Practice Final Even Numbers

Note Title 12/8/2012

2\. 1) Textbook P68 Thm 2.1.

2) P70 Thm 2.3.

4\. 1) linearly independent.

proof: if $a \cdot 1 + b e^{x} + c e^{2x} = 0$,

let $x=0$: $a + b + c = 0$,

take derivative and let $x=0$:

$b + 2c = 0$

take 2^{nd} derivative and let $x=0$:

$b + 4c = 0$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ since $\det \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{pmatrix} = 2 \neq 0$

$\Rightarrow 1, e^{x}, e^{2x}$ are linearly ind.

2) linearly dependent.

$$2 \cdot 1 + 1 \cdot \sin^2 x + (-1)(3 - \cos^2 x) = 0$$

6. $\begin{pmatrix} 1 & -1 & 1 & 2 & 0 \\ 2 & 1 & 1 & -1 & 2 \\ 5 & 0 & 3 & 0 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & 1 & 2 & 0 \\ 0 & 3 & -1 & -4 & 2 \\ 0 & 5 & -2 & -10 & 4 \end{pmatrix}$

$\longrightarrow \begin{pmatrix} 1 & -1 & 1 & 2 & 0 \\ 0 & 3 & -1 & -4 & 2 \\ 0 & -1 & 0 & -2 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & 1 & 2 & 0 \\ 0 & 0 & -1 & -10 & 2 \\ 0 & -1 & 0 & -2 & 0 \end{pmatrix}$

$\longrightarrow \begin{pmatrix} 1 & -1 & 1 & 2 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 10 & -2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & -6 & 2 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 10 & -2 \end{pmatrix}$

let $x_4 = s$, $x_5 = t$, then

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 6s - 2t \\ -2s \\ -10s + 2t \\ s \\ t \end{pmatrix} = \begin{pmatrix} 6 \\ -2 \\ -10 \\ 1 \\ 0 \end{pmatrix} s + \begin{pmatrix} -2 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix} t$$

$\Rightarrow \begin{pmatrix} 6 \\ -2 \\ -10 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}$ is a basis of Ker A

Extend to a basis of $\mathbb{R}^5$:

$$\begin{pmatrix}6\\-2\\-10\\1\\0\end{pmatrix}, \begin{pmatrix}-2\\0\\2\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}.$$

$$\dim \operatorname{Im} A = \dim \mathbb{R}^5 - \dim \operatorname{Ker} A$$
$$= 5 - 2 = 3$$

So A is surjective, and $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$ is a basis of $\operatorname{Im} A$.

8. $M(T)$ has the only eigenvalue 1, so if $M(T)$ is diagonalizable, then $\exists P$, s.t. $P^{-1} M(T) P = \begin{pmatrix}1 & \\ & 1\end{pmatrix}$

P invertible

$\Rightarrow P^{-1}(M(T) - I) P = 0$. By multiplying P on the left and P^{-1} on the right, we have $M(T) - I = 0$. But $M(T) - I = \begin{pmatrix}0 & 1\\ & 0\end{pmatrix}$, contradiction! Hence $M(T)$ is not diagonalizable.

10. M diagonalizable $\Rightarrow \exists P$, P invertible,

$$P^{-1} M P = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}.$$

Let $N = P \begin{pmatrix} \sqrt[3]{\lambda_1} & & \\ & \ddots & \\ & & \sqrt[3]{\lambda_n} \end{pmatrix} P^{-1}$, then $N^3 = M$.

12. 1) $A = \begin{pmatrix} 3 & 2 & -2 \\ 2 & 3 & -2 \\ 6 & 6 & -5 \end{pmatrix}.$

$$\det(\lambda I - A) = \lambda^3 - \lambda^2 - \lambda + 1$$
$$= (\lambda - 1)^2 (\lambda + 1)$$

eigenvalues $\lambda_1 = -1$, $\lambda_2 = \lambda_3 = 1$.

$$(A + I)\vec{v} = \begin{pmatrix} 4 & 2 & -2 \\ 2 & 4 & -2 \\ 6 & 6 & -4 \end{pmatrix} \vec{v} = 0, \quad \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$$

$$(A - I)\vec{v} = \begin{pmatrix} 2 & 2 & -2 \\ 2 & 2 & -2 \\ 6 & 6 & -6 \end{pmatrix} \vec{v} = 0, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$S = (\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3) = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 0 & 0 \\ 3 & 1 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix} = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 0 \end{pmatrix}.$$

$$S^{-1} A S = D$$

2. $$A = \begin{pmatrix} -8 & 5 & 4 \\ -9 & 5 & 5 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\det(\lambda I - A) = \lambda^3 + 3\lambda^2 - 4$$

$$= (\lambda + 2)^2 (\lambda - 1)$$

$$\lambda_1 = 1, \quad \lambda_2 = \lambda_3 = -2$$

$$A + 2I = \begin{pmatrix} -6 & 5 & 4 \\ -9 & 7 & 5 \\ 0 & 1 & 2 \end{pmatrix}$$

$\operatorname{rank}(A + 2I) = 2$, $\dim \ker (A + 2I) = 1$, but eigenvalue $\lambda = 2$ has multiplicity 2,

so A is not diagonalizable.

14. Undetermined. $\begin{pmatrix} 2 & \\ & 6 \end{pmatrix}$ is diagonal,

$\begin{pmatrix} 2 & 1 \\ & 6 \end{pmatrix}$ is not diagonalizable.

see problem 8.

16. $y^{(4)} - 8y^{(2)} + 16y = 0$

auxiliary polynomial $p(t) = t^4 - 8t^2 + 16$

$= (t+2)^2 (t-2)^2$

general solution :

$$y = C_1 e^{-2t} + C_2 t e^{-2t} + C_3 e^{2t} + C_4 t e^{2t}$$

(Thm 2.34, p139)