Explorations 1 - MAT 517/MAE 303

Our goal is to study some methods to find the roots of polynomials, as well as discuss properties of parabolas.

1 First software explorations

Exercise 1 *Write a short description about what each of the following demonstrations do. Your target audience is a reasonable high school student who may not remember some mathematical terms such as "root of a polynomials" or "contour line".*

`http://demonstrations.wolfram.com/PolynomialRootsInTheComplexPlane/`
`http://demonstrations.wolfram.com/TheFundamentalTheoremOfAlgebra/`

2 The roots of polynomials of degree two

The following demonstration shows you how to locate the complex roots of a quadratic equation.
`http://demonstrations.wolfram.com/LocatingTheComplexRootsOfAQuadraticEquation/`

Exercise 2 *Prove that what the demonstration do is (mathematically) correct.*

3 What is a parabola?

Exercise 3
1. *Find the equation of the parabola whose focus is the point $P = (a, b)$ and whose directrix is the line l of equation $y = c$, were a, b and c are real numbers and $b \neq c$.*
2. *In geogebra define sliders labeled with a, b and c. Define a point $P = (a, b)$ a line l of equation $y = c$.*
3. *Find the parabola whose focus is the point P and whose directrix is the line l geometrically (as we did in class in a special case.)*

4. *Plot the curve of the equation you found in 1. The two curves should coincide.*

Exercise 4 *In Geogebra, define a line l (not necessarily horizontal) and a point P and find a parabola with focus P and directrix l. Add explanations to help the user of your Geogebra file understand why the curve that appears is indeed a parabola.*

Exercise 5 *Create three arbitrary points in geogebra and find a polynomial P of degree two that passes through the three of them. (You can use Maple or Mathematica in the process) but the final graph should be a the graph of P in Geogebra, passing through the three ("dragable") points you created.*

Use the Geogebra graph you created to study (and answer) the following question: Given any *three points in the plane, is it there a unique polynomial of degree two whose graph passes through the three of them? (Extra credit: Is it there a unique parabola passing through the three of them?)*

Definition 1 *A* hyperbola *is the set of points in a plane, the absolute value of the difference of whose distances from two fixed points is a constant. The two fixed points are called the* foci.

An ellipse *is the set of points in a plane, the sum of whose distances from to fixed points is constant. Each point is called a* focus.

Definition 2 *A* conic section *is a curve of intersection of a plane with a (right, circular) cone. There are three types of curves that occur this way: the* parabola, ellipse *(including the circle as a special case of ellipse) and* hyperbola. *The curve obtained depends on the inclination of the axis of the cone to the cutting plane.*

See `http://demonstrations.wolfram.com/ConicSectionsTheDoubleCone/` *for a demo of these cases.*

A generator *of a cone is a line lying on the cone. If the cutting plane is parallel to one generator then the curve is a parabola. If the cutting plane is parallel to two generators, the conic section obtained is a hyperbola. If the cutting plane is parallel to no generators, the curve is an ellipse.*

In `http://www.khanacademy.org/math/algebra/conic-sections/conic_basics/v/introduction-to-conic-sections` there is good discussion about conic sections.

Definition 3 *A* conic section (or conic) *is the set of points in a plane whose distances to a fixed point P, and a fixed line l, are in a fixed ratio. The fixed point is the* focus, *the fixed line, is the* directriz *and the ration is the* eccentricity.

Exercise 6 *Create two demonstrations (one for the ellipse, one for the hyperbola) in Geogebra illustrating Definition 1. Indicate all the relevant geometric objects that appear (e.g. a point realizing a distance in the circle, the segments between this point and the fixed point). In the Geogebra doc, include a paragraph explaining your figure. (Hint: using a circle around one of the fixed points might help). Extra credit: Study incident rays passing through the foci in the three conic sections (see Definition 4).*

Definition 4 Law of reflection: *An angle made by an incident ray with a flat mirror is the same as the angle made by the reflected ray. For a curved mirror, angles made by incident rays and reflected rays with the tangent to the mirror (at the point where the incident ray intersects the mirror) must be congruent.*

Exercise 7 *Make an app to illustrate Defintion 3. You do not need to use "dragable" directrix or focus, but of course you can. (Hint: Be wise in your choice of focus and directrix,* $(0, 0)$ *and* $x = c$ *will make calculations esier.) Do not use sliders for the point and the directrix.*

Compare with
`http://demonstrations.wolfram.com/ConicSectionsPolarEquations/`.

4 The roots of polynomials of degree three

The goal of this section is to study how to find the roots of a polynomials of degree three, that is, polynomials of the form

$$Ax^3 + Bx^3 + cX^2 + dX + e,$$

where A, B, C, D and E are real numbers and $A \neq 0$.

Exercise 8 *Using the Wolfram Demostration "Sliding the Roots of Cubics"* `http://demonstrations.wolfram.com/SlidingTheRootsOfCubics/`

1. *Find a few (approximate) values of* p *and* q *for which the polynomial* $x^3 + px + q$ *has double roots. Can you guess what are all those values? (As a guide, note that all the values of* b *and* c *such that the quadratic equation* $x^2 + b \cdot x + c$ *has double roots is the set* $\{(b, c) : b^2 = 4c\}$*.)*

2. *Find a few (approximate) values of* p *and* q *for which the polynomial* $x^3 + px + q$ *has three real roots. Can you guess what are all those values?*

3. *Come back to this problem after reading Subsection 4.1, and state for which values of* p *and* q *the equation* $x^3 + px + q$ *has double roots and for which values has three real roots. Does this answer coincide with what you found in the previous two questions?*

Exercise 9 *Finding the roots of a polynomial of degree three is equivalent to find the roots of polynomials of the form*

$$P(x) = x^3 + ax^2 + bx + c.$$

Can you explain why?

Exercise 10 *Check (using technology) that that by substituting x by $t - \frac{a}{3}$ in the polynomial*

$$P(x) = x^3 + ax^2 + bx + c$$

a polynomial of degree three, with no quadratic term, is obtained. Explain why this implies that in order to find the roots of any polynomial of degree three, it is enough to find the roots of a polynomial of the form $t^3 + pt + q$.

4.1 The Cardano-Tartaglia method

1. In the equation
$$x^3 + P.x + Q = 0, \tag{1}$$
Substitute x by $u + v$, and show that the equation you obtained is satified for all pairs (u, v) such that
$$\begin{aligned} 3 \cdot u \cdot v &= -P \\ u^3 + v^3 &= -Q \end{aligned} \tag{2}$$

2. Solving the system of equations
$$\begin{aligned} u^3 \cdot v^3 &= -P/27 \\ u^3 + v^3 &= -Q \end{aligned} \tag{3}$$
for u and v gives nine pairs (u, v) of solutions. If (u_0, v_0) is one of these nine pairs, is $u_0 + v_0$ a solution of Equation 1)? Justify your answer.

3. Find all the solutions of Equation 1. (Hint: The system 3 will yield a quadratic equation with discriminant Δ. Find the solutions of Equation 1 for each of the cases, $\Delta > 0$, $\Delta = 0$ and $\Delta < 0$.

A detailed explanation of the method to find the roots fo cubit polynomial can be found in:

`http://www.shsu.edu/~kws006/Math_History/4_Calculus_files/SolvingCubics.pdf`

5 Solving a cubic equation with technology

Exercise 11 *Solve the equation* $x^3 + x + 1 = 0$ *with*

1. *Geogebra*
2. *Wolfram-Alpha*
3. *The TI-84 in three ways: using "trace", using the "solver" in Math, using the "calc" key. Explain why the results you obtain in the TI-84 are different are different than those of Wolfram-Alpha.*
4. *In Section 4.1 an algorithm to find the* exact *roots of a cubic equation was given. We just saw that technology also finds roots of cubics. Discuss the pros and cons of each method (technology vs. algebraic).*

6 Netwon's method

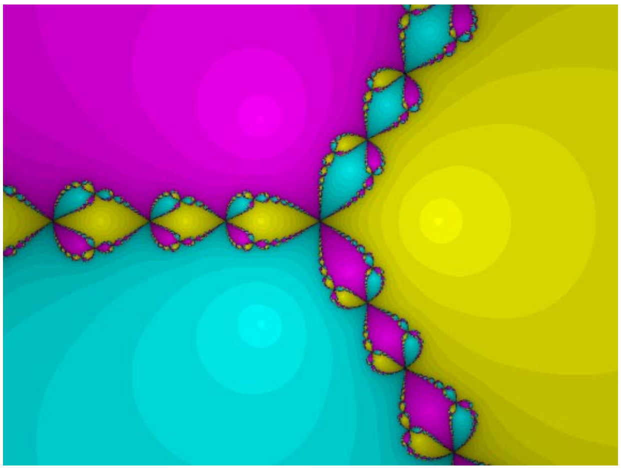

Figure 1: Basins of atraction

To find the roots of polynomials numerically, Newton's method can be applied (see `http://archives.math.utk.edu/visual.calculus/3/newton.5/` for a brief description of the method.)

Do you know the mathematics behind the intricate graphs below?
`http://demonstrations.wolfram.com/ComplexNewtonIterationForACubicPolynomial/`

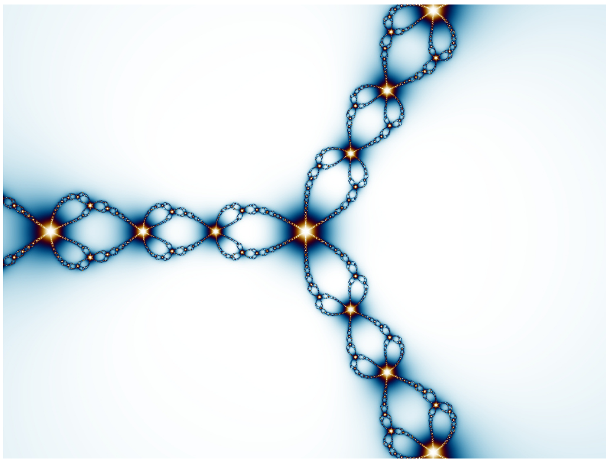

Figure 2: Basins of atraction

Here `http://demonstrations.wolfram.com/LearningNewtonsMethod/` are some examples

Here `http://facstaff.unca.edu/mcmcclur/mathematicaGraphics/Newton/code.html` is the code to generate these pictures in Mathematica.

Exercise 12 *Use Geogebra to explore Newton's method to find roots of quadratic equations.*

7 Extra credit problem

Exercise 13 *(Not with technology) Two trains part simultaneosly, in opposite directions, from A and B. When they cross the train departing from A has traversed 20km more than the other, and arrives to B 45 minutes afterwards. The other train arrives to A one hour and 20 minutes after crossing. What is the distance between A and B?*

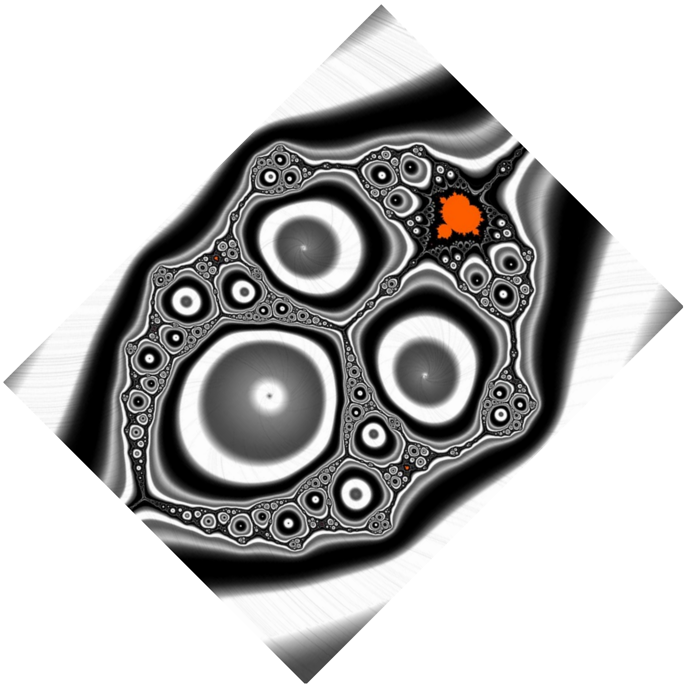

Figure 3: The Mandelbrot set appears in the Basins of atraction of the Newton method for find roots of cubics!