Exercise 3.4 (Homework 7)

28.

Sol:

Let $S = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3]$, then $T(\vec{x}) = A\vec{x}$:

$AS = SB$, B is the matrix of T with respect to $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

$\Rightarrow B = S^{-1}AS$ And $S^{-1} = \begin{pmatrix} \frac{2}{9} & \frac{2}{9} & \frac{1}{9} \\ -\frac{5}{9} & -\frac{4}{9} & -\frac{2}{9} \\ \frac{1}{9} & \frac{1}{9} & -\frac{4}{9} \end{pmatrix}$

$\Rightarrow B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix}$ #

Also: you can do it by direct calculation:

$A\vec{v}_1 = \vec{0}$ $A\vec{v}_2 = 9\vec{v}_2$ $A\vec{v}_3 = 9\vec{v}_3$ $\Rightarrow B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix}$ #

Exercise 4.1

20. Sol:

$a + d = 0 \Rightarrow -a = d$

$\Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

$\Rightarrow$ The space can be represented as:

${}_{\mathbb{R}}V = \operatorname{span}_{\mathbb{R}}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$ #

$\dim_{\mathbb{R}} V = 3$ since $\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ & $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ are linearly independent

32. Sol:

Let $S = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$, then:

$$\left.\begin{aligned} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} &= \begin{bmatrix} x_1+x_3 & x_2+x_4 \\ x_1+x_3 & x_2+x_4 \end{bmatrix} \\ \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} &= \begin{bmatrix} 2x_1 & 0 \\ 2x_3 & 0 \end{bmatrix} \end{aligned}\right\} \text{they are equal}$$

$\Rightarrow \begin{cases} x_1 + x_3 = 2x_1 \\ x_2 + x_4 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = x_3 \\ x_2 = -x_4 \end{cases}$

$\Rightarrow S = \begin{bmatrix} x_1 & x_2 \\ x_1 & -x_2 \end{bmatrix} = x_1\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + x_2\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$

$\Rightarrow$ The space is: ${}_{\mathbb{R}}V = \operatorname{Span}_{\mathbb{R}}\left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \right\}$ #

$\dim_{\mathbb{R}} V = 2$ since $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ & $\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$ are linearly independent.

Homework 5

Section 3.1

8. Sol:

$\vec{x} \in \ker A \Leftrightarrow A\vec{x} = \vec{0}$

Let $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$\Leftrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$\Leftrightarrow x_1 + x_2 + x_3 = 0$

$\Leftrightarrow x_3 = -x_1 - x_2 \Rightarrow \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ -x_1 - x_2 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

$\Rightarrow \ker A = \text{span}_{\mathbb{R}} \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$ #

22. Sol:

$\text{Im } A = \left\{ A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \right\}$

$= \left\{ (x_1 + x_2 + x_3) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \right\}$

$= \text{span}_{\mathbb{R}} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ #

It's a line.

34. Sol:

$A : \mathbb{R}^3 \to \mathbb{R}^3$

$\vec{x} \mapsto A\vec{x}$

to satisfy that $\ker A = \text{span}_{\mathbb{R}} \left\{ \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \right\}$

$\Leftrightarrow A \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} = 0$ & $\text{rank} A = 2$

Then A can be chosen as

$$\begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$ #

Section 3.2

26.

Sol:

Redundant column vector:

$\begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

$\Rightarrow 3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 3 & 6 \\ 1 & 2 & 5 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = 0$

$\Rightarrow \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \in \ker A$ #

Homework 4

Section 2.3 :

34. Sol:

$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \Rightarrow A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \Rightarrow A^3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \Rightarrow A^4 = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$

Geometric meaning:

horizontal shear.

Since $A^n \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + n x_2 \\ x_2 \end{pmatrix}$.

56. Sol:

$X \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} = I_2$

$\Leftrightarrow X = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}^{-1}$

$= -\begin{bmatrix} 5 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$

Section 2.4 :

38.

$A = \begin{bmatrix} 1 & k \\ 0 & -1 \end{bmatrix}$

Use formula:

If $\tilde{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ invertible, $\tilde{A}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$\Rightarrow A^{-1} = \begin{bmatrix} 1 & k \\ 0 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & k \\ 0 & -1 \end{bmatrix}$ #