COMPLEX ANALYSIS-II HOMEWORK: HINTS, COMMENTS, AND FURTHER PROBLEMS

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HW1

1. Cross-ratios and symmetries of the four-punctured spheres

To verify invariance of the cross-ratios under the Möbius group, it is enough to check that the affine maps $z \mapsto az + b$ and the map $z \mapsto 1/z$ preserve the cross-ratios.

Given a tuple $\mathbf{z} = (z_1 z_2 z_3 z_4)$, consider the Möbius transformation

$$M_{\mathbf{z}} : z \mapsto \frac{(z - z_2)(z_3 - z_4)}{(z - z_4)(z_3 - z_2)}$$

that maps $\mathbf{z}$ to $(\tau\, 0\, 1\, \infty)$ with $\tau = [z_1 z_2 z_3 z_4]$. If the tuples $\mathbf{z}$ and $\boldsymbol{\zeta}$ have the same cross-ratios, then consider $M_{\boldsymbol{\zeta}}^{-1} \circ M_{\mathbf{z}}$.

For tuples $(z_1 z_2 z_3\, \infty)$, the cross-ratio becomes a simple *ratio*

$$\frac{z_1 - z_2}{z_3 - z_2}.$$

Letting $(z_1 z_2 z_3)$ be various six permutations of $(\tau\, 0\, 1)$, we obtain six listed values of the cross-ratio.

Any permutation of $(0\, 1,\, \infty)$ is induced by a unique Möbius transformation (which are exactly six listed transformations). This gives a natural isomorphism between the group of symmetries of $\mathbb{C} \smallsetminus \{0,\, 1\}$ and S_3.

Show that this action is conjugate to the group of symmetries of the equilateral triangle in the Euclidean sphere $S^2 \subset \mathbb{R}^3$. [That is, the group of Euclidean rotations of S^2 preserving the triangle.]

The quotient of $\mathbb{C} \smallsetminus \{0,\, 1\}$ modulo this action is the plane. This can be seen is several ways:

- Compactify $\mathbb{C} \smallsetminus \{0,\, 1\}$ to the Riemann sphere and show, by the Riemann-Hurwitz formula, that $\chi(\hat{\mathbb{C}}/S_3) = 2$. Then use the Uniformization Theorem (which is certainly a heavy artillery for this matter).

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• Instead, one can get away with the Riemann Mapping Theorem only. Consider the triangle Δ bounded by the real interval $[0, 1/2]$, vertical interval $[1/2, e^{i\pi/3}]$ and the arc of the circle $\{|z-1|=1\}$. Show that $\Delta \cup \bar{\Delta}$ is a fundamental domain for our group (draw the whole tiling of the sphere). Map Δ to the upper half-plane by the Riemann Mapping, and extend it to the whole sphere by means of the Schwarz Reflections. We obtain a degree six rational map $\phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ automorphic with respect to the action of S_3. It can be identified with the quotient map $\hat{\mathbb{C}} \to \hat{\mathbb{C}}/S_3$.

• You can also try to find an *explicit* algebraic formula for ϕ.

Any permutation $\sigma \in S_4$ transforms the cross-ratio as described, which induces an action $\rho : S_4 \to S_3$ on $\mathbb{C} \smallsetminus \{0, 1\}$ by Möbius transformations. The kernel of this action is the *Klein group* $K = \mathbb{Z}_2 \times \mathbb{Z}_2$: it preserves the cross ratio of *any* tuple $\mathbf{z} = (z_1, \dots, z_4)$. This means that all $\sigma \in K$ are induced by Möbius automorphisms of $\mathbb{C} \smallsetminus \{z, \dots, z_4\}$. So, K is contained in the symmetry group of any configuration of four points. *Realize it as a group of rotations preserving four points* (compare with Problem 1 in HW3).

In general, the symmetry group of a configuration $\{z_1, \dots, z_4\}$ projects onto the stabilizer of τ in S_3. Hence configurations with extra symmetries (up to conformal equivalence) correspond to the fixed points of S_3 acting on $\mathbb{C} \smallsetminus \{0, 1\}$ (or rather, to the orbits of these pts). Thus, there are exactly two special configurations: the orbit $\{1/2, 2, 1\}$ corresponds to the configuration with symmetry group D_8 (the index 2 extension of K), while $e^{\pm i\pi/3}$ corresponds to the one with symmetry group A_4 (the index 3 extension of K). *Realize these groups as rotation groups of the sphere.*

2. Holomorphic index formula and fixed points of rational functions.

In the local coordinate $\zeta = 1/z$ near ∞, the differential

$$\omega = \sum_{n=N}^{-\infty} a_n z^n dz$$

assumes the form

$$- \sum_{n=N}^{-\infty} a_n \zeta^{-(n+2)} d\zeta = - \sum_{m=-(N+2)}^{\infty} a_{-(m+2)} \zeta^m d\zeta. \tag{0.1}$$

So, ω has pole of order $N+2$ at ∞ with residue $-a_{-1}$.

If ω is holomorphic in $\mathbb{C}$ then it has a form $\phi(z)dz$ where ϕ is an entire function. But then ω has a singularity at ∞. (which is a pole iff ϕ is a polynomial). Moreover, by (0.1), this pole has order at least 2.

Take a circle γ surrounding all finite poles. Then

$$\sum_{\text{finite poles}} \operatorname{Res}_a \omega = \frac{1}{2\pi i}\int_\gamma \omega = -\operatorname{Res}_\infty \omega$$

(the "-" sign appeared because γ viewed from ∞ is negatively oriented). So the sum of the residues over all poles (including ∞) vanishes.

In fact,

$$\sum_{poles} \operatorname{Res}_a \omega = 0. \tag{0.2}$$

is true for any meromorphic differential ω on any closed Riemann surface S. To see it, let us surround the poles $a_i \in S$ with small circles γ_i, and let $\Omega = S \smallsetminus \cup D_i$ be the complement of the corresponding disks D_i (bounded by the γ_i). Then ω is holomorphic at hence closed in Ω. By Stokes Theorem,

$$\sum \int_{\gamma_i} \omega = -\int_\Omega d\omega = 0.$$

(Here γ is negatively oriented as viewed from Ω.) But (by definition)

$$\frac{1}{2\pi i}\int_{\gamma_i} \omega = \operatorname{Res}_{a_i} \omega,$$

and (0.2) follows.

Note that *(0.2) implies in this generality that ω cannot have a single simple pole.*

If we conjugate f by a Möbius transformation $\phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, we obtain a degree d rational function $g = \phi \circ f \circ \phi^{-1}$ with $\operatorname{Fix}(g) = \phi(\operatorname{Fix}(f))$. So, without loss of generality w can assume that ∞ is not a fixed point of f, i.e, $\deg P \le \deg Q = d$, where $f = P/Q$. Then $P(z) - zQ(z) = 0$ is a polynomial of degree $d+1$, which gives us $d+1$ fixed pts.

Remark 0.1. A fixed point is *multiple* if is a multiple root of the equation $f(z) - z = 0$. In this case, the multiplier λ is equal to 1 (not 0!).

Also, with this normalization, ∞ is a simple pole for

$$\omega := \frac{dz}{f(z) - z}$$

with residue 1. Finite poles of ω are the fixed pts of f; their residues are $1/(\lambda - 1)$. Applying (0.2), we obtain the Holomorphic Index Formula.

Let $z' = f(z)$ be a germ of analytic map near a fixed point a, and $\zeta = h(z)$ be another local coordinate. "To write f in this local chart" (respecting the notion of a fixed point) means to substitute z with ζ in *both*xs the source and the target, i.e., to let $z = h^{-1}(\zeta)$ and $z' = h^{-1}(\zeta')$. Then $\zeta' = g(\zeta)$, where $g = h \circ f \circ h^{-1}$. Moreover, $b := h(a)$ is the corresponding fixed point for g. Differentiating the equation $g \circ h = h \circ f$ at a, we get:

$$g'(b)\, h'(a) = h'(a)\, f'(a),$$

which implies invariance of the multiplier.

If $\lambda \in \mathbb{D} \setminus \{1\}$ then $\mathrm{Re}\, 1/(\lambda - 1) \leq -1/2$. So, if $d \geq 2$ and all fixed pts have multipliers in $\mathbb{D} \setminus \{1\}$, then

$$\mathrm{Re} \sum \frac{1}{\lambda - 1} \leq -\frac{3}{2}$$

contradicting the Holomorphic Index Formula.

3. Weierstrass $\mathcal{P}$-function

Tile the plane with annuli $A_n = \{n - 1 < |z| \leq n\}$. The number of lattice points in A_n is $O(n)$ while each term of the series E_3 is bounded by n^{-3}. Hence the series is majorated by the convergent series

$$\sum \frac{1}{n^2}.$$

The Weierstrass $\mathcal{P}$-function $\mathbb{T}^2 \to \hat{\mathbb{C}}$ has degree two since 0 is its only pole, and it has order two. It is a degree two Galois branched covering with the $\mathbb{Z}_2$ Galois group generated by the reflection $z \mapsto -z$. The fixed points of this reflection $(0, \omega_1/2, \omega_2/2$, and $(\omega_1 + \omega_2)/2)$ are the critical pts of $\mathcal{P}$. The corresponding critical values are ∞ and the e_i, the roots of $z^3 - g_2 z - g_3$.

HW2

2. Annuli.

Uniqueness of the closed hyperbolic geodesic can be seen as follows: Any closed hyperbolic geodesic lifts to a geodesic in $\mathbb{H}$ invariant under γ_λ, and there is only one such geodesic. Its length is equal to

$$h = |\log \lambda|.$$

There are several way of seeing that $\mathrm{mod}\, C_h$ is a conformal invariant:

1) Since conformal maps are hyperbolic isometries, they preserve the hyperbolic length h of the closed geodesic, and hence preserve $\mathrm{mod}\, C_h = \pi/h$.

2) A conformal equivalence between the annuli lifts to a Möbius transformation $A : \mathbb{H} \to \mathbb{H}$ conjugating the deck transformations γ_{λ_1} and γ_{λ_2} (first on $\mathbb{H}$, and then, by analytic continuation, on the whole $\hat{\mathbb{C}}$). Each of these transformations has two fixed points, one attracting and one repelling, with multipliers λ_i and $1/\lambda_i$. Under the conjugacy, the attracting fixed pt is mapped to the corresponding attracting one. Since conformal conjugacies preserve multipliers of fixed pts, we conclude that $\lambda_1 = \lambda_2$ or $\lambda_1 = 1/\lambda_2$.

3) Working with the round annuli, assume $\phi : A \to A'$ is a conformal isomorphism. *Use the Schwarz Reflection and Removability of Isolated Singularities to extend it to a linear map $z \mapsto \rho z$.*

3. The punctured disk.

The exponential map $z \mapsto e^{2\pi i z}$ provides a covering $\pi : \mathbb{H} \to \mathbb{D}^*$ with the deck group generated by the parabolic Möbius transformation $z \mapsto z + 1$. The hyperbolic metric on $\mathbb{H}$ pushes down to the hyperbolic metric on $\mathbb{D}^*$:

$$d\rho = -\frac{dz|}{|z| \log |z|}$$

The horoballs have infinite diameter but *finite* area. It can be calculated in $\mathbb{D}^*$ using the above expression, or directly in the upper half-plane as

$$\int_0^1 \int_t^\infty \frac{dxdy}{y^2}, \quad t > 0.$$

A closed geodesic in $\mathbb{D}^*$ would lift to a geodesic in $\mathbb{H}$ invariant under the translation $z \mapsto z + 1$, and there are no such.

Here are several ways of seeing that $\mathbb{D}^*$ is not conformally equivalent to an annulus:

1) One has a closed geodesic, the other one does not.

2) Otherwise a hyperboloc deck transformation $z \mapsto \lambda z$ would be conjugate to the parabolic one $z \mapsto z + 1$ by a Möbius map. But they are not as one has two fixed points, while the other one has only one (or, you can use again invariance of the multiplier).

3) By Removability of isolated singularities, a conformal equavalence $\phi : \mathbb{D}^* \to A$ would extend through 0. Then $\phi(0) \in \partial A$, and a neighborhood U of 0 would be mapped onto a neighborhood of $\phi(0)$, so it would not fit inside A.

Any hyperbolic Riemann surface with $\pi_1 = \mathbb{Z}$ is a quotient of $\mathbb{H}$ by a cyclic group $< T >$. If T is hyperbolic, we obtain an annulus, if it is parabolic, we obtain $\mathbb{D}^*$.

However there is also a *parabolic* Riemann surface diffeomorphic to the annulus, namely $\mathbb{C}^*$. [*Prove that it is not conformally equivalent to the above.*]

3. Hyperelliptic surfaces.

The Riemann surface S is obtaied by gluing together two sheets with k slits where $k = [(d+1)/2]$. For even degree d, these slits connect pairs of the roots. In this case, S is unbranched over ∞. For odd degree, one of the slits goes to ∞, and S has a branched pt over there.

Each sheet is a sphere with k holes. Gluing together the boundaries of two holes, we obtain a sphere with $2(k-1)$ holes. Gluing together a pair of the other holes amounts to adding a handle to the sphere, so we obtain a surface of genus $g = k - 1$.

In the even case, the local coordinate on each of the two sheets near ∞ is $\zeta = 1/z$, and $\omega \sim \zeta^{k-2}\, d\zeta$ on each sheet. So, ω has two roots of order $k-2$.

In the odd case, the local coordinate is $\zeta = 1/\sqrt{z}$, and $\omega \sim \zeta^{d-3}\, d\zeta$. So, it has one root of order $d - 3 = 2k - 4$.

In either case, the total number of roots is equal to $2k - 4 = -\chi(S)$, as it should be.

In the case of degree 5 polynomial with real roots, the Abelian integral $\int \omega$ maps each half-plane onto a hexagon with 5 angles $\pi/2$ and one angle $3\pi/2$ (corresponding to ∞). Gluing together two symmetric hexgons, we obtain one sheet of S. As S is obtained by gluing together two sheets, we need four hexagons. [*Explore carefully combinatorics of the gluing!*]

This flat structure has one cone point at ∞ with angle $4 \times 3\pi/2 = 6\pi$. [So, the curvature at this point is $K = 2\pi - 6\pi = -4\pi = 2\pi\,\chi(S)$, as it should be according to Gauss-Bonnet.]

HW3

3. Finite subgroups of $\mathrm{PSL}(2,\mathbb{C})$.

Any finite subgroup G of $\mathrm{GL}(2,\mathbb{C})$ can be realized as a subgoup of $\mathrm{SU}(2)$. Indeed, take any Hermitian product $< x, y >$ in $\mathbb{C}^2$, and average it over the group:

$$(x,y) = \frac{1}{|G|} \sum_{g\in G} < gx, gy > .$$

We obtain a G-invariant Hermitian product. It the orthonormal basis, it assumes the standard form

$$(x,y) = z_1\bar\zeta_1 + z_2\bar\zeta_2,$$

and G becomes a finite subgroup of unitary matrices. It follows that finite subgroups of $\mathrm{PSL}(2,\mathbb{C})$ can be realized as subgroups of $\mathrm{PSU}(2) \approx \mathrm{SO}(3)$ (the group of rotations of S^2).

For instance, in HW1

• The group S_3 of Möbius automorphisms of $\mathbb{C} \setminus \{0,1\}$ can be realized as the group of rotations preserving the equilateral triangle;

• The Klein group $\mathbb{Z}^2 \times \mathbb{Z}^2$ of symetries of a $\mathbb{C}$ with three puctures can be realized as a group of rotation preserving a rectangle or a special tetrahedron (*which one?*);

• The dihedral group D_8 is the group of rotations preserving the square;

• The alternating group A_4 is the group of rotations of the right tetrahedron.

Other two examples are provided by groups of rotations of other four platonic bodies and by groups of rotations of right n-gones.

Is this the full list?

2. Cosine.

Here is three ways to identify the quotient $S := \mathbb{C}/\Gamma$ (where Γ is the dihedral group) (of course, the first one requires much more machinery than the second, while the last one is elementary):

1) S can be obtained by gluing two copies of the half-strip

$$\Pi = \{0 \le x \le 1/2,\ y \ge 0\}$$

along the boundary. It is clearly a topological plane.

The complex structure can be pushed forward from $\mathbb{C}$ to S (never mind Γ has fixed pts). By the Uniformization Theorem, it leaves for S two options: $\mathbb{C}$ or $\mathbb{D}$. But $\mathbb{C}/\Gamma$ is a cusp near $+i\infty$, so S can be compactified to the Riemann sphere. Hence it is $\mathbb{C}$.

2) Consider the Riemann mapping $\phi : \Pi \to \mathbb{H}$ and extend it to the whole complex plane by means of the Schwarz reflection.

3) Since $\cos : \mathbb{C} \to \mathbb{C}$ has exactly the same group of symmetries, it can be idetified with the quotient in question. (Of course, the above Riemann mapping ϕ, if appropriately normalized, is just cos restricted to Π.)

Since the $T_n : z \mapsto nz$ maps the lattice $\mathbb{Z}$ into itself and commutes with the involution $z \mapsto -z$, it descends to a continuous map $P_n : S \to S$. It is obviously holomorphic outside the critical values of cos. At the critical values P_n is holomorphic by Removability of isolated singularitis (or by using explicitly the local coordinate $\sqrt{z}$ on S). Hence it becomes a polynomial $P_n : \mathbb{C} \to \mathbb{C}$ (once S is identified with $\mathbb{C}$).

The P_n are called *Chebyshev polynomials*.

The critical values of $P_n \circ \cos$ are equal to the critical values of $\cos 2\pi nz$, which are ± 1. Hence the critical values of P_n are contained in $\{\pm 1\}$. However, for $n = 2$, there is only one critical value, -1.

Explicitly, the P_n can be found from the Moivre formula:

$$\cos nz = \Re(\cos z + i \sin z)^n.$$

HW4

3. Schwarzian derivative.

To verify the chain rule, it is convenient to normalize both f and g (by composing with translations) so that $z = 0$, and $f(0) = g(0) = 0$, and then further bring f to the normal form

$$f(z) = z + \frac{Sf(0)}{6} z$$

by post-composing it with a Möbius transformation.

Solutions of non-linear equation can blow up in finite time! (Consider $z' = z^2$.) So, even though the Schwarzian equation $Sf = g$ has local solutions in the domain of analyticity of g, it is not obvious that it can be extended along any path. However, *linear equations can always be solved globally*, so the relation of the S-equation to the linear equation saves the day.

A fundamental system of solutions for the linear equation $z'' + \frac{c}{2} z = 0$ is $\sin \omega z$ and $\cos \omega z$ (with $\omega = \sqrt{c/2}$). Hence the general solution of $Sf = c$ is

$$f(z) = \frac{a \operatorname{tg} \omega z + b}{c \operatorname{tg} \omega z + d}.$$

3. Hyperbolic metric on plane domains.

We know explicitly the universal covering $\lambda : \mathbb{D} \to \mathbb{C} \smallsetminus \{0, 1\}$: It is a modular function whose group of deck transformations is the congruence subgroup $\Gamma(2)$. The latter is generated by $T : z \mapsto z + 2$ and $S : z \mapsto \dfrac{z}{2z+1}$. Moreover, the upper half-plane $\mathbb{H}_1 = \{\operatorname{Im} z > 1\}$ is *completely invariant* under the cyclic group $< T >$, meaning that $T^{\pm 1}(\mathbb{H}_1) = \mathbb{H}_1$, while $R(\mathbb{H}_1) \cap \mathbb{H}_1 = \emptyset$ for any $R \neq T^n$, $n \in \mathbb{Z}$. It follows that $\lambda(\mathbb{H}_1) \approx \mathbb{H}_1 / < T > \approx \mathbb{D}^*$. Hence the hyperbolic metric on $\lambda(\mathbb{H}_1)$ behaves as the one on the punctured disk.

The second part follows from the Schwarz Lemma applied to the embedding $U \to \hat{\mathbb{C}} \smallsetminus \{a, b, c\}$, where a, b and c are three pts in the complement of U.

HW6

3. Meromorphic functions on E.

If f is even then $f(z)$ depends only on $\mathcal{P}(z)$, so $f(z) = R(\mathcal{P}(z))$ for some function $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. This function is continuous everywhere and meromorphic outside the critical values of $\mathcal{P}$. By removability of singularities, it is meromorphic everywhere.

If f is odd then $f/\mathcal{P}'$ is even, and the previous applies.

HW7

3. Symmetries of hyperbolic Riemann surfaces.

Since automorphisms of hyperbolic Riemann surfaces are hyperbolic isometries, they form an equicontinuous family of maps $S \to S$. This implies compactness (even in the uniform topology). [Hyperbolicity is important here: the result is false for $S = \hat{\mathbb{C}}$.]

Any automorphism $A : S \to S$ lifts to the unifersal covering giving a Möbius transformation $\tilde{A} : \mathbb{H} \to \mathbb{H}$. Moreover, for any deck transforamtion $\gamma \in \Gamma \approx \pi_1(S)$, we have:

$$\tilde{A} \circ \gamma \circ \tilde{A}^{-1} = A_*(\gamma),$$

where A_* is an automorphism induced in π_1. If A is homotopic to id then A_* is an inner automorphism, which can be normalized to be id. Then $\tilde{A}$ commutes with Γ.

Any element $\gamma \in \Gamma$ is hyperbolic (since S does not have cusps) and hence has two fixed points. As $\tilde{A}$ commutes with γ, it either fixes or permutes these pts. Since $\tilde{A}$ is equivariantly homotopic to id, it must fix the pts.

Since Γ is not cyclic, there are two elements in it that do not share the fixed points. So, $\tilde{A}$ fixes at least three points and hence $\tilde{A} = \mathrm{id}$.

It follows that $\mathrm{Aut}(S)$ is discrete and hence finite.