MAT 552. HOMEWORK 3
SPRING 2014
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1. Prove that the differential, assuming it exists, of the exponential map $exp : T_e(G) \to G$ at $0 \in T_e(G)$ is equal to identity map. Hint: use the definition of exp through the theory of one-parametric subgroups.

Definition 1. Let $H \subset G$ be a closed subgroup of a topological group. We introduce a topology into the set of right cosets G/H as follows. Let Σ be a complete system of neighborhoods (a basis) of the space G and let $U \in \Sigma$. Denote by U^* the set of all cosets of the form Hx, where $x \in U$. For the system Σ^* of neighborhoods of the space G/H we take the totality of all sets of the form U^*, where U is an arbitrary element of Σ. The topological space G/H thus obtained we shall call the space of right cosets of the subgroup in the group G. Analogously we define the space of left cosets.

2. Let G be a topological group, H one of its closed subgroups and G/H the space of cosets. We associate with every element $x \in G$ the element $X = f(x)$ of the space G/H which is the coset containing the element x. Show that $f : G \to G/H$ is a continuous open mapping.

3. Let G be a topological group and N a closed normal subgroup. The set G/N of cosets is an abstract group , and at the same time the set G/N is a topological space . Show that the group operations in G/N are continuous in the topological space G/N.

Definition 2.

(1) A topological space R is called connected if it cannot be decomposed into the sum of two non-null and non-intersecting closed sets A and B. The same definition can be given in still another form: a topological space R is connected if it cannot be decomposed into the sum of two non-null and non- intersecting open sets A and B.

(2) A subset of a space R is called connected if it cannot be decomposed into the sum of two non-null and non-intersecting sets A and which are such that $(\overline{A} \cap \overline{B}) \cap M = \emptyset$.

4. Verify the following statements:

(1) Let Δ be the totality of connected subsets of the space R which have a point a in common. Then the union M of all the sets contained in Δ is connected.

(2) Let a be a point of a topological space R. Then there exists in R a maximal connected subset K which contains the point a. The set K is a maximal set in the sense that every connected subset of the space R which contains a is in K. The set K is always closed and is called the component of the point a in the space R.

(3) If $f : R \to R'$ be onto map of topological spaces with connected R, then R' is connected.

Definition 3. In case the space of the topological group G is connected, the component of the identity of the group G coincides with G, and the group itself is said to be connected. If, on the other hand, the component of the identity of the group G contains only the identity, the group G is called a zero-dimensional or totally-disconnected group.

5. Verify the following statements:

(1) Let G be a topological group, and let N be the component of the point e in the topological space G . Then N is a closed normal subgroup of G.

(2) Let G be a topological group and N the component of the identity in G. Then $G/N = G^*$ is a totally-disconnected group.