Problem Set #14

Solution to section 4.5 # 5

Solution: We have seen that when you parallel transport a vector around a closed loop, the vector does not come back to itself. I tried to argue in class that the amount that the vector is rotated, the *defect angle*, is a measure of the total Gaussian curvature in the region bounded by the loop. Moreover, you can use this idea to compute the curvature at a point, p, by computing the defect angle for any infinite family of shrinking loops containing p.

This problem makes that idea explicit by asking you to perform the calculation on a sphere. We take p to be the north pole use as our family of curves the parallels, C_ϕ. For each C_ϕ, we start with a vector tangent to the curve and parallel transport it around the circle.

In order to compute the defect angle, we follow the procedure of example 1 on page 243. The crucial observation is the following: the covariant derivative of a vector field depends only on the curve, the vector field, and the tangent planes to the surface along the curve. In particular, if two surfaces are tangent to each other along the curve, then the covariant derivatives for the two surfaces will be equal. Therefore, in order to compute the defect angle, we can replace the sphere with a cone which is tangent to C_{ϕ_0}. The advantage of this is that the cone is locally isometric to the plane, where the covariant derivative is easy to compute. I will not repeat their calculation here. I will only use the result, which is that the defect angle is

$$\Delta\phi = 2\pi - \theta = 2\pi - 2\pi\sin\psi = 2\pi - 2\pi\sin(\pi/2 - \phi_0) = 2\pi(1 - \cos\phi_0)$$

(It is worth noting that the concept of parallelism is, in fact, older than the covariant derivative. It was originally defined by precisely this procedure: replace the surface with a flat one which is tangent to the orginal surface, unroll the flat surface, and compute the parallel transport in the plane.)

In order to complete the problem, we still need to compute the area, A, bounded by the parallel, C_ϕ. There are many ways to do this. For example, it can be done with elementary calculus. However, for the sake of exposition, I will show how to use the area formulas from section 2.5. We begin by parameterizing the sphere by

$$\mathbf{x}(\theta, \phi) = (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$$

so that the first fundamental form is given by

$$\mathfrak{g} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\phi \end{pmatrix}$$

We defined the area of a region, R, to be

$$A = \iint_R dA = \iint_{\mathbf{x}^{-1}(R)} \sqrt{\det\mathfrak{g}}\, d\theta d\phi$$

which, when R is the portion of the upper hemisphere bounded by C_{ϕ_0} evaluates to

$$A = \int_0^{\phi_0}\int_0^{2\pi} \sin\phi d\theta d\phi = 2\pi(1 - \cos\phi_0)$$

Finally,

$$\lim_{R \to p} \frac{\Delta \phi}{A} = \lim_{\phi_0 \to 0} \frac{2\pi(1 - \cos \phi_0)}{2\pi(1 - \cos \phi_0)} = 1$$

as expected.