HW7

This is due Monday, April 14

1. Let $f(x), g(x)$ be primitive non constant polynomial in $\mathbb{Z}[x]$, such that the greatest common divisor $(f(m), g(m)) > 1$ for infinitely many positive integers m. Construct an example to show that such polynomials exist with g.c.d$(f(x), g(x)) = 1$ in the polynomial sense.

2. Find a minimal polynomial of the following algebraic numbers 7, $\sqrt[3]{7}$, $(1+\sqrt[3]{7})/2$, $1+\sqrt{2}+\sqrt{3}$. Which of these are algebraic integers.

3. By definition the degree of an algebraic number α is the degree of its minimal polynomial. Prove that if α is algebraic of degree n then $-\alpha, \alpha^{-1}, \alpha - 1$ are also of degree n.

4. Let $i \in \mathbb{C}$, $i^2 = -1$ be the imaginary unit. Let $\alpha = a_i + a_2 i$ be an algebraic number, where a_1, a_2 are real. Does it follow that a_1, a_2 are algebraic numbers? If α is an algebraic integer, would a_1, a_2 necessarily be algebraic integers?

5. The norm $N(\alpha)$ of an element $\alpha = a_1 + \sqrt{m}a_2 \in \mathbb{Q}(\sqrt{m})$ where $a_1, a_2 \in \mathbb{Q}$ is defines by the formula $N(\alpha) = a_1^2 - ma_2^2$. Prove that the following assertion is false in $\mathbb{Q}(\sqrt{-1}) = \mathbb{Q}(i)$: If $N(\alpha) \in \mathbb{Z}$ then α is an algebraic integer.

6. If a polynomial $f(x)$ with integral coefficients factors into a product $g(x)h(x)$ of two polynomials with coefficients in $\mathbb{Q}$, prove that there is a factoring $f(x) = g_1(x)h_1(x)$ with integral coefficients.