Midterm I

MAT127C Fall 2023

Tue Oct 3, 8:30-9:50 pm, Frey 100

Name: (please print)	**ID #**:
Your lecture: (see list below)	

Lecture 01	MW 4:00- 5:20pm Harriman Hall 108	Insung Park
Lecture 02	TuTh 10:00-11:20am Earth&Space 131	Mikhail Movshev
Lecture 03	TuTh 5:30- 6:50pm Earth&Space 131	Charles Cifarelli

No notes, books or calculators. You must show your reasoning, not just the answer. Answers without justification will get only partial credit.

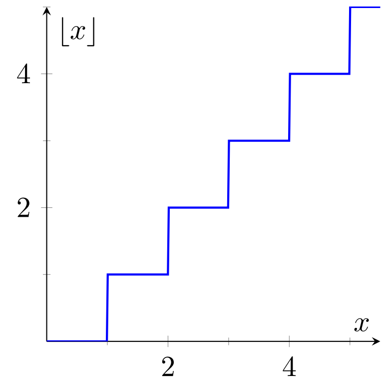
Please cross out anything that is not part of your solution — e.g., some preliminary computations that you didn't need. All answers should be simplifies if possible — e.g., sin(0) should be replaced by 0. However, unless instructed, do not replace exact answers by approximate ones — e.g. do not replace $\sqrt{2}$ by 1.41

	1 20pt	2 20pt	3 20pt	4 20pt	5 20pt	**Total** 100pts
Grade						

Problem 1. Determine the limit

$$\lim_{n\to\infty} \frac{\ln\lfloor \ln(n)\rfloor}{n}$$

provided it is defined. Otherwise, give an explanation for its non-existence. In the given expression, $x \to \lfloor x \rfloor$ represents the step function, denoting the integral part of x.



Solution. Evidently $\ln(x)$ is increasing function and $\lfloor x \rfloor$ satisfies

$$\lfloor x \rfloor \le x, x > 0. \tag{1}$$

From this we deduce that $0 \le \frac{\ln\lfloor \ln(n)\rfloor}{n} \le \frac{\ln\ln(n)}{n}, n \ge 10$ Determine the limit

$$\lim_{n\to\infty} \frac{\ln\ln(n)}{n} = \lim_{x\to\infty} \frac{\ln\ln(x)}{x}$$

provided it is defined.

Using l'Hôpital's Rule, we can differentiate the numerator and denominator with respect to x:

$$\lim_{x\to\infty} \frac{\ln\ln(x)}{x} = \lim_{x\to\infty} \frac{\frac{d}{dx}(\ln\ln(x))}{\frac{d}{dx}(x)}$$

Now, let's find the derivatives:

$$\lim_{x\to\infty} \frac{\frac{1}{\ln(x)} \cdot \frac{d}{dx}(\ln(x))}{1} = \lim_{x\to\infty} \frac{\frac{1}{\ln(x)} \cdot \frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x\ln(x)}$$

As x approaches infinity, the denominator $x\ln(x)$ also approaches infinity, and the limit becomes:

$$\lim_{x\to\infty} \frac{1}{x\ln(x)} = 0$$

Therefore, the limit of $\frac{\ln\ln(n)}{n}$ as n approaches infinity is defined and equals 0. By squeeze theorem $\lim_{n\to\infty} \frac{\ln\lfloor\ln(n)\rfloor}{n} = 0$

$\square$

Problem 2.

1. Write decimal expansion for $\frac{1}{11}$. Show your work. Perform geometric series summation to show that this decimal expansion converges to $\frac{1}{11}$.

2. Sum the series

$$\sum_{n=0}^{\infty} \frac{1+(-2)^{(-1)^n \cdot n}}{3^n}. \tag{2}$$

Solution. **1.**

$$\begin{array}{r|ccccccccc}
 & & 0 & . & 0 & 9 & 0 & 9 & 0 & 9 \\ \hline
11 & 1 & : & 0 & . & 0 & 0 & 0 & 0 & 0 \\
 & & 0 & 0 & & 0 & 0 & 0 & 0 & 0 \\ \cline{3-5}
 & & & & 1 & 0 & 0 & 0 & 0 & 0 \\
 & & & & 1 & 0 & 0 & 0 & 0 & 0 \\ \cline{5-7}
 & & & & & & 0 & 0 & 0 & 0 \\
 & & & & & & 0 & 0 & 0 & 0 \\ \cline{7-9}
 & & & & & & & 0 & 0 & 0
\end{array}$$

. Let's consider the decimal $0.0909090909\ldots$. We can express it as an infinite geometric series:

$$0.0909090909\ldots = \frac{9}{10} \cdot \frac{1}{10} + \frac{9}{10^2} \cdot \frac{1}{10} + \frac{9}{10^3} \cdot \frac{1}{10} + \ldots \tag{3}$$

The first term is $a = \frac{9}{10} \cdot \frac{1}{10}$ and the common ratio is $r = \frac{1}{10}$.

Now, let's calculate the sum of the infinite geometric series using the formula:

$$S = \frac{a}{1-r}$$

Substituting in the values:

$$S = \frac{\frac{9}{10} \cdot \frac{1}{10}}{1 - \frac{1}{10}}$$

Simplify further:

$$S = \frac{\frac{9}{100}}{\frac{9}{10}}$$

Now, divide:

$$S = \frac{\frac{9}{100}}{\frac{9}{10}} = \frac{\frac{9}{100}}{\frac{90}{100}} = \frac{9}{100} \cdot \frac{100}{90} = \frac{1}{11}$$

Therefore, the sum of the geometric series $0.0909090909\ldots$ is equal to $\frac{1}{11}$, which confirms that $0.0909090909\ldots = \frac{1}{11}$.

2. We break the sum according to the parity of n:

$$\sum_{n=0}^{\infty} \frac{1 + (-2)^{(-1)^n \cdot n}}{3^n} = \sum_{n=0}^{\infty} \frac{1}{3^n} + \sum_{n=0}^{\infty} \frac{(-2)^{(-1)^n \cdot n}}{3^n} =$$

$$\sum_{n=0}^{\infty} \frac{1}{3^n} + \sum_{n=0}^{\infty} \frac{(-2)^{(-1)^{2n} \cdot 2n}}{3^{2n}} + \sum_{n=0}^{\infty} \frac{(-2)^{(-1)^{2n+1} \cdot (2n+1)}}{3^{2n+1}}$$

$$= \sum_{n=0}^{\infty} \frac{1}{3^n} + \sum_{n=0}^{\infty} \frac{(-2)^{2n}}{3^{2n}} + \sum_{n=0}^{\infty} \frac{(-2)^{-(2n+1)}}{3^{2n+1}}$$

The first sum is $3/2$ the second $9/5$ the third is $-6/5$. The answer is $\frac{21}{10}$

□

Problem 3. Determine convergence of the series

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{1+(-1)^n \cdot n} \tag{4}$$

Solution. Break the sum according to the parity of n

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{1+(-1)^n \cdot n} = \sum_{n=1}^{\infty} \frac{1}{1+2n} - \sum_{n=1}^{\infty} \frac{1}{1-(2n+1)} =$$

$$\sum_{n=1}^{\infty} \frac{1}{2n} + \sum_{n=1}^{\infty} \frac{1}{2n+1} = \sum_{n=2}^{\infty} \frac{1}{n}$$

We see that our series is equivalent to harmonic series, which diverges by the integral test.

□

Problem 4.

$$A = \sum_{n=1}^{\infty} \frac{\cos(\sin(n))}{n^4}$$

Estimate the error of approximation $R_N = A - \sum_{n=1}^{N} \frac{\cos(\sin(n))}{n^4}$ for $N = 10$.

Solution.

$$R_N = A - \sum_{n=1}^{N} \frac{\cos(\sin(n))}{n^4} = \sum_{n=N+1}^{\infty} \frac{\cos(\sin(n))}{n^4} \tag{5}$$

Function $\cos(\sin(n))$ satisfies

$$-1 \le \cos(\sin(n)) \le 1, \tag{6}$$

so

$$\sum_{n=N+1}^{\infty} \frac{-1}{n^4} \le R_N = \sum_{n=N+1}^{\infty} \frac{\cos(\sin(n))}{n^4} \le \sum_{n=N+1}^{\infty} \frac{1}{n^4}$$

$\sum_{n=N+1}^{\infty} \frac{1}{n^4}$ satisfies

$$\sum_{n=N+1}^{\infty} \frac{1}{n^4} \le \int_{N+1}^{\infty} \frac{dx}{x^4} \Rightarrow$$

$$\sum_{n=N+1}^{\infty} \frac{1}{n^4} \le \frac{1}{3(N+1)^3}$$

Thus $-\frac{1}{3(N+1)^3} \le R_N \le \frac{1}{3(N+1)^3}$

□

Problem 5.

1. Find a power series representation (at 0) of the function

$$f(x) = -\int_0^x \frac{\log(1-t)}{t} dt$$

and compute its radius of convergence.

2. Determine whether the series are absolutely convergent, conditionally convergent, or divergent at the end-points of the interval of convergence.

Solution. **1.** To find the power series representation of the function $f(x) = -\int_0^x \frac{\log(1-t)}{t}\, dt$ centered at 0, we'll first find the Taylor series expansion of $\log(1-t)$ centered at 0:

$$\ln(1-t) = -\sum_{n=1}^{\infty} \frac{t^n}{n}$$

Now, let's integrate term by term to obtain the series representation of $f(x)$:

$$\begin{aligned} f(x) &= -\int_0^x \frac{\log(1-t)}{t}\, dt \\ &= \int_0^x \sum_{n=1}^{\infty} \frac{t^n}{n} \cdot \frac{1}{t}\, dt \\ &= \sum_{n=1}^{\infty} \int_0^x \frac{t^{n-1}}{n}\, dt \\ &= \sum_{n=1}^{\infty} \left[\frac{t^n}{n^2}\right]_0^x \\ &= \sum_{n=1}^{\infty} \frac{x^n}{n^2} \end{aligned}$$

So, the power series representation of $f(x)$ centered at 0 is:

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

To determine the radius of convergence, we can use the ratio test. The ratio test states that a power series $\sum_{n=0}^{\infty} a_n x^n$ converges if the following limit exists and is less than 1:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = L|x|$$

In this case, $a_n = \frac{1}{n^2}$, so we have:

$$\begin{aligned}
L &= \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \\
&= \lim_{n\to\infty}\left|\frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}}\right| \\
&= \lim_{n\to\infty}\left|\frac{n^2}{(n+1)^2}\right| \\
&= \lim_{n\to\infty}\left|\frac{n^2}{n^2+2n+1}\right| \\
&= \lim_{n\to\infty}\left|\frac{1}{1+\frac{2}{n}+\frac{1}{n^2}}\right| \\
&= 1
\end{aligned}$$

Since $L = 1$, the radius of convergence is $R = \frac{1}{L} = 1$.

So, the power series representation of $f(x)$ centered at 0 is $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$, and it converges for $-1 < x < 1$.

2. At $x = 1$ the power series become $\sum_{n=1}^{\infty} \frac{1}{n^2}$ at $x = -1$- $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ The first series is convergent because is p-series with $p = 2$. It is automatically absolutely convergent. The second series satisfies $|(-1)^n/n^2| \leq 1/n^2$. So it is absolutely convergent $\Rightarrow$ convergent.

□