

MAT 311: Number Theory Spring 2008

Solutions to HW9

NOTE: in this assignment d stands for a positive integer which NOT a perfect square!

1. Prove that $x^2 - 80y^2 = -1$ has no solutions in positive integers. Generalize the argument that for any integer k the equation $x^2 - (k^2 - 1)y^2 = -1$ has no solutions in positive integers.
We will do the general case. Observe that $\sqrt{k^2 - 1} = \langle k - 1, 1, 2(k - 1) \rangle$. Since the period is even, by Theorem 7.25 of NZM. For $k = 9$, we get the initial equation. □
2. Given $\sqrt{18} = \langle 4, \overline{4}, \overline{8} \rangle$, find the least positive solution of $x^2 - 18y^2 = \pm 1$ (if any).
 $x^2 - 18y^2 = -1$ has NO positive solutions again by Theorem 7.25 in NZM. $x^2 - 18y^2 = -1$, however, does have solutions, and even infinitely many of them. Since the period $r = 2$, the solutions are $(x, y) = (h_{2n+1}, k_{2n+1})$, $n = 0, 1, 2, \dots$, by the same theorem. Observe that the sequences h_n, k_n are non-decreasing for $n \geq 1$ by construction, and therefore (h_1, k_1) should be the *smallest* solution. A simple computation gives $(h_1, k_1) = (17, 4)$. □
3. Prove that if d is divisible by a prime p with $p \equiv 3 \pmod{4}$, then $x^2 - dy^2 = -1$, has no solutions.
Assume for a contradiction that such a solution does exist. By quadratic reciprocity, $\left(\frac{3}{p}\right) = -1$, that is, -1 is not a QR mod p . But, taking the equation mod p , we get $x^2 = -1 \pmod{p}$, i.e. -1 is a QR mod p , a contradiction. □
4. Suppose that N is a non-zero integer. Prove that if $x^2 - dy^2 = N$ has a solution then it has infinitely many solutions.
By Theorem 7.26 of NZM, all the integral solutions of $x^2 - dy^2 = 1$ is given by (x_n, y_n) which are defined by $x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n$ where (x_1, y_1) are the least positive solutions of that equation, whose existence is granted by Theorem 7.25. Now, let (r, s) be a solution of $x^2 - dy^2 = N$. Then by the identity $N = 1 \cdot N = (r^2 - ds^2)(x_n^2 - dy_n^2) = (rx_n - dsy_n)^2 - d(ry_n - sx_n)^2 = X_n^2 - dY_n^2$ we get infinitely many solutions (X_n, Y_n) for $X^2 - dY^2 = N$. □
5. If k is a positive integer, prove that there are infinitely many integral solutions of $x^2 - dy^2 = 1$ with $k | y$.
If d is not a perfect square, $d' = dk^2$ is neither. Thus by Theorem 7.26 and 7.25, there are infinitely many solutions of the equation $X^2 - d'Y^2 = 1$. But since $X^2 - d'Y^2 = X^2 - d(kY)^2 = 1$, the equation $x^2 - dy^2 = 1$ has solutions $(x, y) = (X, kY)$. Clearly, $k | y$. □
6. Prove that $n^2 + (n + 1)^2$ is a perfect square for infinitely many n .
Let $x = 2n + 1$, $y = \sqrt{n^2 + (n + 1)^2}$. Then $x^2 - 2y^2 = -1$. Since $\sqrt{2} = \langle 1, \overline{2} \rangle$ has odd period, it has infinitely many integral solutions (x, y) by Theorem 7.25. But that precisely means that $n^2 + (n + 1)^2$ is a square for infinitely many n . □