H.W. #8

2-) $$\int_0^{\pi/2} \cos^5 x\,dx \overset{\cos^2 x = 1-\sin^2 x}{=} \int_0^1 (1-u^2)(1-u^2)\,du \;;\; \begin{matrix} u = \sin x \\ du = \cos x\,dx \end{matrix} \longrightarrow \begin{cases} x = \frac{\pi}{2} \to u = 1 \\ x = 0 \to u = 0 \end{cases}$$

$$\int_0^1 (1-2u^2+u^4)\,du = u - \frac{2u^3}{3} + \frac{u^5}{5}\Big|_0^1 = 1 - \frac{10}{15} + \frac{3}{15} = \frac{8}{15}$$

8-) $$\int_0^{\pi/4} \tan^2 x \sec^4 x\,dx \overset{\sec^2 x = \tan^2 x + 1}{=} \int_0^1 u^2(u^2+1)\,du \;;\; \begin{matrix} u = \tan x \\ du = \sec^2 x\,dx \end{matrix} \to \begin{cases} x = 0 \to u = 0 \\ x = \frac{\pi}{4} \to u = 1 \end{cases}$$

$$= \int_0^1 (u^4+u^2)\,du = \frac{u^5}{5} + \frac{u^3}{3}\Big|_0^1 = \frac{1}{5} + \frac{1}{3} = \frac{8}{15}$$

10-) Using $\begin{matrix} x = \sec\theta \\ dx = \sec\theta\tan\theta\,d\theta \end{matrix} \to \int \frac{\sqrt{x^2-1}}{x^4}\,dx = \int \frac{\overbrace{\sqrt{\sec^2\theta - 1}}^{\tan\theta}}{\sec^3\theta}\tan\theta\,d\theta = \int \frac{\tan^2\theta}{\sec^3\theta}\,d\theta = \int \frac{\sin^2\theta}{\cos^2\theta}\cdot\cos^3\theta\,d\theta$

$$= \int \sin^2\theta\cos\theta\,d\theta = \int u^2\,du = \frac{u^3}{3} + C = \frac{\sin^3\theta}{3} + C$$

Using $x = \sec\theta$ &

$\to \sin\theta = \frac{\sqrt{x^2-1}}{x} \longrightarrow \boxed{\frac{(x^2-1)^{3/2}}{3x^3} + C}$

3-) $$\int_0^1 \frac{x-1}{x^2+3x+2}\,dx \to \left(\frac{x-1}{x^2+3x+2} = \frac{A}{x+2} + \frac{B}{x+1}\right)(x+2)(x+1)$$

$$\Rightarrow x - 1 = A(x+1) + B(x+2)$$

$$x - 1 = x(A+B) + (A+2B)$$

Using $A+B = 1$ & $A+2B = -1 \to -A = -3 \to A = 3$, and $B = -2$

$$\int_0^1 \frac{3}{x+2}\,dx - \int_0^1 \frac{2}{x+1}\,dx = 3\ln|x+2|\Big|_0^1 - 2\ln|x+1|\Big|_0^1 = 3(\ln 3 - \ln 2) - 2\ln 2 = 3\ln 3 - 5\ln 2$$

28.) $\int_0^1 \frac{x^3-4x-10}{x^2-x-6}\,dx$

$$\begin{array}{r} x+1 \quad + \frac{3x-4}{x^2-x-6} \\ x^2-x-6 \overline{)\, x^3+0x^2-4x-10} \\ -(x^3-x^2-6x) \\ \hline 0 \quad x^2+2x-10 \\ -(x^2-x-6) \\ \hline 0 \quad +3x-4 \end{array}$$

$$\underbrace{\int_0^1 (x+1)\,dx}_{\underbrace{\frac{x^2}{2}+x\Big|_0^1}_{\frac{3}{2}}} + \underbrace{\int_0^1 \frac{3x-4}{x^2-x-6}\,dx}$$

$$\left(\frac{3x-4}{x^2-x-6} = \frac{A}{x-3} + \frac{B}{x+2}\right)(x-3)(x+2)$$

$$3x-4 = A(x+2) + B(x-3)$$

$$3x-4 = x(A+B) + (2A-3B) \to \overbrace{A+B=3}^{B=3-A} \;\&\; 2A-3B=-4$$

Plugging in, we get $2A-3(3-A)=-4 \to 5A=5 \to A=1$ and $B=2$

$$\int_0^1 \frac{3x-4}{x^2-x-6}\,dx = \int_0^1 \frac{1}{x-3}\,dx + \int_0^1 \frac{2}{x+2}\,dx = \ln|x-3|\Big|_0^1 + 2\ln|x+2|\Big|_0^1 = \ln 2 - \ln 3 + 2(\ln 3 - \ln 2)$$

$$= \ln 3 - \ln 2$$

$$\int_0^1 \frac{x^3-4x-10}{x^2-x-6}\,dx = \frac{3}{2} + \ln 3 - \ln 2$$

or

$$= \boxed{\frac{3}{2} + \ln\left(\frac{3}{2}\right)}$$

5.9 # 2,18

2-)(a) From the concavity of $y=f(x)$, we see that $\begin{cases} L_n\text{: left end approx.} \rightarrow \text{overestimate } (0.9540)\ L_n \\ R_n\text{: right end approx.} \rightarrow \text{underestimate } (0.7811)\ R_n \end{cases}$

We can easily distinguish T_n from M_n by remembering (pg 413) that:

trapezoidal approx.: $T_n = \frac{R_n+L_n}{2} = \frac{0.9540+0.7811}{2} \approx 0.8675$

Since $R_n<M_n<L_n$ and $R_n<T_n<L_n$, we conclude that:

$T_n = 0.8675$ & $M_n = 0.8632$

(b) It should be clear that M_n, T_n are much closer to $\int_0^2 f(x)dx$ compared with L_n, R_n.

- If $f(x)$ is concave up from $x=0$ to $x=2$, then: $M_n < \int_0^2 f(x)dx < T_n$
- If $f(x)$ is concave down from $x=0$ to $x=2$, then $T_n < \int_0^2 f(x)dx < M_n$

Since our graph ($y=f(x)$) is concave up:

$$0.8632 < \int_0^2 f(x)dx < 0.8675$$

18-) (a-) Using eq. on pg. 413:

$$T_8 = \frac{1}{16}\left[f(0)+2f(\tfrac{1}{8})+2f(\tfrac{1}{4})+2f(\tfrac{3}{8})+2f(\tfrac{1}{2})+2f(\tfrac{5}{8})+2f(\tfrac{3}{4})+2f(\tfrac{7}{8})+f(1)\right], \quad a=0 \rightarrow x_i = i\Delta x,\ \Delta x = \frac{1}{8}$$

$$= \frac{1}{16}\left[1+2\cos(\tfrac{1}{64})+2\cos(\tfrac{1}{16})+\cdots+\cos 1\right]$$

$T_8 \approx 0.9023$

Using midpoint rule ($\bar{x}_i = \frac{1}{2}(x_{i-1}+x_i)$):

$$M_8 = \frac{1}{8}\left[f(\tfrac{1}{16})+f(\tfrac{3}{16})+f(\tfrac{5}{16})+f(\tfrac{7}{16})+f(\tfrac{9}{16})+\cdots+f(\tfrac{15}{16})\right]$$

$M_8 \approx 0.9036$

(b-) We want to apply eq. 3 for errors, so:

Using chain rule for $f(x)=\cos(x^2) \rightarrow f'(x) = -2x(\sin(x^2)) \xrightarrow{\text{product rule}} f''(x) = \overline{-2\sin(x^2)-4x^2\cos(x^2)}^{\,-(2\sin x^2+4x^2\cos x^2)}$

$|f''(x)| = 2(\sin x^2 + 2x^2\cos x^2)$

Using $a=0, b=1$ → For $0\le x\le 1$ → $\boxed{x^2\le 1}$ → $\boxed{\sin x^2 \le 1 \text{ \& } \cos x^2 \le 1}$

So we now have: $|f''(x)| \le 2(1+2\cdot 1)=6 \rightarrow K=6$ (from eq. 3)

From trapezoid rule: $(a=0, b=1, n=8)$

$$|E_T| \le \frac{K(b-a)^3}{12n^2} = \frac{6(1-0)^3}{12(8)^2} = \frac{1}{128} \to |E_T| \le 0.00781$$

From midpt. rule:

$$|E_M| \le \frac{K(b-a)^3}{24n^2} = \frac{1}{2}\left(\frac{1}{128}\right) = \frac{1}{256} \to |E_T| \le 0.00391$$

c-) From trap. rule: $$\frac{6(1)^3}{12n^2} < 0.00001 \to n^2 > \frac{1}{0.00002} \to n > 223.61 \to \overbrace{n \ge 224}^{\text{optional}}$$

From mdpt. rule $$\frac{6(1)^3}{24n^2} < 0.00001 \to n^2 > \frac{1}{4}(10^5) \to n > 158.11 \to \overbrace{n \ge 159}^{\text{optional.}}$$