1 Semifields

Definition 1 *A semifield* $\mathbb{P} = (\mathbb{P}, \oplus, \cdot)$*:*

1. $(\mathbb{P}, \cdot)$ *is an abelian (multiplicative) group.*

2. $\oplus$ *is an auxiliary addition: commutative, associative, multiplication distributes over* $\oplus$*.*

Exercise 1 *Show that semi-field* $\mathbb{P}$ *is torsion-free as a multiplicative group. Why doesn't your argument prove a similar result about fields?*

Exercise 2 *Show that if a semi-field contains a neutral element* 0 *for additive operation and* 0 *is multiplicatively absorbing*

$$0a = a0 = 0$$

then this semi-field consists of one element

Exercise 3 *Give two examples of non injective homomorphisms of semi-fields*

Exercise 4 *Explain why a concept of kernel is undefined for homorphisms of semi-fields.*

A semi-field $Trop_{\min}$ as a set coincides with $\mathbb{Z}$. By definition $a \underset{Trop}{\cdot} b = a + b$, $a \oplus b = \min(a, b)$. Similarly we define $Trop_{\max}$.

Exercise 5 *Show that* $Trop_{\min} \cong Trop_{\max}$

Let $\mathbb{Z}[u_1, \ldots, u_n]_{\geq 0}$ be the set of nonzero polynomials in $u_1, \ldots, u_n$ with non-negative coefficients.

A free semi-field $\mathbb{P}(u_1, \ldots, u_n)$ is by definition a set of equivalence classes of expression $\frac{P}{Q}$, where $P, Q \in \mathbb{Z}[u_1, \ldots, u_n]_{\geq 0}$.

$$\frac{P}{Q} \sim \frac{P'}{Q'}$$

if there is P'', Q'', a, a' such that $P'' = aP = a'P', Q'' = aQ = a'Q'$.

Exercise 6 *Show that for any semi-field $\mathbb{P}'$ and a collection $v_1, \dots, v_n$ there is a homomorphism*

$$\psi : \mathbb{P}(u_1, \dots, u_n) \to \mathbb{P}', \psi(u_i) = v_i$$

Let k be a ring. Then $k[\mathbb{P}]$ is the group algebra of the multiplicative group of the semi-field $\mathbb{P}$.

2 Cluster algebras - foundations

Definition 2 *$B = (b_{ij})$ is an $n \times n$ integer matrix is skew-symmetrizable if there exists a diagonal matrix D with positive diagonal entries such that DBD^{-1} is skew-symmetric*

Exercise 7 *Show that B is skew-symmetrizable iff there exist positive integers $d_1, \dots, d_n$ such that $d_i b_{ij} = -d_j b_{ji}$ for all i and j.*

Definition 3 *An exchange matrix is a skew-symmetrizable $n \times n$ matrix $B = (b_{ij})$ with integer entries*

Let F be purely transcendental extension (of transcendental degree n) of the field of fractions $\mathbb{Q}(\mathbb{P})$ of $\mathbb{Q}[\mathbb{P}]$.

Definition 4 *A labeled seed is a triple (x, y, B), where*

- *B is an $n \times n$ exchange matrix,*
- *$y = (y_1, \dots, y_n)$ is a tuple of elements of $\mathbb{P}$ called coefficients, and*
- *$x = (x_1, \dots, x_n)$ is a tuple (or cluster) of algebraically independent (over $\mathbb{Q}(\mathbb{P})$) elements of F called cluster variables*

A pair (y, B) is called a Y-seed.

Definition 5 *Let $B = (b_{ij})$ be an exchange matrix. Write $[a]_+$ for $\max(a, 0)$. The mutation of B in direction k is the matrix*

$$b'_{ij} = \begin{cases} -b_{ij}, & \text{if } k \in \{i, j\} \\ b_{ij} + \operatorname{sign}(b_{kj})[b_{ik} b_{kj}]_+, & \text{otherwise} \end{cases}$$

Exercise 8 *Show that $\mu_k(B)$ is an exchange matrix,e.g. it is skew-symmetrizable.*

Exercise 9 *Show that matrix mutation can be equivalently defined by*

$$b'_{ij} = \begin{cases} -b_{ij}, \quad \text{if } k \in \{i,j\} \\ b_{ij} + [b_{ik}]_+ b_{kj} + b_{ik}[b_{kj}]_+, \text{ otherwise} \end{cases}$$

Definition 6 *Let (y, B) be a Y-seed. The mutation of (y, B) in direction k is the Y-seed $(y', B') = \mu_k(y, B)$, where $B' = \mu_k(B)$ and y' is the tuple $(y'_1, \ldots, y'_n)$ given by*

$$y'_j = \begin{cases} y_k^{-1}, \quad \text{if } j = k \\ y_j y_k^{[b_{kj}]_+} (y_k \oplus 1)^{-b_{kj}}, \text{if } j \neq k \end{cases}$$

Definition 7 *Let (x, y, B) be a labeled seed. The mutation of (x, y, B) in direction k is the labeled seed $(x', y', B') = \mu_k(x, y, B)$, where (y', B') is the mutation of (y, B) and where x' is the cluster $(x'_1, \ldots, x'_n)$ with $x'_j = x_j$ for $j \neq k$, and*

$$x'_k = \frac{y_k \prod x_i^{[b_{ik}]_+} + \prod x_i^{[-b_{ik}]_+}}{(y_k \oplus 1) x_k}$$

Exercise 10 *Show that each mutation μ_k is an involution on labeled seeds.*

Applying several mutations $\mu_{i_1} \cdots \mu_{i_l}$ to a labelled seed (x, y, B) we get a new labelled seed. Let $\Delta_n(x, y, B)$ be the set of all such seeds.

Definition 8 *A cluster algebra $A(x, y, B)$ is a subalgebra in $\mathbb{Q}(\mathbb{P})(x_1, \ldots, x_n)$ generated by all cluster variables in $\Delta_n(x, y, B)$.*

Definition 9 *Let $\widetilde{B}$ be $(m+n) \times n$ matrix, such that the top $n \times n$ matrix is skew-symmetrizable and $\widetilde{x} = (x_1 \ldots, x_n, x_{n+1}, \ldots, x_{n+m})$. Then we say that $(\widetilde{x}, \widetilde{B})$ is a labelled seed for a cluster algebra of geometric type. Collection $(x_1, \ldots, x_n)$ is known as exchangeable variables; $(x_{n+1}, \ldots, x_{n+m})$ as frozen variables or "coefficients". Notation: $(u_1, \ldots, u_m)$ is occasionally used for frozen variables.*

Let $\widetilde{x}' = \mu_k(\widetilde{x})$, $\widetilde{B}' = \mu_k(\widetilde{B})$, $k = 1, \ldots, n$, Then $\mu_k(\widetilde{B})$ is defined as in $n \times n$ case; $x'_j = x_j, j \neq k$

$$x'_k = \frac{\prod x_i^{[b_{ik}]_+} + \prod x_i^{[-b_{ik}]_+}}{x_k}$$

Definition 10 *Let $\Delta_n(x, B)$ be the set of mutations of geometric seed (x, B). By definition cluster algebra of geometric type as as a subalgebra in $\mathbb{Q}(x_1, \ldots, x_{n+m})$ generated by cluster variables in $\Delta_n(x, B)$.*

Exercise 11 *Let $\mathbb{P}$ a tropical semi-field on n generators $y_1, \ldots, y_n$. Show that the homomorphism of fields $\phi : \mathbb{Q}(x_1, \ldots, x_n, y_1, \ldots, y_n) \to \mathbb{Q}(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})$ identical on $x_1, \ldots, x_n$ and on $y_1, \ldots, y_n$ defined by the formula:*

$$\phi(y_j) = \prod_{i=1}^{m} x_{n+i}^{b_{n+i,j}}$$

is compatible with mutations.

Exercise 12 *Consider the cluster algebra of geometric type defned by the initial labeled seed given by $x = (x_1, x_2, u_1, u_2, u_3)$ and*

$$B = \begin{pmatrix} 0 & 2 \\ -1 & 0 \\ -1 & 0 \\ 1 & 0 \\ 1 & 2 \end{pmatrix}$$

Compute all cluster variables generating this cluster algebra.

3 Root systems

Definition 11 *Given a nonzero vector α in Euclidean space V, the reflection in the hyperplane orthogonal to α is σ_α, given by*

$$\sigma_\alpha(x) = x - 2\langle \frac{\alpha}{\sqrt{\langle \alpha, \alpha \rangle}}, x \rangle \cdot \frac{\alpha}{\sqrt{\langle \alpha, \alpha \rangle}} = x - 2\frac{\langle \alpha, x \rangle}{\langle \alpha, \alpha \rangle} \qquad (1)$$

Define $\alpha^\vee = 2\frac{\alpha}{\langle \alpha, \alpha \rangle}$. Then $\sigma_\alpha(x) = \langle \alpha^\vee, x \rangle \alpha$

Definition 12 *A root system is a collection Φ of nonzero vectors (called roots) in a real vector space V such that:*

1. *Φ is finite,*
2. *$0 \notin \Phi$ and Φ spans V,*
3. *For each root β, the reflection σ_β permutes Φ,*
4. *Given a line L through the origin, either $L \cap \Phi$ is empty or $L \cap \Phi = \{\pm\beta\}$ for some β (reduced system condition),*
5. *$\langle \alpha^\vee, \beta \rangle \in \mathbb{Z}$, for each $\alpha, \beta \in \Phi$. (crystallographic condition).*

Definition 13 *Two root systems $\Phi \subset V$ and $\Phi' \subset V'$ are isomoprhic if there is an isometry $f : V \to V$ with $f(\Phi) = \Phi'$.*

Exercise 13 *Describe all not necessarily reduced finite one-dimensional crystallographic root systems up to an isomorphism.*

Exercise 14 *Let θ be an angle between vectors α, β. Show that $\langle \alpha^\vee, \beta \rangle \langle \beta^\vee, \alpha \rangle = 4\cos^2\theta$ and find possible values of θ, $\langle \alpha^\vee, \beta \rangle$, $\langle \beta^\vee, \alpha \rangle$ and $4\cos^2\theta$ for vectors in a finite crystallographic root system.*

Exercise 15 *Let α, β be two non proportional vectors in a finite crystallographic root system Φ. Show that if $\langle \alpha, \beta \rangle < 0$ then $\alpha + \beta \in \Phi$. If $\langle \alpha, \beta \rangle > 0$ then $\alpha - \beta \in \Phi$.*

Definition 14 *Let α, β be a pair of linearly independent roots. A subset $\{\gamma \in \Phi | \gamma = \beta + k\alpha (k \in \mathbb{Z})\}$ of a root system Φ is called an α-series of roots, containing β. In particular if $\beta - \alpha \notin \Phi$ then $\beta + \alpha \in \Phi$ iff $\langle \beta, \alpha \rangle < 0$.*

Exercise 16 *An α-series of roots, containing β has a form $\{\beta + k\alpha | -p \leq k \leq q\}$, where $p, q \geq 0$ and $p - q = \langle \alpha^\vee, \beta \rangle$.*

Definition 15 ***Exercise 17*** *We define a collection $\Phi^\vee = \{\alpha^\vee | \alpha \in \Phi\} \subset V$. Prove that $\Phi^\vee$ is a root system.*

(Direct sums). Let Φ and Φ' be root systems in V and V', respectively. Then $\Phi \cup \Phi'$ is a root system in the vector space $V \oplus V'$. A root system is reducible if it can be written as such an (orthogonal) direct sum, and irreducible otherwise.

Definition 16 *Let Φ be a root system. Then the Weyl group of Φ is the group generated by σ_α for all $\alpha \in \Phi$.*

Exercise 18 *Is the Weyl group well-defined (i.e., do isomorphic root systems give isomorphic Weyl groups?).*

Exercise 19 *Is the Weyl group of a finite root system finite?*

Exercise 20 *What are the Weyl groups of the four crystallographic root systems in $\mathbb{R}^2$?*

Exercise 21 *Find a root system having the symmetric group on four letters, S_4, as its Weyl group.*

Definition 17 *Let $\Phi \subset V$ be a root system, and choose $v \in V$. Define $\Phi^+(v) = \{\alpha \in \Phi | \langle \alpha, v \rangle > 0\}$. We say that v is regular if $\Phi = \pm\Phi^+(v)$, and singular otherwise. If v is regular, we call $\Phi^+(v)$ a positive system for Φ.*

Exercise 22 *Why does a regular v exist?*

Let v be regular we set $\Phi^+ = \Phi^+(v)$. In general Φ^+ depends on the choice of v.

Definition 18 *The set $\Pi(\Phi^+) \subset \Phi^+$ is formed by elements α that can not be presented as a sum $\alpha = \beta\gamma, \beta\gamma \in \Phi^+$.*

Exercise 23 *Show that any* $\alpha \in \Phi^+$ *can be written in the form* $\alpha = \sum_{\beta \in \Pi(\Phi^+)} c_\beta \beta$, *where* c_β *are nonnegative integers.*

Exercise 24 *If* $\alpha, \beta \in \Pi(\Phi^+)$ *and* $\alpha \neq \beta$, *then* $\alpha - \beta \neq \Phi$ *and* $\langle \alpha, \beta \rangle \leq 0$.

Exercise 25 *Let* $\alpha_1, \ldots, \alpha_k$ *be a set of vectors in* V *such that* $\langle \alpha_i, \alpha_j \rangle \leq 0, i \neq j$. *Suppose we have a nontrivial linear combination with positive* c_i, c'_j:

$$\sum_{r=1}^{k} c_r \alpha_{i_r} - \sum_{r'=1}^{l} c'_{r'} \alpha_{j_{r'}} = 0$$

with all $i_1, \ldots, i_k, j_1, \ldots, j_l$ *distinct. Then*

1. $\sum_{r=1}^{k} c_r \alpha_{i_r} = \sum_{r=1}^{l} c'_r \alpha_{j_r} = 0$.
2. $\langle \alpha_{i_r}, \alpha_{j_{r'}} \rangle = 0, r = 1, \ldots, k, r' = 1, \ldots, l$

Exercise 26 *Let* $\alpha_1, \ldots, \alpha_k \in V$ *be a set of linearly independent vectors. Show that theres is* $\beta \in V$ *such that* $\langle \alpha_i \beta \rangle > 0, i = 1, \ldots, k$

Definition 19 *The* n*-th Catalan number* C_n *is the number of full binary planar trees with* $n+1$ *leaves.*

Exercise 27 *Prove the formula*

$$C_n = \frac{(2n)!}{n!(n+1)!}$$

Exercise 28 *Prove the Ptolemy's theorem:let* Δ_{ABCD} *be a quadrilateral whose vertices lie on a common circle. Then*

$$|AC||BD| = |AB||CD| + |BC||AD|$$