MAT 311: Number Theory
Spring 2008

Solutions to HW8

1. Evaluate the infinite continued fractions:
(a) $\langle 2, 2, \dots \rangle = 1 + \sqrt{2}$
(b) $\langle 2, 1, 2, 1, \dots \rangle = 1 + \sqrt{3}$
(c) $\langle 1, 2, 1, 2, \dots \rangle = (1 + \sqrt{3})/2$
(d) $\langle 1, 3, 1, 3, \dots \rangle = (3 + \sqrt{21})/6$ □

2. For $n \geq 1$, prove that $k_n/k_{n-1} = \langle a_n, \dots, a_1 \rangle$. Find a similar formula for h_n/h_{n-1}.
Recall that those two sequences h_n and k_n are defined in (7.6) in NZM.
(1) Proof follows from induction: For $n = 1$, $k_1/k_0 = a_1/1 = a_1 = \langle 1 \rangle$, as required. Assume now that $k_n/k_{n-1} = \langle a_n, \dots, a_1 \rangle$. Then $k_{n+1}/k_n = (a_{n+1}k_n + k_{n-1})/k_n = a_{n+1} + 1/(k_n/k_{n-1}) = a_{m+1} + 1/\langle a_n, \dots, a_1 \rangle = \langle a_{n+1}, \dots, a_1 \rangle$, by induction.
(2) It is tempting to guess the formula for h_n/h_{n-1} to be $\langle a_n, \dots, a_0 \rangle$, but this is not exactly true if $a_0 = 0$: indeed, it doesn't even make sense, because 0 is not allowed to be anywhere in the simple cont'd fraction except for the first slot; that is $a_j \neq 0$ for $j \neq 0$. But this is easy to overcome: if a_0 happens to be equal to 0, then the formula becomes $h_n/h_{n-1} = \langle a_n, \dots, a_2 \rangle$. The proof is similar, and is done by induction. □

3. Expand each of the following as infinite continued fractions:
Using the formulae we discovered in Problem 1, we see that
(a) $\sqrt{2} = \langle 1, 2, 2, \dots \rangle$
(b) $\sqrt{2} - 1 = \langle 0, 2, 2 \dots \rangle$
(c) $\sqrt{2}/2 = \langle 0, 1, 2, 2, \dots \rangle$
(d) $1/\sqrt{3} = \langle 0, 1, 1, 2, 1, 2, 1, 2, 1, \dots \rangle$ □

4. Given two irrational numbers a, b with identical convergents $r_0, \dots, r_n$, prove that $a_j = b_j$ for $j = 0, \dots, n$.
Induction on n: if $n = 0$, then $r_0 = h_0/k_0 = a_0/1 = b_0/1$, so $a_0 = b_0$, as required. Now assume that $a_j = b_j$ for $j = 0, \dots, m$, for some $m < n$. We want to prove that $a_{m+1} = b_{m+1}$. Indeed, since $m + 1 \leq n$, the convergents $r_{m+1} = h_{m+1}/k_{m+1}$ are the same for both a and b, hence we have: $(a_{m+1}h_m + h_{m-1})/(a_{m+1}k_m + k_{m-1}) = (b_{m+1}h_m + h_{m-1})/(b_{m+1}h_m + h_{m-1})$. Simplifying this gives: $a_{m+1}(h_m k_{m-1} - h_{m-1}k_m) = b_{m+1}(h_m k_{m-1} - h_{m-1}k_m)$. Now we can cancel the expression in parantheses, because they are NON-ZERO: Indeed it is equal to $(-1)^{m+1}$ by Theorem 7.5 in NZM.

5. Let ξ be an irrational number. Prove that for $n \geq 1$, $\xi - h_n/k_n = (-1)^n k_n^{-2}(\xi_{n+1} + \langle 0, a_n, a_{n-1}, \dots, a_1 \rangle)^{-1}$.

By (7.9), $\xi - h_n/k_n = (-1)^n/(k_n(\xi_{n+1}k_n + k_{n-1}))$. Multiplying and dividing it by k_n^{-2} gives: $= (-1)^n k_n^{-2}/(\xi_{n+1} + 1/\langle a_n, \dots, a_1 \rangle))^{-1}$ by Problem 2. But $1/\langle a_n, \dots, a_1 \rangle)^{-1} = \langle 0, a_n, a_{n-1}, \dots, a_1 \rangle$. This completes the proof. □

6. Let $a_0, a_1, \dots, a_n$ and $b_0, b_1, dots, b_{n+1}$ be positive integers. State the condition for $\langle a_0, \dots, a_n \rangle < \langle b_0, \dots, b_{n+1} \rangle$.
Case 1: Assume $a_j = b_j$ for $j = 0, \dots, n$. Then we have the inequality iff n is even.
Case 2: Assume Case 1 does not hold, i.e. there is some ℓ such that $a_j = b_j$ for $j = 0, \dots, \ell - 1$ but $a_\ell = b_\ell$.
Case 2.1: Assume $\ell < n$. If ℓ is even, inequality holds iff $b_\ell > a_\ell$. If ℓ is odd, inequality holds iff $b_\ell < a_\ell$.
Case 2.2: Assume $\ell = n$. If ℓ is even, and if $b_{n+1} > 1$, then the inequality holds iff $a_n > b_n$. If ℓ is even, but if $b_{n+1} = 1$, then the inequality holds iff $a_n > b_n + 1$. If ℓ is odd, inequality holds iff $a_n \leq b_n$. □