MAT313 Fall 2008

Practice Midterm II

Problem 1. Let X be a set of lines through 0 in $\mathbb{R}^2$. The group $SL_2(\mathbb{R})$ acts on X by linear transformations. Let H be the stabilizer of a line defined by the equation $y = 0$.

1. Describe the set of matrices H.

   Denote the set of lines by $\mathbf{P} = \{L_{m,n}\}$, where $L_{m,n}$ is equal to $\{(x, y)|mx + ny = 0\}$. Note that $L_{km,kn} = L_{m,n}$ if $k \neq 0$. Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, $ad - bc = 1$ be an element of $SL_2(\mathbb{R})$. Then $gL_{m,n} = \{(x, y)|m(ax + by) + n(cx + dy) = 0\} = L_{ma+nc,mb+nd}$. By definition the line $\{(x, y)|y = 0\}$ is equal to $L_{0,1}$. Then $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) L_{0,1} = L_{c,d}$. The line $L_{c,d}$ coincides with $L_{0,1}$ if $c = 0$. Then the stabilizer $Stab(L_{0,1})$ coincides with $H = \{\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)\} \subset SL_2(\mathbb{R})$.

2. Describe the orbits of H in X. How many orbits are there?

   We already know one trivial H-orbit $O = \{L_{0,1}\}$. The complement $\mathbf{P}\backslash\{L_{0,1}\}$ is equal to $\{L_{m.n}|m \neq 0\} = \{L_{1.n/m}|m \neq 0\} = \{L_{1.t}\}$. The line $L_{1,0}$ is an element of $\mathbf{P}\backslash\{L_{0,1}\}$. Its H orbit is equal to $\{gL_{1,0}|g \in H\} = \{\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) L_{1,0}\} = \{L_{a,b}|a \neq 0\} = \{L_{1,b/a}|a \neq 0\} = \mathbf{P}\backslash\{L_{0,1}\}$.

   We conclude that H consists of two H-orbits $\{L_{0,1}\}$ and $\mathbf{P}\backslash\{L_{0,1}\}$.

3. Identify X with the set of cosets of $SL_2(\mathbb{R})$.

   The action of $SL_2(\mathbb{R})$ on $\mathbf{P}$ is transitive. This is because $gL_{0,1} = L_{a,b}$ with $(a, b) \neq (0, 0)$. For any pair (a, b), defined up to a multiplicative constant there is g such that $gL_{0,1} = L_{a,b}$. This is because for any such a pair (a, b) the equation $ad - bc = 1$ always has a solution.

Problem 2.

1. Describe all elements of order eight in $\mathbb{Q}/\mathbb{Z}$. a is an element of order 8 in $\mathbb{Q}/\mathbb{Z}$ iff a representative $\tilde{a} \in \mathbb{Q}$ such that $\tilde{a} \in a = \tilde{a} + \mathbb{Z}$ satisfies $8\tilde{a} \in \mathbb{Z}$. Then $\tilde{a} = s/8 + k, 0 \leq s < 8, k \in \mathbb{Z}$. Element $s_1/8 + k_1$ and $s_2/8 + k_3$ define the same element in $\mathbb{Q}/\mathbb{Z}$ id $s_1 = s_2$. We conclude that there are precisely eight elements of order 8 in $\mathbb{Q}/\mathbb{Z}$.

2. Find all elements of infinite order in $\mathbb{Q}/\mathbb{Z}$.

   Any rational number $a = p/q$ satisfies $qa \in \mathbb{Z}$. Thus $\mathbb{Q}/\mathbb{Z}$ contains no elements of infinite order.

3. Identify $\mathbb{Q}/\mathbb{Z}$ with a subgroup of $\mathbb{C}^*$

   Define a homomorphism $\psi : \mathbb{Q} \to \mathbb{C}^*$ by the formula $\psi(a) = exp(2\pi i a)$. The image coincides with group of roots of unity. The kernel is the set of integers. By the first isomorphism theorem $\mathbb{Q}/\mathbb{Z}$ is isomorphic to the group of unity, i.e. the set of solution of the equations $z^k = 1, k \in \mathbb{Z}$ in the complex numbers.

Problem 3.

Give an example of a non commutative group with a normal subgroup of index $p-1$, where p is prime

$H = \{\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\} \subset \mathrm{SL}_2(\mathbb{Z}_p)$. There is a homomorphism $\psi : H \to \mathbb{Z}_p^*$ defined by the formula $\psi(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}) = a$. Its kernel is equal to $K = \{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\}$. Since $|\mathbb{Z}_p^*| = p-1$, the index of K is $p-1$.

Problem 4. Give an example of a non commutative group that contains a subgroup of prime order.

The group K from Problem 3.

Problem 5. Let G be the group of quaternions, i.e., $G = \{1, -1, i, j, k, (-1)i, (-1)j, (-1)k\}$. The elements $1, -1$ are central and satisfy $-1^2 = 1$. In addition $i^2 = j^2 = k^2 = -1$, $(-1)ji = ij = k, (-1)ki = ik = (-1)j, (-1)jk = kj = (-1)i$.

Find orders of all elements in $G/Z(G)$, Is $G/Z(G)$ isomorphic to $\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2$. Why?

Define a homomorphism $G \to Z_2 + Z_2$ by the formula $\psi(1) = \psi(-1) = (0,0), \psi(i) = \psi(-1i) = (1,0), \psi(j) = \psi(-1j) = (0,1), \psi(k) = \psi(-1k) = (1,1)$. The kernel of this homomorphism is $\{1, -1\}$. The homomorphism is surjective. So $G/Z(G)$ is isomorphic to $\mathbb{Z}_2 + \mathbb{Z}_2$. The latter is not isomorphic to $\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2$, because the groups have different orders.

Problem 6 Give the definition of a factor group.

Let G be a group with a normal subgroup H. Define the group structure on the set of cosets $gH|g \in G$ by the formula $g_1H \times g_2H = g_1g_2H$. The set of cosets with this group structure is the factor(quotient) group G/H.