathbf{c}(a_2 a_3 a_2 a_3 \bar{a}_1 \bar{a}_1 \bar{a}_1), (2)$ ;  $\mathbf{c}(a_3 a_1 \bar{a}_2 a_3 a_1 \bar{a}_2 a_3 a_1 \bar{a}_2 \bar{a}_2 \bar{a}_2), (2)$ ;  $\mathbf{c}(\bar{a}_2 \bar{a}_2 a_1 a_1 a_1 a_1 \bar{a}_2 a_1 a_1 a_1 a_1), (8)$ ;  $\mathbf{c}(\bar{a}_2 a_3 a_4 a_4 a_4 a_4 a_1 \bar{a}_2 a_3 a_1), (4)$ ;  $\mathbf{c}(a_3 a_1 a_3 a_1 a_2 a_2) (1)$ .  $\square$

Using Examples 5.7 and 5.8, we can construct examples of classes with cobracket zero which are not powers of simple classes, in any surface of genus at least one (with or without boundary) except the (closed) torus. Observe that, every free homotopy class on the torus is a multiple of a simple class, and so the cobracket is identically zero.

### 5.3 Open problems

We could not find examples of non-simple free homotopy classes in the pair of pants with cobracket zero. This leaves open the following.

**Question.** *Let  $\Sigma$  be two sphere with three or more punctures. Is every class on  $\Sigma$  with cobracket zero a multiple of a simple class?*

When we told Turaev about the above examples, he asked the following question:

**Question.** *Are there further “secondary” operations necessary to detect simple loops?*

Our computations suggest an answer for Turaev’s question:

**A conjectural characterization of simple curves.** *A primitive reduced cyclic word  $\mathcal{V}$  has a simple representative if and only if  $[\mathcal{V}, \bar{\mathcal{V}}] = 0$ .*

By running our program, we could see that for a surface of genus two and one puncture, as well as for the pair of pants, for all the reduced cyclic words  $\mathcal{V}$  of length at most fifteen, the bracket of  $\mathcal{V}$  with  $\overline{\mathcal{V}}$  has exactly twice as many terms as the minimal number of self-intersection points of  $\mathcal{V}$ . Hence we have the following.

**Theorem 5.9.** (1) *On the sphere with three punctures all the cyclic words with at most sixteen letters, except the multiples of three peripheral curves, have non-zero cobracket.*

(2) *On the torus with two punctures all the cyclic words  $\alpha$  with at most fifteen letters have the property that twice the minimal number of self-intersection points equals the number of terms of the bracket  $[\alpha, \overline{\alpha}]$  in the natural basis.*

**Question.** *Let  $\mathcal{V}$  be a primitive reduced cyclic word. Is the number of terms of the bracket of  $\mathcal{V}$  with  $\overline{\mathcal{V}}$  equal to twice the minimal number of self-intersection points of  $\mathcal{V}$ ?*

Some examples we have computed suggest that even a more general result may hold:

**Question.** *Let  $n$  and  $m$  be two different non-zero integers and let  $\mathcal{V}$  be a primitive reduced cyclic word. Is the number of terms of the bracket of  $\mathcal{V}^n$  with  $\mathcal{V}^m$  equal to  $2|m.n|$  multiplied by the minimal number of self-intersection points of  $\mathcal{V}$ ?*

## A Definition of Lie bialgebra

Let  $V$  denote a vector space. In order to recall the definition of Lie bialgebra we need two auxiliary linear maps  $\omega: V \otimes V \otimes V \longrightarrow V \otimes V \otimes V$  and  $s: V \otimes V \longrightarrow V \otimes V$  defined by  $\omega(\mathcal{U} \otimes \mathcal{V} \otimes \mathcal{W}) = \mathcal{W} \otimes \mathcal{U} \otimes \mathcal{V}$  and  $s(\mathcal{V} \otimes \mathcal{W}) = \mathcal{W} \otimes \mathcal{V}$  for each triple of cyclic words  $\mathcal{U}, \mathcal{V}, \mathcal{W} \in V$ .

A *Lie algebra* on a vector space  $V$  is given by a linear map  $\langle \cdot, \cdot \rangle: V \otimes V \longrightarrow V$  such that  $\langle \cdot, \cdot \rangle \circ s = -\langle \cdot, \cdot \rangle$  (*skew symmetry*) and  $\langle \cdot, \cdot \rangle (\text{Id} \otimes \langle \cdot, \cdot \rangle) (\text{Id} + \omega + \omega^2) = 0$  (*Jacobi identity*). A *Lie coalgebra* on  $V$  is given by a linear map  $\Delta: V \longrightarrow V \otimes V$  such that  $s \circ \Delta = -\Delta$  (*co-skew symmetry*) and  $(\text{Id} + \omega + \omega^2) (\text{Id} \otimes \Delta) \Delta = 0$  (*cojacobi identity*).  $(V, \langle \cdot, \cdot \rangle, \Delta)$  is a *Lie bialgebra* if  $(V, \langle \cdot, \cdot \rangle)$  is a Lie algebra,  $(V, \Delta)$  is a Lie coalgebra and the compatibility equation  $\Delta \langle a, b \rangle = \langle \Delta a, b \rangle + \langle a, \Delta b \rangle$  holds for every  $a, b \in V$  where  $\langle a, b \otimes c \rangle = -\langle b \otimes c, a \rangle = \langle a, b \rangle \otimes c + b \otimes \langle a, c \rangle$ .  $(V, \langle \cdot, \cdot \rangle, \Delta)$  is an *involutive Lie bialgebra* if  $(V, \langle \cdot, \cdot \rangle, \Delta)$  is a Lie bialgebra and  $\langle \cdot, \cdot \rangle \circ \Delta = 0$  on  $\mathbb{V}$

**Remark A.1.** According to [6], the equation  $\langle \cdot, \cdot \rangle \circ \Delta = 0$  is the infinitesimal analogue of having an antipodal map on a Hopf algebra with square equal to the identity.  $\square$

## B The Lie bialgebra of curves on a surface

Let  $\mathbb{W}$  denote the vector space generated by all (trivial and non-trivial) free homotopy classes of loops on a surface and let  $\mathbb{W}_0$  denote the vector subspace generated by the class of the trivial loop. In [7], Goldman defined a Lie algebra structure on  $\mathbb{W}$ . Turaev proved that this structure passes to the quotient  $\mathbb{W}/\mathbb{W}_0$  and that there exists a cobracket on  $\mathbb{W}/\mathbb{W}_0$  which makes the whole structure a Lie bialgebra on  $\mathbb{W}/\mathbb{W}_0$ . Notice that the quotient vector space  $\mathbb{W}/\mathbb{W}_0$  is canonically isomorphic to  $\mathbb{V}$ . If  $w \in \mathbb{W}$ , we denote by  $\{w\}_0$  the equivalence class of  $\mathbb{W}/\mathbb{W}_0$  containing  $w$ .

We now recall the constructions of Goldman and Turaev. Let  $\mathcal{V}$  and  $\mathcal{W}$  be two non-trivial free homotopy classes of curves. Choose representatives,  $A$  of  $\mathcal{V}$  and  $B$  of  $\mathcal{W}$  in general position. Hence the intersection of  $A$  and  $B$  consists in a finite number of double points,  $p_1, p_2, \dots, p_m$ . To each of such points  $p_i$  we assign a free homotopy class and a sign: the class that contains the loop that starts at  $p_i$ , runs around  $A$ , and then around  $B$ , and the sign obtained by comparing the orientation of the surface with the orientation given by the branches of  $A$  and  $B$  (in that order) coming out of  $p_i$  (positive if it coincides, negative otherwise). The bracket  $\langle \mathcal{V}, \mathcal{W} \rangle$  is defined to be the sum of all the signed free homotopy classes. In symbols,

$$\langle \mathcal{V}, \mathcal{W} \rangle = \sum_{p \in A \cap B} \text{sign}_p(A, B) \{ \text{class}(A \cdot_p B) \}_0$$

where  $\text{sign}_p(A, B)$  denotes the sign,  $A \cdot_p B$  means the usual multiplication of based loops at  $p$  and  $\text{class}(X)$  denotes the conjugacy class of a loop  $X$ .

In order to define the cobracket, for each non-trivial free homotopy class  $\mathcal{W}$ , choose a representative  $A$  in general position with respect to itself. Thus, its self-intersection points are finitely many double points  $q_1, q_2, \dots, q_m$ . To each of these points  $q_i$  one associates an ordered pair of free homotopy classes  $(\mathcal{W}_1^{q_i}, \mathcal{W}_2^{q_i})$  as follows. Firstly, order the two branches of  $A$  coming out of  $q_i$  in such a way that they define the same orientation as the surface.  $\mathcal{W}_1^{q_i}$  is the conjugacy class of the loop that starts on the first branch, and runs along  $A$  until it arrives to  $q_i$  again. Analogously,  $\mathcal{W}_2^{q_i}$  is the conjugacy class of the loop that starts at the second branch and runs along  $A$  until it finds  $q_i$  again.

The cobracket of  $\mathcal{W}$ , is given by the formula:

$$\Delta(\mathcal{W}) = \sum_{q_1, q_2, \dots, q_m} \{ \mathcal{W}_1^{q_i} \}_0 \otimes \{ \mathcal{W}_2^{q_i} \}_0 - \{ \mathcal{W}_2^{q_i} \}_0 \otimes \{ \mathcal{W}_1^{q_i} \}_0.$$

**Proposition B.1.** *The Goldman-Turaev Lie bialgebra is involutive, i.e.,  $\langle , \rangle \circ \Delta = 0$ .*

*Proof.* Let  $\mathcal{W}$  be a free homotopy class and let  $\alpha$  be a geometric representative of  $\mathcal{W}$  in general position.  $\langle , \rangle \circ \Delta(\mathcal{W})$  is a sum of free homotopy classes over certain ordered pairs  $(p, q)$  of self-intersection points of  $\alpha$ . Denote this set of pairs of self-intersection points by  $D_\alpha$ . Thus,  $(p, q) \in D_\alpha$  if and only if the arcs of the circle between the two

preimages of  $p$  contain exactly one preimage of  $q$ . Hence for each  $(p, q) \in D_\alpha$ , there are four arcs,  $\alpha_1, \alpha_2, \alpha_3$  and  $\alpha_4$  of  $\alpha$  such that  $\alpha_1$  and  $\alpha_3$  go from  $p$  to  $q$ ,  $\alpha_2$  and  $\alpha_4$  go from  $q$  to  $p$  and  $\alpha$  runs along  $\alpha_1$  then  $\alpha_2$ ,  $\alpha_3$  and finally, along  $\alpha_4$ .

Clearly,  $(p, q) \in D_\alpha$  if and only if  $(q, p) \in D_\alpha$ . We claim that for each  $(p, q) \in D_\alpha$ , the two terms of  $\langle \cdot, \cdot \rangle \circ \Delta(\mathcal{W})$  corresponding to  $(p, q)$  cancel with the two terms corresponding to  $(q, p)$ .

The two terms of  $\Delta(\mathcal{W})$  corresponding to  $p$  are:  $s_{1,3}(\text{class}(\alpha_1\alpha_2) \otimes \text{class}(\alpha_3\alpha_4) - \text{class}(\alpha_3\alpha_4) \otimes \text{class}(\alpha_1\alpha_2))$ , where  $s_{1,3} = 1$  if the orientation given by the tangent vector of  $\alpha_1$  at  $p$  and the tangent vector of  $\alpha_3$  at  $p$  coincides with the orientation of the surface and  $s_{1,3} = -1$  otherwise.

Now, the terms of bracket of the above linear combination corresponding to  $q$  are  $2s_{1,3}s_{2,4}\text{class}(\alpha_4\alpha_3\alpha_2\alpha_1)$  where  $s_{2,4} = 1$  if the orientation given by the tangent vector of  $\alpha_2$  at  $q$  and the tangent vector of  $\alpha_4$  at  $q$  coincides with the orientation of the surface and  $s_{2,4} = -1$  otherwise.

The same computation for  $(q, p)$  gives  $-2s_{1,3}s_{2,4}\text{class}(\alpha_4\alpha_3\alpha_2\alpha_1)$ , so the claim is proved and so also the proposition. ■

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