# Solution to practice final:



1) • $$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \underbrace{\begin{pmatrix} -3 & 0 & -1 \\ 3 & 2 & 3 \\ 2 & 0 & 0 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} -3-\lambda & 0 & -1 \\ 3 & 2-\lambda & 3 \\ 2 & 0 & -\lambda \end{vmatrix}$$

$$= (-3-\lambda) \begin{vmatrix} 2-\lambda & 3 \\ 0 & -\lambda \end{vmatrix} - \begin{vmatrix} 3 & 2-\lambda \\ 2 & 0 \end{vmatrix}$$

$$= -(\lambda+3)\lambda(\lambda-2) - 2(\lambda-2)$$

$$= -(\lambda-2)\left[\lambda^2 + 3\lambda + 2\right]$$

$$= -(\lambda-2)(\lambda+2)(\lambda+1)$$

$\Rightarrow$ eigenvalues are $\lambda = 2$, $-2$ and $-1$.

# Solution to practice final

## 11 (continued)

• $\lambda = 2$;

$$\begin{pmatrix} -5 & 0 & -1 \\ 3 & 0 & 3 \\ 2 & 0 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{array}{c} \\ \times(-1) \\ \\ \end{array}\begin{pmatrix} +5 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{array}{c} \\ \times(-5) \\ \\ \end{array}\begin{pmatrix} 5 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & -4 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} x_1 = 0 \\ x_3 = 0 \end{cases}$$

$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector of $A$ for $\lambda = 2$.

(one could have seen this by looking at the matrix $A - 2I$ above, which had its second column full of 0's).

Solution to practice final:

11 (continued):

- $\lambda = -2$

$$\begin{pmatrix} -1 & 0 & -1 \\ 3 & 4 & 3 \\ 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\overset{(-3)}{\Longleftrightarrow} \begin{pmatrix} 1 & 0 & 1 \\ 3 & 4 & 3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Longleftrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_3 \\ 0 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Hence $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ is an eigenvector of $A$ for $\lambda = -2$.

# solution to practice midterm

## 11 (continued)

$\underline{\lambda = -1}$ :

$$\begin{pmatrix} -2 & 0 & -1 \\ 3 & 3 & 3 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\times(-\frac{1}{2})$ $\Leftrightarrow$

$$\begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$2x_2 + x_3 = 0 \Rightarrow x_2 = -\frac{1}{2} x_3$$

$$2x_1 + x_3 = 0 \Rightarrow x_1 = -\frac{1}{2} x_3$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix}$$

Hence $\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$ for instance is an eigenvector of $A$ for $\lambda = -1$.

# Solution to practice midterm:

## 11 (continued)

Hence the gen. solution is:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = Ae^{2t}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + Be^{-2t}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + Ce^{-t}\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$$

## 11 (second part)

Same ~~ODE as~~ system as 10, but using eigenvalues:

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}}_{A}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = (\lambda-1)(\lambda-3)+1$$
$$= \lambda^2 - 4\lambda + 4$$
$$= (\lambda-2)^2$$

$$(A-2I)\,\underline{x} = 0$$

$$\Rightarrow \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Hence $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector of $A$ with $\lambda = 2$.

Next, we need to find a $v_2 \neq 0$ s.t.

$(A - 2I) v_2 = v_1$, i.e.

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\Rightarrow \quad x_1 = -x_2 - 1$

$$\Rightarrow \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

Hence we can take say $v_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

Hence the gen. solution is:

$$\underline{x} = A e^{2t} v_1 + B e^{2t} (t v_1 + v_2)$$

$$= e^{2t} \left( (A + Bt) v_1 + B v_2 \right)$$

$$\boxed{\underline{x} = e^{2t} \begin{pmatrix} A + Bt - B \\ -(A + Bt) \end{pmatrix}}$$

which is what we got for problem 10, for $A \mapsto -A$ & $B \mapsto -B$.