Problem Set #3

Solutions

Problem: Let $\alpha(s)$ be a regular curve, parameterized by arclength, such that $\kappa(s) \neq 0$ and $\tau(s) \neq 0$ for all s.

(a) Prove that if α lies on the sphere of radius r, centered at p, then

$$\frac{\tau}{\kappa} = \left(\frac{\kappa'}{\tau\kappa^2}\right)'$$

(b) Prove that the center of the sphere, p, satisfies

$$p = \alpha(s) + \frac{1}{\kappa(s)}N(s) + \frac{\kappa'(s)}{\tau(s)\kappa^2(s)}B(s)$$

for all s.

(c) Prove the converse of part (a).

Solution: Assume that α lies on the surface of the sphere. That means that the distance between any point on the curve, $\alpha(s)$, and the center, p, is equal to r. Symbolically,

$$(\alpha(s) - p) \cdot (\alpha(s) - p) = r^2$$

for all s.

Notice that the equation that we are trying to derive depends on the second derivative of κ. Since curvature is, itself, computed in terms of the second derivative of α, the right hand side of the desired equation depends on the fourth derivative of α. Therefore, no matter what we do, we must compute at least four derivatives.

Differentiating once with respect to s, we get $\alpha'(s) \cdot (\alpha(s) - p) + (\alpha(s) - p) \cdot \alpha'(s) = 0$, which simplifies to

$$T \cdot (\alpha(s) - p) = 0 \tag{1}$$

Differentiating again, we get $T' \cdot (\alpha(s) - p) + T \cdot T = 0$. Using the Frenet equation $T' = \kappa N$, and the fact that T is a unit vector, this reduces to

$$N \cdot (\alpha(s) - p) = -\frac{1}{\kappa} \tag{2}$$

Differentiating a third time, we have $N' \cdot (\alpha(s) - p) + N \cdot T = (-1/\kappa)'$. The left hand can be simplified by substituting $N' = -\kappa T - \tau B$, and by using the fact that N and T are orthogonal, to get

$$(-\kappa T - \tau B) \cdot (\alpha(s) - p) = \left(-\frac{1}{\kappa}\right)' = \frac{\kappa'}{\kappa^2}$$

We can use equation (1) to further simplify this equation down to

$$B \cdot (\alpha(s) - p) = -\frac{\kappa'}{\kappa^2\tau} \tag{3}$$

Differentiating a fourth, and final, time, we get $B' \cdot (\alpha(s) - p) + B \cdot T = -(\kappa'/(\kappa^2\tau))'$. Using the Frenet equations once again, and the fact that B and T are orthogonal, this reduces to

$$\tau N \cdot (\alpha(s) - p) = -\left(\frac{\kappa'}{\kappa^2\tau}\right)'$$

Substituting equation (2) into the left hand side finishes part (a).

In order to answer part (b), notice that equations (1–3) give the projections of the vector $\alpha(s) - p$ onto the orthonormal basis $\{T, N, B\}$. We can reconstruct any vector from its projections, so

$$\alpha(s) - p = 0T - \frac{1}{\kappa}N - \frac{\kappa'}{\kappa^2\tau}B$$

which, once rearranged, is the answer. (If you are unsatisfied by this, write $\alpha(s) - p = aT + bN + cB$ and solve for the coefficients a, b, and c, as we did in the proof of the Frenet–Serrat Theorem.)

The proof of part (c) is not yet written.