Solutions to 3.2 problems:

4) $\frac{g}{2} + \frac{h}{3} = \cos^2 x + \sin^2 x = 1$

Multiply by 17 on both sides, you get:

$17 = f = \frac{17}{2} g + \frac{17}{3} h$

$\Rightarrow \boxed{f - \frac{17}{2} g - \frac{17}{3} h = 0}$

9) $W(e^x, \cos x, \sin x)$

$$= \begin{vmatrix} e^x & \cos x & \sin x \\ e^x & -\sin x & \cos x \\ e^x & -\cos x & -\sin x \end{vmatrix}$$

$= e^x (1 + 0 + 1)$

$= 2e^x$

<u>Proposition</u>: suppose $f_1, \dots, f_k$ are k smooth functions on an interval I. If $W(f_1, \dots, f_k)$ is different from 0 for some $t \in I$ (remember $W(f_1, \dots, f_k)$ is a function from I to $\mathbb{R}$), then $f_1, \dots, f_k$ are linearly independent over I.

3.2 problems (continued)

9) (continued)

$W(e^x, \cos x, \sin x)(0) = 2 \neq 0$

Hence f, g and h are lin. independent over $\mathbb{R}$.

16) $y^{(3)} - 5y'' + 8y' - 4y = 0$

$y(0) = 1, \quad y'(0) = 4, \quad y''(0) = 0$

$y_1 = e^x, \quad y_2 = e^{2x}, \quad y_3 = xe^{2x}$

General solution of this homogeneous 3rd order linear ODE:

$y = Ae^x + Be^{2x} + Cxe^{2x}$

$y(0) = 1 \Rightarrow A + B = 1$

$y(0) = 4 \Rightarrow A + 2B + C = 4$

$y' = Ae^x + 2Be^{2x} + Ce^{2x} + 2Cxe^{2x}$

$y'' = Ae^x + 4Be^{2x} + 2Ce^{2x} + 2Ce^{2x} + 4Cxe^{2x}$

$y''(0) = 0 \Rightarrow A + 4B + 4C = 0$

so we have to solve

$x(-1)\ x(-1)$

$$\begin{cases} A + B = 1 \\ A + 2B + C = 4 \\ A + 4B + 4C = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} A + B = 1 \\ B + C = 3 \\ 3B + 4C = -1 \end{cases} \quad \times(-3)$$

$$\Leftrightarrow \begin{cases} A + B = 1 \\ B + C = 3 \\ C = -10 \end{cases}$$

Hence $C = -10$

$B = 13$

$A = -12$

Hence $\boxed{y = -12e^{x} + 13e^{2x} - 10xe^{2x}}$

30) $y_1 = x$, $y_2 = x^2$

$x^2 y'' - 2xy' + 2y = 0$

$y_1' = 1$

$y_1'' = 0$

$\Rightarrow x^2 y_1'' - 2x y_1' + 2y_1 = -2x + 2x = 0$

similarly, one can check that

$x^2 y_2'' - 2x y_2' + 2y_2 = 0$

Hence the hypotheses of theorem 3 are not satisfied. QED

3.2 problems (continued)

$$W(x, x^2) = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix}$$

$$= x \begin{vmatrix} 1 & x \\ 1 & 2x \end{vmatrix}$$

$$= x(2x - x)$$

$$= x^2$$

$W(x, x^2)(1) = 1 \neq 0$.

Hence the wronskian is different from 0 at $x = 1$, which implies that y_1, y_2 are linearly independent over $\mathbb{R}$.

This does not contradict theorem 3, because the ODE is not of the form

$$y^{(n)} + p_1(x) y^{(n-1)} + \dots + p_{n-1}(x) y' + p_n(x) y = 0$$

To make it of this form, we need to divide by x^2:

$$y'' - \frac{2}{x} y' + \frac{2}{x^2} y = 0$$

and we see that $P(x) = -\frac{2}{x}$ and $Q(x) = \frac{2}{x^2}$ are not defined (even less are they continuous) at $x = 0$.