MAT127 Fall 2023

Practice Midterm II

The updated time and location of the test:

Wed Nov 8 8:30-9:50 pm, Eng 145

Exam will cover sections 8.7-8.8, 7.1-7.3

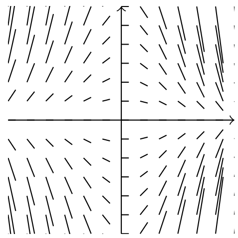
You will be allowed to use calculators. The actual test will contain 5 problems (some multipart)

Problem 1. Match the differential equation with its direction field.

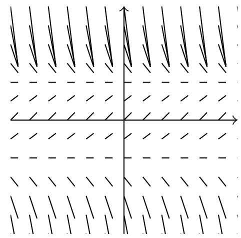
(1) $y' = (y^2 + 1)/(x - 1)$

(2) $y' = -xy$

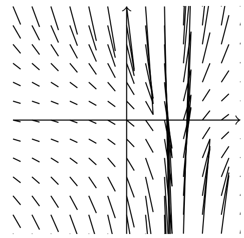
(3) $y' = 1 - y^2$



(A)



(B)



(C)

FIGURE 1. Direction fields

Solution. Among the above equations, only (3) is x-independent in the right-hand side. This indicates that the direction field exhibits a translational symmetry in the x-direction. This specific property is solely reflected in picture (1b).

Equation (1) exhibits the characteristic of having an infinite slope at $x = 1$. We observe that picture (1c) precisely captures this attribute.

To summarize the correspondence:

$$(1) \Rightarrow (1c),$$
$$(2) \Rightarrow (1a),$$
$$(3) \Rightarrow (1b).$$

□

An *equilibrium* solution $y(x)$ of a differential equation $y' = G(x, y)$ is a constant function $y(x) = a$ such that $0 = G(x, a)$. for all x.

Problem 2. Find equilibrium solutions of the following differential equations

(1) $y' = x(y^2 - 2)$
(2) $y' = \cos(x)\cos(2y)$
(3) $y' = x^2 - y^2$

Solution. (1) To find the equilibrium solutions of the differential equation $\frac{dy}{dx} = x(y^2 - 2)$, we set $\frac{dy}{dx}$ equal to zero and solve for y. Equilibrium solutions are the values of y for which the derivative is zero.

$$0 = x(y^2 - 2)$$

Now, we have two cases:

Case 1: $x = 0$

If $x = 0$, then the equation becomes:

$$0 = 0(y^2 - 2)$$

This equation is satisfied for any value of y. But the line $x = 0$ is not a graph of a function.

Case 2: $y^2 - 2 = 0$

If $y^2 - 2 = 0$, then: $y^2 = 2$ Taking the square root of both sides:

$$y = \pm\sqrt{2}$$

So, the equilibrium solutions are: $y = \sqrt{2}$ or $y = -\sqrt{2}$.

(2) To find the equilibrium solutions of the differential equation $\frac{dy}{dx} = \cos(x)\cos(2y)$, we set $\frac{dy}{dx}$ equal to zero and solve for y. Equilibrium solutions are the values of y for which the derivative is zero.

$$0 = \cos(x)\cos(2y)$$

Now, we need to consider the solutions for $\cos(x) = 0$ and $\cos(2y) = 0$:

1. When $\cos(x) = 0$, this occurs at $x = \frac{\pi}{2} + k\pi$, where k is an integer. In this case, the value of y can be anything, so we don't restrict y in this case. We discard this because the line $x = \frac{\pi}{2} + k\pi$ is not a graph of a function.

2. When $\cos(2y) = 0$, this occurs at $2y = \frac{\pi}{2} + k\pi$, where k is an integer. This implies $y = \frac{\pi}{4} + \frac{k\pi}{2}$.

So, the equilibrium solutions are: For $y = \frac{\pi}{4} + \frac{k\pi}{2}$, where k is an integer, x can be any real number.

(3) To find the equilibrium solutions of the differential equation $\frac{dy}{dx} = x^2 - y^2$, we set $\frac{dy}{dx}$ equal to zero and solve for y. Equilibrium solutions are the values of y for which the derivative is zero. $0 = x^2 - y^2$

Now, we can factor this equation:$0 = (x + y)(x - y)$

This equation has two cases:

Case 1: $x + y = 0$

If $x + y = 0$, then $y = -x$. But $\frac{dy}{dx} = \frac{d(-x)}{dx} = -1 \neq 0$

Case 2: $x - y = 0$

If $x - y = 0$, then $y = x$.But $\frac{dy}{dx} = \frac{d(x)}{dx} = 1 \neq 0$

So, in this case there are no equilibrium solution:

□

Problem 3. Estimate the maximum error when approximating the following functions with the indicated Taylor polynomial of degree n centered at a, on the given interval.

(1) $f(x) = x\sin(x), n = 4, a = 0, -1 \leq x \leq 1$
(2) $f(x) = x\ln(x), n = 3, a = 1, 0.5 \leq x \leq 1.5$
(3) $f(x) = x^{2/3}, n = 3, a = 1, 0.8 \leq x \leq 1.2$

Solution. (1) Since $n = 4$ we must find the fifth derivative of $x\sin(x)$.

To find the fifth derivative of $f(x) = x\sin(x)$, we start with the previous derivatives:

The first derivative is: $f'(x) = \sin(x) + x\cos(x)$.

The second derivative is: $f''(x) = 2\cos(x) - x\sin(x)$.

The third derivative is: $f'''(x) = -3\sin(x) - x\cos(x)$.

The fourth derivative is: $f^{(4)}(x) = x\sin(x) - 4\cos(x)$.

So, the fifth derivative is $f^{(5)}(x) = 5\sin(x) + x\cos(x)$. We apply Taylor's inequality

(1) $$|R_n(x)| \leq \frac{M_{n+1}}{(n+1)!}|x-a|^{(n+1)}$$

where $M_{n+1} = \max_x f^{(n+1)}(x)$. In our case $M_5 = \max_{-1\leq x\leq 1}|f^{(5)}(x)| = \max_{-1\leq x\leq 1}|5\sin(x)+ x\cos(x)| \leq \max_{-1\leq x\leq 1}|5\sin(x)| + \max_{-1\leq x\leq 1}|x\cos(x)| \leq 5+1 = 6$. We conclude that

$$|R_4(x)| \leq \frac{6}{(4+1)!}|x|^{4+1} \leq \max_{-1\leq x\leq 1}\frac{6}{5!}|x|^5 \leq \frac{6}{5!} = \frac{1}{120}$$

(2) Since $n = 3$ we must find the fourth derivative of $x\ln(x)$.

To compute the fourth derivative of $x\ln(x)$, we'll find the first three derivatives step by step.

The original function is $f(x) = x\ln(x)$.

The first derivative is:

$$\begin{aligned} f'(x) &= \frac{d}{dx}(x\ln(x)) \\ &= \ln(x) + x\cdot\frac{1}{x} \\ &= \ln(x) + 1. \end{aligned}$$

The second derivative is:

$$\begin{aligned} f''(x) &= \frac{d}{dx}(\ln(x)+1) \\ &= \frac{d}{dx}(\ln(x)) + \frac{d}{dx}(1) \\ &= \frac{1}{x} + 0 \\ &= \frac{1}{x}. \end{aligned}$$

Now, the third derivative is:

$$f'''(x) = \frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}\left(x^{-1}\right) = -x^{-2} = -\frac{1}{x^2}.$$

So, the fourth derivative of $x\ln(x)$ is $\frac{d}{dx}(-\frac{1}{x^2}) = 2\frac{1}{x^3}$. We are going to apply Taylor's inequality (1) with $n = 3$

$M_4 = \max_{0.5\le x\le 1.5} |f^{(4)}(x)| = \max_{0.5\le x\le 1.5} |2\frac{1}{x^3}| = 2\frac{1}{0.5^3} = 16$. We used that $2\frac{1}{x^3}$ is a decreasing function.

$$|R_3(x)| \le \frac{M_4}{(4)!}|x-1|^4 \le \frac{16}{4!}\max_{0.5\le x\le 1.5}|x-1|^4 \le \frac{16}{24}0.5^4 = \frac{1}{24}$$

(3) Since $n = 3$ we must find the fourth derivative of $x^{2/3}$.

To compute the fourth derivative of $x^{2/3}$, we'll find the first four derivatives step by step.

The first derivative is:

$$f'(x) = \frac{d}{dx}(x^{2/3}) = \frac{2}{3}x^{2/3-1} = \frac{2}{3}x^{-1/3}.$$

The second derivative is:

$$f''(x) = \frac{d}{dx}\left(\frac{2}{3}x^{-1/3}\right) = \frac{2}{3}\frac{d}{dx}(x^{-1/3}) = \frac{2}{3}\left(-\frac{1}{3}\right)x^{-1/3-1} = -\frac{2}{9}x^{-4/3}.$$

The third derivative is:

$$\begin{aligned} f'''(x) &= \frac{d}{dx}\left(-\frac{2}{9}x^{-4/3}\right) \\ &= -\frac{2}{9}\frac{d}{dx}(x^{-4/3}) \\ &= -\frac{2}{9}\left(-\frac{4}{3}\right)x^{-4/3-1} \\ &= \frac{8}{27}x^{-7/3}. \end{aligned}$$

Now, the fourth derivative is:

$$\begin{aligned} f''''(x) &= \frac{d}{dx}\left(\frac{8}{27}x^{-7/3}\right) \\ &= \frac{8}{27}\frac{d}{dx}(x^{-7/3}) \\ &= \frac{8}{27}\left(-\frac{7}{3}\right)x^{-7/3-1} \\ &= -\frac{56}{81}x^{-10/3}. \end{aligned}$$

So, the fourth derivative of $x^{2/3}$ is $-\frac{56}{81}x^{-10/3}$.

We are going to apply Taylor's inequality (1) with $n = 3$

$M_4 = \max_{0.8\leq x\leq 1.2}|f^{(4)}(x)| = \max_{0.8\leq x\leq 1.2}|-\frac{56}{81}x^{-10/3}| = \frac{56}{81}\max_{0.8\leq x\leq 1.2}|x^{-10/3}| = \frac{56}{81}\times 0.8^{-10/3}$. We used that $x^{-10/3}$ is a decreasing function.

$$|R_3(x)| \leq \frac{M_4}{(4)!}|x-1|^4 \leq \frac{56}{81}\times 0.8^{-10/3}\frac{1}{4!}\max_{0.8\leq x\leq 1.2}|x-1|^4 \leq \frac{56}{81}\frac{1}{4!}0.8^{-10/3}0.2^4 \sim 0.0000969717$$

□

Problem 4.

(1) Find the general solution of the differential equation: $y' = \frac{y(y+1)}{x(x-1)}$

(2) Find a solution of $(y + x^2y)\frac{dy}{dx} = 1$ such that $y(0) = 2$

(3) The problem $y' = x^{3/2}y^2 - x^{3/2} - 2xy^2 + 2x$ may or may not be separable. If it is, then decompose the problem as $y' = F(x)G(y)$ and write formulae for F, G, followed by solving for all solutions y (left as implicit to save time). Otherwise, explain in detail why it fails to be separable, and don't solve for y.

Solution. (1) To solve the differential equation $y' = \frac{y(y+1)}{x(x-1)}$, we'll use separation of variables.

Separating the variables:

$$\frac{dy}{dx} = \frac{y(y+1)}{x(x-1)} \tag{2}$$
$$\frac{dy}{y(y+1)} = \frac{dx}{x(x-1)}$$

Now, integrate both sides:

$$\int \frac{1}{y(y+1)}\,dy = \int \frac{1}{x(x-1)}\,dx$$

On the left side, we can use partial fraction decomposition to simplify:

$$\int \left(\frac{1}{y} - \frac{1}{y+1}\right)\,dy = \int \left(-\frac{1}{1-x} - \frac{1}{x}\right)\,dx = \int \left(\frac{1}{x-1} - \frac{1}{x}\right)\,dx$$

Integrate each term:

$$\ln|y| - \ln|y+1| = -\ln|x| + \ln|1-x| + C$$

Where C is the constant of integration.

Now, we can simplify further:

$$\ln\left|\frac{y}{y+1}\right| = \ln\left|\frac{1-x}{x}\right| + C$$

Remove the natural logarithms:

$$\frac{y}{y+1} = \pm\frac{1-x}{x}e^C$$

Since $\pm e^C$ is just another constant, let's call it K:

$$\frac{y}{y+1} = K\frac{1-x}{x}$$

Now, solve for y:

$$y = \frac{K(1-x)}{K(x-1)+x}$$

So, the solution to the differential equation is:

$$y = \frac{K(1-x)}{K(x-1)+x} \tag{3}$$

We need to exercise caution in our approach. In equation (2), we divided both sides of the equation by $y(y+1)$ to separate the variables. However, we made an assumption that $y(y+1)$ is not identically zero.

But what if it is?

In this case, it implies that either $y(x) = 0$ or $y(x) = -1$.

The question arises: Are these functions solutions to the initial equations? The answer is yes; these are stationary solutions.

These solutions correspond to the limiting cases where K takes specific values:

1. When $y(x) = 0$, this corresponds to $K = 0$. 2. When $y(x) = -1$, this corresponds to $K = \infty$.

(2) We rewrite the equation in the form $y' = \frac{1}{(x^2+1)y}$ and see that it is a separable equation.

$$\int y dy = \int \frac{dx}{x^2+1}$$

$\Rightarrow y^2/2 = \arctan(x) + c \Rightarrow y = \pm\sqrt{2\arctan(x) + 2c}$

$2 = y(0) = \pm\sqrt{2\arctan(0) + 2c} \Rightarrow c = 2$ and $y(x) = \sqrt{2\arctan(x) + 4}$.

(3) $y' = F(x)G(y) = (x^{3/2} - 2x)(y^2 - 1)$

$$\frac{1}{2}\log|1-y| - \frac{1}{2}\log|y+1| = \int -\frac{dy}{2(y+1)} - \frac{dy}{2(1-y)} = \int \frac{dy}{y^2-1} = \int (x^{3/2} - 2x)dx = \frac{2x^{5/2}}{5} - x^2 + C$$

$$\frac{1}{2}\log|\frac{1-y}{y+1}| = \frac{2x^{5/2}}{5} - x^2 + C$$
$$\frac{1-y}{y+1} = K\exp(\frac{4}{5}x^{5/2} - 2x^2)$$

□

Problem 5.

(1) The function $P(t)$ models the number of bees (in thousands) in a colony at time t (in years). Suppose the function $P(t)$ satisfies the differential equation

$$\frac{dP}{dt} = 2(1 - 2\sin(t))P$$

The colony initially has 500 bees. Use Euler's method, with three steps, to find the approximate number of bees (in thousands) in the farm after one year. Fill in the table with the appropriate values of t and your approximations.

t (in years)	0			1
$P(t)$ (in thousands)				

(2) Find the general solution of $\frac{dP}{dt} = 2(1 - 2\sin(t))P$

(3) Use the differential equation $\frac{dP}{dt} = 2(1 - 2\sin(t))P$ to find the exact value of t during the first year at which the number of bees in the colony has a maximum.

Solution. (1) The function $P(t)$ models the number of bees (in thousands) in a colony at time t (in years). The differential equation is given as:

$$\frac{dP}{dt} = 2(1 - 2\sin(t))P$$

The colony initially has 500 bees, which means that at $t = 0$, $P(0) = 500$.

We want to find the approximate value of $P(t)$ after one year, using Euler's method with three steps, each with a step size of $\Delta t = 1/3$ years.

We will use the following table to fill in the values:

t (in years)	0	1/3	2/3	1
$P(t)$ (in thousands)	500			

Now, we will use Euler's method to estimate $P(t)$ at $t = 1/3$:

Euler's Method:

$$P(t + \Delta t) = P(t) + \frac{dP}{dt}\Delta t$$

For $t = 0$:

$$P(1/3) = P(0) + \frac{dP}{dt}(0) \cdot (1/3)$$

Now, we need to calculate $\frac{dP}{dt}$ at $t = 0$:

$$\frac{dP}{dt}(0) = 2(1 - 2\sin(0))P(0) = 2(1 - 0) \cdot 500 = 1000$$

So, we can calculate $P(1/3)$:

$$P(1/3) = 500 + 1000 \cdot (1/3) = 500 + 333.33 = 833.33$$

Now, we will use Euler's method to estimate $P(t)$ at $t = 2/3$:

For $t = 1/3$:

$$P(2/3) = P(1/3) + \frac{dP}{dt}(1/3) \cdot (1/3)$$

We have already calculated $P(1/3)$ and $\frac{dP}{dt}(1/3) = 2(1 - 2\sin(1/3))P(1/3) = 2(1 - 2\sin(1/3))833.33 = 576.015$

$$P(2/3) = 833.33 + 576.015 \cdot (1/3) = 833.33 + 192.005 = 1025.34$$

Finally, we will use Euler's method to estimate $P(t)$ at $t = 1$:

For $t = 2/3$:

$$P(1) = P(2/3) + \frac{dP}{dt}(2/3) \cdot (1/3)$$

We have already calculated $P(2/3)$ above and

$$\frac{dP}{dt}(2/3) = 2(1 - 2\sin(2/3))P(2/3) = 2(1 - 2\sin(2/3))1025.34 = -485.477$$

.

$$P(1) = 1025.34 - 485.477 \cdot (1/3) = 944.44 - 161.826 = 782.614$$

Now, we have the approximate values of $P(t)$ at $t = 0$, $t = 1/3$, $t = 2/3$, and $t = 1$. The table can be filled as follows:

t (in years)	0	1/3	2/3	1
$P(t)$ (in thousands)	500	833.33	1025.34	782.614

So, after one year, there are approximately 1166.67 thousand bees in the colony.

(2) We have the differential equation:

$$\frac{dP}{dt} = 2(1 - 2\sin(t))P$$

To solve this equation, we'll use separation of variables. Let's begin by isolating the variables P and t:

$$\frac{dP}{P} = 2(1 - 2\sin(t))dt$$

Now, we can integrate both sides:

$$\int \frac{dP}{P} = \int 2(1 - 2\sin(t))dt$$

Integrating, we get:

$$\ln |P| = 2t + 4\cos(t) + C_1$$

Where C_1 is the constant of integration. To remove the absolute value, we can express P as:

$$P = \pm e^{2t+4\cos(t)+C_1}$$

Next, we can simplify this further by using the properties of exponents and constants:

$$P = \pm e^{C_1} e^{2t} e^{4\cos(t)}$$

Let $C = \pm e^{C_1}$ (a new constant). Then, we have the general solution:

$$P(t) = Ce^{2t}e^{4\cos(t)}$$

This is the general solution to the differential equation. Observe that if $P(t_0) = 0$ for some $t_0 \Rightarrow C = 0 \Rightarrow P(t) = 0$ for all t.

(3) In order to get $dP/dt = 0$ we need either $2(1 - 2\sin(t)) = 0$ or $P = 0$. Considering that the bee population, represented by $P(t)$, never reaches zero during the first year, as demonstrated in the previous question, it follows that at the maximum point, the expression $2(1 - 2\sin(t))$ equals zero. This occurs when $\sin(t) = 1/2$. During the first year it is at $t = \pi/6 = 0.523$ years.

□

Problem 6. $f(x) = e^{-x^3/3}$

(1) Find the tenth derivative $f^{(10)}(x)|_{x=0}$
(2) Find the sixth derivative $f^{(6)}(x)|_{x=0}$.
(3) Find the Taylor polynomial for $\log(1 + x)e^{-x^3/3}$ of degree three at $x = 0$.

Solution. We identify Taylor expansion formula

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + \dots \tag{4}$$

with the series

$$\sum_{n=0}^{\infty} \frac{\left(-x^3/3\right)^n}{n!} = \sum_{n=0}^{\infty}(-1)^n \frac{x^{3n}}{3^n n!} \tag{5}$$

obtained by substituting $-x^3/3$ into $\exp(x) = \sum_{n=0}^{\infty} x^n/n!$. The series (5) involves only powers of x which are multiples of tree. By comparing coefficients of (4) and (5) conclude that $f^{3n+1}(0) = f^{3n+2}(0) = 0$.

(1) So $f^{10}(0) = 0$.
(2) We also see that $\frac{f^{(3n)}(0)}{(3n)!} = (-1)^n \frac{1}{3^n n!}$ and $f^{(3n)}(0) = (-1)^n \frac{(3n)!}{3^n n!}$. $f^{(6)}(0) = f^{(3\times 2)}(0) = (-1)^2 \frac{(3\times 2)!}{3^2 \times 2!} = \frac{1\cdot\dots\cdot 6}{1\cdot 2\times 3^2} = 1 \cdot 2 \cdot 4 \cdot 5 = 40$

(3) We the above computations write down first two terms of the Taylor expansion for $e^{-x^3/3} = 1 - x^3/3 + \cdots$. $g(x) = \ln(1+x)$ has the following two derivative: $g'(x) = (1+x)^{-1}$,$g''(x) = -(1+x)^{-2}$, $g''(x) = 2(1+x)^{-3}$ and $g(0) = 0, g'(0) = 1, g''(0) = -1, g'''(0) = 2$. So the Taylor polynomial for $\ln(1+x)$ is $x - \frac{1}{2!}x^2 + \frac{2}{3!}x^3 + \cdots = x - \frac{1}{2}x^2 - \frac{1}{3}x^3 + \cdots$. We conclude that $e^{-x^3/3}\ln(1+x) = (1 - x^3/3 + \cdots) \times (x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots) = x - \frac{x^2}{2} - \frac{x^3}{3} + \cdots$. In this formula we are neglecting terms of degree greater than 3.

□

Problem 7. A tank with a capacity of 500 gal originally contains 200 gal of water with 100 lb of salt in solution. Water containing 1 lb salt per gallon is entering at a rate of 3 gal/min, and the mixture is allowed to flow out of the tank at a rate of 2 gal/min. Let $x(t)$ be the amount of salt in the tank at time t. Derive a differential equation for $x(t)$. Include a initial condition and the time for which the model is valid.YOU DO NOT HAVE TO SOLVE THE IVP

Solution. Let $x(t)$ be the amount of salt in the tank at the time t and $V(t)$ is the volume of the water in tank. Note that $V(t) = 200 + t$. So the differential equation we need is

$$x'(t) = \text{rate in} - \text{ rate out } = 3 - \frac{2x(t)}{V(t)} = 3 - \frac{2x(t)}{200+t} \text{ and } x(0) = 100.$$

□