1.6.12 Solve

$$xyy' = y^2 + x\sqrt{4x^2+y^2}$$

$$\frac{1}{xy}\, xyy' = \frac{1}{xy}\left[y^2 + x\sqrt{4x^2+y^2} \right]$$

$$y' = \frac{y}{x} + \frac{x}{y}\sqrt{4+\frac{y^2}{x^2}}$$

Let $u = \frac{y}{x}$. Then $y' = u'x + u$

$$u'x + u = u + \frac{1}{u}\sqrt{4+u^2}$$

$$\frac{uu'}{\sqrt{4+u^2}} = \frac{1}{x}$$

$$\sqrt{4+u^2} = \ln|x| + C$$

$$4 + u^2 = \left[\ln|x| + c\right]^2$$

$$4 + \frac{y^2}{x^2} = \left[\ln|x| + c\right]^2$$

$$4x^2 + y^2 = x^2\left[\ln|x| + c\right]^2$$

1.6.34 Verify exactness, then solve.

$$(2xy^2 + 3x^2)\,dx + (2x^2y + 4y^3)\,dy = 0$$

Exact:

$$\frac{\partial}{\partial y}(2xy^2 + 3x^2) = 4xy$$

$$\frac{\partial}{\partial x}(2x^2y + 4y^3) = 4xy$$

Solution:

$$F(x,y) = \int 2xy^2 + 3x^2\,dx$$

$$= x^2y^2 + x^3 + g(y)$$

$$\frac{\partial}{\partial y}F(x,y) = 2x^2y + g'(y) = 2x^2y + 4y^3$$

$$\Rightarrow g'(y) = 4y^3 \Rightarrow g(y) = y^4 + c$$

$$F(x,y) = x^2y^2 + x^3 + y^4 + c$$

$$dF(x,y) = 0 \Rightarrow \boxed{x^2y^2 + x^3 + y^4 = \tilde{C}}$$

For some constant $\tilde{C}$.

1.6.38 Verify exactness, then solve.

$$(x+\tan^{-1}y)\,dx + \frac{x+y}{1+y^2}\,dy = 0$$

$$\left.\begin{aligned} \frac{\partial}{\partial y}(x+\tan^{-1}y) &= \frac{1}{1+y^2} \\ \frac{\partial}{\partial x}\left(\frac{x+y}{1+y^2}\right) &= \frac{1}{1+y^2} \end{aligned}\right\} \Rightarrow \text{exact}$$

Solution:

$$F(x,y) = \int x+\tan^{-1}y\,dx$$

$$= \tfrac{1}{2}x^2 + x\tan^{-1}y + g(y)$$

$$\frac{\partial}{\partial y}F(x,y) = \frac{x}{1+y^2} + g'(y) = \frac{x+y}{1+y^2}$$

$$\Rightarrow g'(y) = \frac{y}{1+y^2} \Rightarrow g(y) = \tfrac{1}{2}\ln(1+y^2) + c$$

$$\Rightarrow F(x,y) = \tfrac{1}{2}x^2 + x\tan^{-1}y + \tfrac{1}{2}\ln(1+y^2) + c$$

$$dF = 0 \Rightarrow \boxed{\begin{array}{l}\tfrac{1}{2}x^2 + x\tan^{-1}y + \tfrac{1}{2}\ln(1+y^2) = \tilde{c} \\ \text{For some constant } \tilde{c}\end{array}}$$

1.6.54 Solve (assuming $y, y' > 0$)

$$yy'' = 3(y')^2$$

Let $y' = u$

Then $y'' = \frac{du}{dy}\frac{dy}{dx} = \frac{du}{dy} \cdot u$

Thus,

$$y \cdot \frac{du}{dy} \cdot u = 3u^2$$

$$y\frac{du}{dy} = 3u \qquad \text{(since } u \neq 0\text{)}$$

$$\frac{1}{u}\frac{du}{dy} = 3\frac{1}{y}$$

$$\ln u = 3\ln y + A \qquad \text{(since } u, y > 0\text{)}$$

$$u = By^3$$

$$y' = By^3$$

$$y^{-3}y' = B$$

$$-\frac{1}{2}y^{-2} = Bx + C$$

$$y^{-2} = \tilde{B}x + \tilde{C}$$

$\cancel{y = (\tilde{B}x + \tilde{C})^{-1/2}}$ ~~for some constants~~ $\cancel{\tilde{B}}$ ~~and~~ $\cancel{\tilde{C}}$.

$y^2 = (\tilde{B}x + \tilde{C})^{-1}$ for some constants $\tilde{C}$ and $\tilde{B}$.

1.6.55

Show that the substitution $v = ax + by + c$ transforms $\frac{dy}{dx} = F(ax + by + c)$ into a seperable equation.

According to the definition on page 32, the above equation will be separable if we can write it as

$$\frac{dv}{dx} = g(x) \cdot h(v)$$

for some functions g and h.

$$v = ax + by + c \implies \frac{dv}{dx} = a + b\frac{dy}{dx}$$

$$\implies \frac{dy}{dx} = \frac{1}{b}\left(\frac{dv}{dx} - a\right)$$

Thus $\frac{dy}{dx} = F(ax+by+c)$ becomes

$$\frac{1}{b}\left(\frac{dv}{dx} - a\right) = F(v)$$

$$\frac{dv}{dx} = bF(v) + a$$

$$= (bF(v) + a) \cdot 1$$

$$= 1 \cdot (bF(v) + a)$$

$$= g(x) \cdot h(v)$$

where $g(x) = 1$ and $h(v) = bF(v) + a$.