Problem 7.

- Show that the functions $\cos(x) + 3\sin(x)$ and $\sin(x)$ are linearly independent functions on the real line by looking at their Wronskian.

- Find the general solution of the following equation using the method of variation of parameter

$$y'' + y + 1 + x = 0$$

- $$W = \det \begin{bmatrix} \cos x + 3\sin x & \sin x \\ -\sin x + 3\cos x & \cos x \end{bmatrix} = \cos^2 x + 3\sin x \cos x - (-\sin^2 x + 3\sin x \cos x)$$
$$= 1 \quad \text{for all } x.$$

- $r^2 + 1 = 0 \Rightarrow r = \pm i \Rightarrow y_1 = \cos x, \; y_2 = \sin x$

$$\begin{cases} u_1' \cos x + u_2' \sin x = 0 \\ -u_1' \sin x + u_2' \cos x = -x - 1 \end{cases}$$

$$u_1' = (x+1)\sin x \Rightarrow u_1 = \int x\sin x + \sin x \, dx$$
$$= -x\cos x + \int \cos x dx - \cos x$$
$$= \sin x - \cos x - x\cos x$$

$$u_2' = -x\cos x - \cos x \Rightarrow u_2 = \int -x\cos x \, dx - \int \cos x dx$$
$$= -x\sin x + \int \sin x dx - \sin x$$
$$= -x\sin x - \cos x - \sin x$$

$y_p = u_1 y_1 + u_2 y_2 = -1 - x$ (using trig identities)

$$\boxed{y = -1 - x + c_1 \cos x + c_2 \sin x.}$$