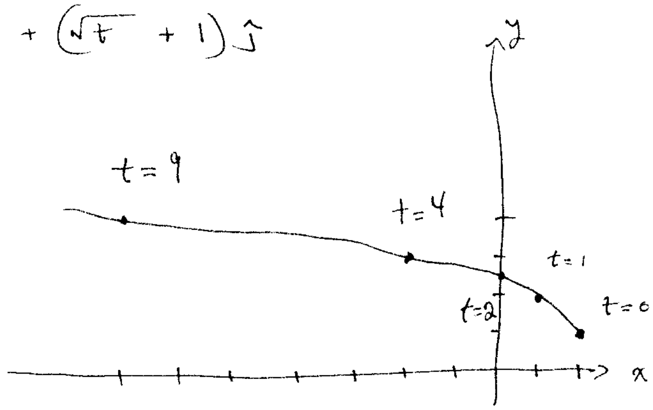
Review Problem solutions

1 – 4, 6, 7 are on the previous practice midterm
6 and 7 are labelled as
12, 13 respectively.

5. $r(t) = (2-t)\hat{i} + (\sqrt{t} + 1)\hat{j}$



t	$\vec{r}(t)$
0	(2, 1)
1	(1, 2)
2	$(0, \sqrt{2}+1)$
4	(-2, 3)
9	(-7, 4)

8. $x = t^2 \quad y = t^3$ from $(1,1)$ to $(4,8)$. find arc length:
first find range for t:

$(1,1) \Rightarrow \begin{cases} 1 = t^2 \\ 1 = t^3 \end{cases} \Rightarrow t = 1$

$(4,8) \Rightarrow \begin{cases} 4 = t^2 \\ 8 = t^3 \end{cases} \Rightarrow t = 2$

$\frac{dx}{dt} = 2t$

$\frac{dy}{dt} = 3t^2$

$$\text{arclength} = \int_{t=1}^{2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_{t=1}^{2} \sqrt{4t^2 + 9t^4}\, dt = \int_{t=1}^{2} t\sqrt{4 + 9t^2}\, dt$$

Substitute
$u = 4 + 9t^2$
$du = 18t\,dt$

$$= \frac{1}{18}\int_{t=1}^{2} \sqrt{u}\, du$$

$$= \left(\frac{1}{18}\right)\left(\frac{2}{3}\right) u^{3/2} \Big|_{t=1}^{2}$$

$$= \left(\tfrac{1}{27}\right)(4 + 9t^2)^{3/2} \Big|_{t=1}^{2} = \frac{1}{27}\left[40^{3/2} - 13^{3/2}\right]$$

9. $f(x,y) = \begin{cases} x+y^2 & x>0 \text{ + } y>0 \\ 0 & \text{else} \end{cases}$

sketch level curves, ie $f(x,y)=c$ for various constants, c.

if $c=0$ $f(x,y)=0 \Rightarrow (x,y)$ is any point outside of first quadrant.

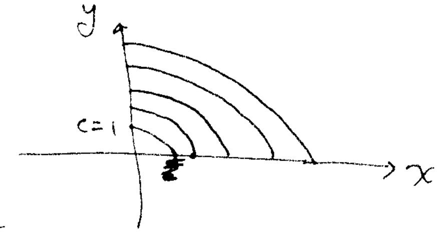
$c=1$, $f(x,y)=1$

$\Rightarrow x+y^2=1 \Rightarrow$ parabola

$x=1-y^2$

1 is x-intercept

restrict to first quadrant



$f(x,y)=c$

$x+y^2=c$

$x=c-y^2$ $\Rightarrow$ parabola restricted to first quadrant

c is x-intercept

10.1. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x+y}{x-y}$

if $y=x$ then

we have

$\displaystyle\lim_{x\to 0} \frac{x+x}{x-x}$ which does not exist

So the original limit does not exist.

10.2 $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$

these two limits do not agree so the original limit does not exist.

if $y=0$

then $\displaystyle\lim_{x\to 0} \frac{0}{x^2+0} = 0$

but if $y=x$

then $\displaystyle\lim_{x\to 0} \frac{x^2}{x^2+x^2} = \frac{1}{2}$

10.3 $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2} = \lim_{r\to 0} \frac{r^4\cos^2\theta\sin^2\theta}{r} = \lim_{r\to 0} r^3\cos^2\theta\sin^2\theta = 0$

~~$x = r\sin\theta$~~

$x = r\cos\theta,\ y = r\sin\theta$

11.1 $f(x,y) = \sin(\ln(x+y)\cos(xy))$

Find gradient $\nabla f(x,y) = f_x(x,y)\hat{i} + f_y(x,y)\hat{j}$

$f_x(x,y) = \cos(\ln(x+y)\cos(xy)) \cdot \frac{\partial}{\partial x}(\ln(x+y)\cos(xy))$

$= \cos(\ln(x+y)\cos(xy)) \left[\frac{\cos(xy)}{(x+y)} - \sin(xy)\cdot\frac{\partial}{\partial x}(xy) \ln(x+y) \right]$

$= \cos(\ln(x+y)\cos(xy)) \left[\frac{\cos(xy)}{x+y} - y\sin(xy)\ln(x+y) \right]$

by symmetry

$f_y(x,y) = \cos(\ln(x+y)\cos(xy)) \left[\frac{\cos(xy)}{x+y} - x\sin(xy)\ln(x+y) \right]$

so $\nabla f(x,y) = \left[\cos(\ln(x+y)\cos(xy))\right]\left[\left(\frac{\cos(xy)}{x+y} - y\sin(xy)\ln(x+y)\right)\hat{i} + \left(\frac{\cos(xy)}{x+y} - x\sin(xy)\ln(x+y)\right)\hat{j}\right]$

11.2. $g(x,y) = \dfrac{\sqrt{x+y+z}}{1+x^2+y^2+z^2}$ $\qquad \nabla g(x,y) = g_x(x,y)\hat{i} + g_y(x,y)\hat{j}$

$g_x(x,y) = \dfrac{\frac{1}{2}(x+y+z)^{-1/2}(1+x^2+y^2+z^2) - (x+y+z)^{1/2}(2x)}{(1+x^2+y^2+z^2)^2}$

$g_y(x,y) = \dfrac{\frac{1}{2}(x+y+z)^{-1/2}(1+x^2+y^2+z^2) - (x+y+z)^{1/2}(2y)}{(1+x^2+y^2+z^2)^2}$

12 $z = e^{2r}\sin(3\theta)$, $r = st - t^2$, $\theta = \sqrt{s^2+t^2}$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial s} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial s}$$

$$= \left(2e^{2r}\sin(3\theta)\right)(t) + \left(3e^{2r}\cos(3\theta)\right)\left(\frac{\partial}{\partial s}\left(s^2+t^2\right)^{1/2}\right)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial r}\cdot\frac{\partial r}{\partial t} + \frac{\partial z}{\partial \theta}\cdot\frac{\partial \theta}{\partial t}$$

$$= \left(2e^{2r}\sin(3\theta)\right)(s-2t) + \left(3e^{2r}\cos 3\theta\right)\left(t\left(s^2+t^2\right)^{-1/2}\right)$$

13. $F(x,y,z) = x^2 + zy + y^2 + zx^2 + z^3$

$F(x, y, g(x,y)) = 0$

$$g_x(x,y,z) = -\frac{F_x(x,y,z)}{F_z(x,y,z)} = -\frac{(2x+2z)}{y+3z^2+x^2}$$

$$g_y(x,y,z) = \frac{-F_y(x,y,z)}{F_z(x,y,z)} = \frac{-(z+2y)}{y+x^2+3z^2}$$

14. 1. Normal vector is $\nabla f(x_0, y_0)$ when it is non-zero

$\nabla f(x,y) = (2x+3)\,\hat{\imath} + (1-3y^2)\,\hat{\jmath}$ $\qquad f(x,y) = x^2+3x+y-y^3$

normal to level curves.

14.2 critical points : $\nabla f(x,y) = (0,0)$

$2x+3 = 0$ & $1-3y^2 = 0$

$x = -\frac{3}{2}$ $\qquad y = \pm\sqrt{\frac{1}{3}}$

$d = f_{xx}(x,y)f_{yy}(x,y) - \left(f_{xy}(x,y)\right)^2$

$= (2)(-6y) - 0$

$= -12y$

$f_{xx}(x,y) = 2$

$f_{yy}(x,y) = -6y$

$f_{xy}(x,y) = 0$

14.2 (cont.)

critical points: $\left(-\frac{3}{2}, +\sqrt{1/3}\right)$

$d\left(-\frac{3}{2}, +\sqrt{1/3}\right) = -12\left(\sqrt{\frac{1}{3}}\right) < 0$

So there is a saddle point at $\left(-\frac{3}{2}, \sqrt{1/3}\right)$

$\left(-\frac{3}{2}, -\sqrt{1/3}\right)$

$d(-3/2, -\sqrt{1/3}) = -12(-\sqrt{1/3}) > 0$

& $f_{yy}(-3/2, -\sqrt{1/3}) > 0$

so there is a relative minimum at $(-3/2, -\sqrt{1/3})$